Heap Games, Numeration Systems and Sequences

Aviezri S. Fraenkel

Department of Applied Mathematics and Computer Science
Weizmann Institute of Science
Rehovot 76100, Israel
fraenkel@wisdom.weizmann.ac.il
http://www.wisdom.weizmann.ac.il/~ fraenkel

Abstract. We propose and analyse a 2-parameter family of 2-player games on two heaps of tokens, and present a strategy based on a class of sequences. The strategy looks easy, but is actually hard. A class of exotic numeration systems is then used, which enables us to decide whether the family has an efficient strategy or not. We introduce yet another class of sequences, and demonstrate its equivalence with the class of sequences defined for the strategy of our games.

1. An Example

Given a two-player game played on two heaps (piles) of finitely many tokens. There are two types of moves: I. Take any positive number of tokens from *one* heap, possibly the entire heap. II. Take from *both* heaps, k from one and l from the other, with, say, $k \le l$. Then the move is constrained by the condition $0 < k \le l < 2k+2$, which is equivalent to $0 \le l-k < k+2,\ k > 0$. The player making the last move (after which both heaps are empty) wins, and the opponent loses.

A position q in a game of this sort is called a P-position, if the *P*revious player can win, i.e., the player who moved to q. It is an N-position, if the *N*ext player can win, i.e., the player moving from q. The position $(0,0)$ (two empty heaps) is a P-position, since the first player cannot even make a move, so the second wins by default. The next P-position is $(1,4)$: if Jean takes an entire heap, then Gill takes the other and wins. If Jean takes any part of the larger heap, Gill can take the balance of both heaps. Lastly, Jean cannot remove both heaps, and if she takes from both heaps, then Gill takes the balance and wins.

Table 1 lists the first few P-positions. The reader will do well to try and construct the next few entries of the table before reading on.

If S is any finite subset of nonnegative integers, denote by $\operatorname{mex} S$ the least nonnegative integer in the complement of S, i.e., the least nonnegative integer not occurring in S. Note that the mex of the empty set is 0. The term mex, used in

Table 1. The first few P-positions.

n	A_n	B_n
0	0	0
1	1	4
2	2	8
3	3	12
4	5	18
5	6	22
6	7	26
7	9	32
8	10	36
9	11	40
10	13	46
11	14	50
12	15	54
13	16	58

[BCG1982], stands for Minimum EXcluded value. The structure of Table 1 is made explicit by:

$$A_n = \text{mex}\{A_i, B_i : i < n\},\ B_n = 2(A_n + n) \quad (n \geq 0).$$

This is a special case of Theorem 1 below, in the proof of which we also see that if $A = \bigcup_{n=1}^{\infty} A_n$, $B = \bigcup_{n=1}^{\infty} B_n$, then A and B are *complementary*, i.e., $A \cup B =$ set of all positive integers, and $A \cap B = \emptyset$.

Given any two heaps of our game, containing x and y tokens with $x \leq y$. The complementarity of A and B implies that either $x = A_n$ or $x = B_n$ for some n. Hence Table 1 has to be computed only up to the encounter of x. Moreover, it is not hard to see that $n \leq x$, and if $x = A_n$, then $x/2 < n$, so the table has to be computed up to at most $\Omega(x)$, which implies a strategy computation linear in x, which looks good.

The trouble with this strategy is the same as that of the simple-minded primality-testing algorithm for a given integer m: divide m by the integers $\leq \sqrt{m}$, and if none of them divides m, then m is prime. This algorithm is linear in m. The problem in both cases is of course that the input length is the log of the input numbers x, y and m, rather than x, y and m themselves.

The two algorithms mentioned above, that for the strategy computation, and that for primality testing are thus actually exponential in the input length.

The central question we address here is whether games of the type considered above have a polynomial strategy, or whether their best strategies are necessarily exponential. Before that we define in §2 the family of games precisely, introduce a family of sequences, and formulate and prove the winning strategy in terms of these sequences.

In §3 we present an argument against polynomiality of the games, and in §4 we introduce a numeration system that turns out to be relevant to our games. The connection between the games and the numeration system is made explicit in §5. This enables us to decide the games' polynomiality question in §6. Yet another class of sequences is introduced in §7, where we prove equivalence between the two classes of sequences. In the final §8 we summarize our results, give motivation and present a few open problems.

2. A Family of Heap Games and their Winning Strategies

Denote by $\mathbb{Z}^0$ and $\mathbb{Z}^+$ the set of nonnegative integers and positive integers respectively. Our family of heap games depends on two parameters $s, t \in \mathbb{Z}^+$. Given are two heaps of finitely many tokens. There are two types of moves: I. Take any positive number of tokens from a single heap, possibly the entire heap. II. Take $k > 0$ and $l > 0$ from the two heaps, say $0 < k \le l$. This move is constrained by the condition

$$0 < k \le l < sk + t, \tag{1}$$

which is equivalent to $0 \le l - k < (s-1)k + l, \quad k \in \mathbb{Z}^+$.

The example presented in §1 is the special case $s = t = 2$. Denote by $\mathcal{P}$ the set of all P-positions.

Theorem 1. $\mathcal{P} = \bigcup_{i=0}^{\infty}\{(A_i, B_i)\}$, *where*

$$A_n = \text{mex}\{A_i, B_i : 0 \le i < n\}, \; B_n = sA_n + tn \quad (n \in \mathbb{Z}^0). \tag{2}$$

Proof. Let $A = \bigcup_{n=1}^{\infty} A_n$, $B = \bigcup_{n=1}^{\infty} B_n$. Then A, B are complementary with respect to $\mathbb{Z}^+$: $A \cup B = \mathbb{Z}^+$ follows from the mex property. Suppose that $A_m = B_n$. Then $m > n$ implies that A_m is the mex of a set containing $B_n = A_m$, a contradiction. If $m \le n$, then $B_n = sA_n + tn \ge sA_m + tm > A_m$, another contradiction. Thus $A \cap B = \emptyset$. We will also need the fact that A_n and B_n are strictly increasing sequences, which is clear from their definition.

Let $W = \bigcup_{i=0}^{\infty}(A_i, B_i)$. It evidently suffices to show two things: I. A player moving from some $(A_n, B_n) \in W$ lands in a position not in W. II. Given any position (x, y) not in W, there is a move to some $(A_n, B_n) \in W$.

I. A move of the first type from $(A_n, B_n) \in W$ clearly leads to a position not in W, since A_n and B_n are strictly increasing, so they have no repeating terms. Suppose that a move of the second type from $(A_n, B_n) \in W$ produces a position $(A_m, B_m) \in W$. Then $m < n$. For $k = A_n - A_m$, $l = B_n - B_m$, we have

$$l = sA_n + tn - sA_m - tm = s(A_n - A_m) + t(n - m) \ge sk + t,$$

which contradicts condition (1).

II. Let (x, y) with $x \le y$ be a position not in W. Since A and B are complementary, every positive integer appears exactly once in exactly one of A and B. Therefore we have either $x = B_n$ or else $x = A_n$ for some $n \ge 0$.

Case (i): $x = B_n$. Then move $y \to A_n$.

Case (ii): $x = A_n$. If $y > B_n$, then move $y \to B_n$. So suppose that $A_n \le y < B_n$. If $y < sA_n + t$, move $(x, y) \to (0, 0)$, which satisfies (1) with $k = A_n$, $l = y$. So let $y \ge sA_n + t$. Put $m = \lfloor (y - sA_n)/t \rfloor$, and move $(x, y) \to (A_m, B_m)$, where $\lfloor x \rfloor$ denotes the largest integer $\le x$. This move is legal, since (a) $m < n$, (b) $y > B_m$, (c) $A_n - A_m \le y - B_n < s(A_n - A_m) + t$. Indeed,

(a) $y - sA_n < B_n - sA_n = tn$, so $m = \lfloor (y - sA_n)/t \rfloor \le (y - sA_n)/t < n$;
(b) $m \le (y - sA_n)/t$, so $y \ge tm + sA_n = B_m + s(A_n - A_m) > B_m$;
(c) $m > ((y - sA_n)/t) - 1$, so $y < tm + sA_n + t$; by (b), $y - B_m \ge A_n - A_m$, hence

$$A_n - A_m \le y - B_m < tm + sA_n + t - sA_m - tm = s(A_n - A_m) + t,$$

and (1) is satisfied. ■

The *statement* of Theorem 1 enables one to decide whether any given position (x, y) is a P-position or N-position, and the *proof* clearly indicates a winning move from any N-position. These two things together constitute a winning strategy for the game.

However, as was pointed out in §1 after Table 1, the strategy is exponential (the inequalities for x hold for all $s, t \in \mathbb{Z}^+$, not just for the special example considered there). But only the construction of the table needs exponential time and, in fact, exponential space. The rest of the algorithm is polynomial.

3. An Argument Against Polynomiality

Suppose that there exist real numbers α and β such that for the A_n and B_n defined in Theorem 1, $A_n = \lfloor n\alpha \rfloor$, and $B_n = \lfloor n\beta \rfloor$ for all $n \in \mathbb{Z}^0$. A simple density argument then shows that α and β must satisfy $\alpha^{-1} + \beta^{-1} = 1$, hence $1 < \alpha < 2 < \beta$, and α, β are in fact irrational.

A strategy based on this observation can be applied to any given game position (x, y). We have

$$x = \lfloor n\alpha \rfloor \iff x < n\alpha < x + 1$$
$$\iff \frac{x}{\alpha} < n < \frac{x+1}{\alpha} \iff \left\lfloor \frac{x+1}{\alpha} \right\rfloor = \left\lfloor \frac{x}{\alpha} \right\rfloor + 1.$$

Therefore either $x = \lfloor n\alpha \rfloor = A_n$ where $n = \lfloor (x+1)/\alpha \rfloor$, or else, by complementarity, $x = \lfloor n\beta \rfloor = B_n$, where $n = \lfloor (x+1)/\beta \rfloor$. We have thus reduced the situation to that considered in cases (ii) and (i) in the proof of Theorem 1, and hence the strategy presented in that proof can be followed. For example, if $x = \lfloor n\alpha \rfloor = A_n$ and $s\lfloor n\alpha \rfloor + t \le y < s\lfloor n\alpha \rfloor + tn = \lfloor n\beta \rfloor$, then for $m = \lfloor (y - s\lfloor n\alpha \rfloor)/t \rfloor$, we move $(x, y) \to (\lfloor m\alpha \rfloor, \lfloor m\beta \rfloor) \in \mathcal{P}$. For implementing this strategy, α has to be computed to a precision of $O(\log x)$ digits, and its storage requires $O(\log x)$ words, which is linear in the input size of x (given in binary, say). Thus this strategy *is* polynomial. See also the remark at the end of the previous section.

Is it far-fetched to hope for the existence of such real numbers α and β? Well, for the special case $s = t = 1$ our games reduce to Wythoff's game, for which such real numbers indeed exist, namely $\alpha = (1 + \sqrt{5})/2$, $\beta = (3 + \sqrt{5})/2$. This was already shown in [Wyt07]. See also [Cox53], [YaYa67]. In [Fra82] a generalization of Wythoff's game was proposed, namely the case of any $t \in \mathbb{Z}^+$, but $s = 1$. Also for this case these numbers exist, namely, $\alpha = (2 - t + \sqrt{t^2 + 4})/2$, $\beta = \alpha + t$.

We now show, however, that for $s > 1$, such real numbers cannot exist!

Theorem 2. *For A_n, B_n as defined in Theorem* 1, there exist real numbers α, γ, β, δ such that $A_n = \lfloor n\alpha + \gamma \rfloor$ and $B_n = \lfloor n\beta + \delta \rfloor$ for all $n \in \mathbb{Z}^0$, if and only if $s = 1$.

Proof. Since the sequence $\{A_n\}$ is strictly increasing,

$$B_{n+1} - B_n = sA_{n+1} + t(n+1) - sA_n - tn = s(A_{n+1} - A_n) + t \geq s + t \geq 2.$$

Since $B_n = sA_n + tn$, the sequence $\{B_n\}$ is nonempty. Since A and B are complementary, we thus cannot have $A_{n+1} - A_n = 1$ for all n. Therefore there exists n such that $A_{n+1} - A_n \geq 2$. Hence there is n for which $B_{n+1} - B_n \geq 2s + t \geq 3$. It follows that there is n for which $A_{n+1} - A_n = 1$. Since for all $n \in \mathbb{Z}^0$ we have $B_{n+1} - B_n \geq 2$, there can be no n for which $A_{n+1} - A_n > 2$. Hence $A_{n+1} - A_n \in \{1, 2\}$ and $B_{n+1} - B_n \in \{s + t, 2s + t\}$ for all $n \geq 0$.

Given a nondecreasing sequence of integers $S = a_1, a_2, \ldots$, the *spectrum* question is whether there exist real numbers α, γ such that $S = \lfloor n\alpha + \gamma \rfloor$. The spectrum terminology is used in [GLL78], where it is shown, for the *homogeneous* case ($\gamma = 0$), that if the prefix M_r of length r of S is "nearly linear", then it is the beginning of a spectrum. If it is, we'll say that M_r is *spectral*. In [BoFr81], necessary and sufficient conditions are given for M_r to be spectral in the (possibly) nonhomogeneous case. See also [BoFr84].

Let

$$\underline{d}(M_r) = \max_{1 \leq i < k \leq r} \frac{a_k - a_{k-i} - 1}{i}\ , \quad \overline{d}(M_r) = \min_{1 \leq i < k \leq r} \frac{a_k - a_{k-i} + 1}{i}\ .$$

One of the necessary and sufficient conditions for M_r to be spectral given in [BoFr81] is that $\underline{d}(M_r) < \overline{d}(M_r)$.

Put $a_k = B_{n+1}$, $a_{k-1} = B_n$. For any portion of length r of the sequence $\{B_n\}$ for which both the difference $2s + t$ and $s + t$ occurs, we then have, where we use the larger difference for $\underline{d}$ and the smaller for $\overline{d}$,

$$\underline{d}(M_r) \geq a_k - a_{k-1} - 1 = 2s + t - 1, \quad \overline{d}(M_r) \leq a_k - a_{k-1} + 1 = s + t + 1.$$

So a necessary condition for M_r to be spectral is $2s + t - 1 < s + t + 1$, which holds if and only if $s < 2$, i.e., $s = 1$. For $s = 1$ the sequence $\{B_n\}$ is indeed a spectrum, as remarked above.

The structure of $\{B_n\}$ implies that in $\{A_n\}$ there are runs of 1s of length $s + t - 2$, and $2s + t - 2$. An argument analogous to the above then leads to the necessary condition $2s + t - 3 < s + t - 1$, which again leads to $s = 1$, for which case indeed $\{A_n\}$ is a spectrum. ■

Thus the question whether our heap games have a polynomial strategy or not is still open for $s > 1$.

4. A Class of Exotic Numeration Systems

For u_{-1} a constant, $u_0 = 1$ and b_1, b_2 integers satisfying $b_1 \geq b_2 \geq 1$, consider the linear recurrence $u_n = b_1 u_{n-1} + b_2 u_{n-2}$ ($n \geq 1$). We can consider the $u_0, u_1, \ldots$ as bases of a numeration system with digits $d_i \in \{0, \ldots, b_1\}$. But then an integer such as u_n has two representations: u_n itself, and $b_1 u_{n-1} + b_2 u_{n-2}$. Since we would like to have uniqueness of representation, it is natural to require that $d_i = b_1 \implies d_{i-1} < b_2$ $(i \geq 1)$. It turns out that under this condition every positive integer m indeed has a unique representation. This is a special case of Theorem 2 in [Fra85]. Moreover, the greedy algorithm of repeatedly dividing m or its remainder by the largest u_i not exceeding this remainder, yields this unique representation. The case $b_1 = b_2 = 1$ gives a binary representation known as the Zeckendorf representation [Zec72].

Example. We consider the case $u_{-1} = \frac{1}{2}$, $(b_1, b_2) = (3, 2)$. Then $u_1 = 4$, $u_2 = 14$, $u_3 = 50$, $u_4 = 178$, $\ldots$. The representations of the integers 1 to 60 in this numeration system are displayed in Table 2.

A question we just might ask at this point is whether there is any connection between Tables 1 and 2. If we scan the first few entries of both, we may be tempted to conclude that the entries under A_n in Table 1 all have representation ending in no 0. But then 14 is a counterexample, whose representation ends in two 0s. Also it appears that the B_n all have representation ending in a single 0. But 50, with representation 1000 is a counterexample, in fact, the only counterexample in the range of the two tables.

However, there is no counterexample, as far as the two tables go, to the following two remarkable, æsthetically pleasing, properties:

a. All the A_n have representations ending in an *even* number of 0s, and all the B_n have representations ending in an *odd* number of 0s.

b. For every $(A_n, B_n) \in \mathcal{P}$, the representation of B_n is the "left shift" of the representation of A_n.

Thus $(1, 4)$ of Table 1 has representation $(1, 10)$, and $(6, 22)$ has representation $(12, 120)$: 10 is the "left shift" of 1, 120 the left shift of 12. We remark that the second part of **a** is not independent; it follows from its first part, since A and B are complementary.

In the next section we state these properties in a precise manner and prove their validity.

5. Wedding Numeration Systems with Heap Games

For fixed $s, t \in \mathbb{Z}^+$, put $u_{-1} = 1/s$, $u_0 = 1$, and let $u_n = (s + t - 1)u_{n-1} + su_{n-2}$ $(n \geq 1)$. Denote by $\mathcal{U}$ the numeration system with bases u_0, u_1, $\ldots$ and digits $d_i \in \{0, \ldots, s + t - 1\}$ such that $d_{i+1} = s + t - 1 \implies d_i < s$ $(i \geq 0)$. Every positive integer has a unique representation over $\mathcal{U}$, as mentioned in the previous section.

Notation and Definitions.

(a) For every $m \in \mathbb{Z}^0$ write $R(m)$ for the representation of m over $\mathcal{U}$.

(b) Denote by $LR(m)$ the "left shift" of $R(m)$, i.e., if $R(m) = \sum_{i=0}^{n} d_i u_i$, then $LR(m) = \sum_{i=0}^{n} d_i u_{i+1}$.

(c) A positive integer m is *EVen-taILed* (for short: *evil*), if $R(m)$ ends in an even (possibly 0) number of 0s. It is *Odd-taiLeD* (for short: *old*), if $R(m)$ ends in an odd number of 0s. It is convenient to let 0 be both evil and old.

(d) Put $q = s - 1$, and $r = s + t - 1$. Then the above recurrence has the form $u_{-1} = 1/s$, $u_0 = 1$, $u_n = ru_{n-1} + su_{n-2}$ $(n \geq 1)$; and the representation with digits $d_i \in \{0, \ldots, r\}$ satisfies $d_{i+1} = r \implies d_i \leq q$ $(i \geq 0)$.

Table 2. Representation of first few integers in $\mathbb{Z}^+$.

50	14	4	1	n	14	4	1	n
	2	0	3	31			1	1
	2	1	0	32			2	2
	2	1	1	33			3	3
	2	1	2	34		1	0	4
	2	1	3	35		1	1	5
	2	2	0	36		1	2	6
	2	2	1	37		1	3	7
	2	2	2	38		2	0	8
	2	2	3	39		2	1	9
	2	3	0	40		2	2	10
	2	3	1	41		2	3	11
	3	0	0	42		3	0	12
	3	0	1	43		3	1	13
	3	0	2	44	1	0	0	14
	3	0	3	45	1	0	1	15
	3	1	0	46	1	0	2	16
	3	1	1	47	1	0	3	17
	3	1	2	48	1	1	0	18
	3	1	3	49	1	1	1	19
1	0	0	0	50	1	1	2	20
1	0	0	1	51	1	1	3	21
1	0	0	2	52	1	2	0	22
1	0	0	3	53	1	2	1	23
1	0	1	0	54	1	2	2	24
1	0	1	1	55	1	2	3	25
1	0	1	2	56	1	3	0	26
1	0	1	3	57	1	3	1	27
1	0	2	0	58	2	0	0	28
1	0	2	1	59	2	0	1	29
1	0	2	2	60	2	0	2	30

We mention that in [BCG82, Ch. 4], "evil number" is used for a number whose binary expansion contains an even number of 1s (*even weight* in coding theory language).

Lemma 1. *For $m \in \mathbb{Z}^+$, let $R(m) = \sum_{i=0}^{n} d_i u_i$.*

(i) Suppose that for some $k \in \mathbb{Z}^0$, the tail of $R(m)$ has digits

$$d_{2k}d_{2k-1}d_{2k-2}\dots d_3d_2d_1d_0 = d_{2k}rq\dots rqrq,$$

where $d_{2k} \in \{0, \ldots, q\}$ and $d_{2k} = q \implies d_{2k+1} < r$. Then $R(m+1) = (d_{2k} + 1)u_{2k} + \sum_{i=2k+1}^{n} d_i u_i$, so $m+1$ is evil.

(ii) Suppose that for some $k \in \mathbb{Z}^0$, the tail of $R(m)$ has digits

$$d_{2k+1}d_{2k}d_{2k-1}\ldots d_2d_1d_0 = d_{2k+1}rq\ldots rqr,$$

where $d_{2k+1} \in \{0, \ldots, q\}$ and $d_{2k+1} = q \implies d_{2k+2} < r$. Then $R(m+1) = (d_{2k+1} + 1)u_{2k+1} + \sum_{i=2k+2}^{n} d_i u_i$, so $m+1$ is old.

Note. We point out the special case $k = 0$, where for (i), $d_0 \le q$ and $d_0 = q \implies d_1 < r$; for (ii), $d_1 \le q$ and $d_1 = q \implies d_2 < r$.

Proof. We note that the hypothesis on d_{2k} for (i) implies that $(d_{2k} + 1)u_{2k} + \sum_{i=2k+1}^{n} d_i u_i$ is a legal representation over $\mathcal{U}$. Similarly for (ii).

(i) By adding and subtracting u_0, we get

$$m = (qu_0 + ru_1) + (qu_2 + ru_3) + \cdots + (qu_{2k-2} + ru_{2k-1}) + d_{2k}u_{2k} + \sum_{i=2k+1}^{n} d_i u_i = (d_{2k} + 1)u_{2k} + \sum_{i=2k+1}^{n} d_i u_i - 1.$$

Thus $m + 1 = (d_{2k} + 1)u_{2k} + \sum_{i=2k+1}^{n} d_i u_i = R(m+1)$.

(ii) Adding and subtracting $su_{-1} = 1$, gives

$$m = ru_0 + (qu_1 + ru_2) + (qu_3 + ru_4) + \cdots + (qu_{2k-1} + ru_{2k}) + d_{2k+1}u_{2k+1} + \sum_{i=2k+2}^{n} d_i u_i = (d_{2k+1} + 1)u_{2k+1} + \sum_{i=2k+2}^{n} d_i u_i - 1.$$

Thus $m + 1 = (d_{2k+1} + 1)u_{2k+1} + \sum_{i=2k+2}^{n} d_i u_i = R(m+1)$. ■

Lemma 2. *Consider the set of pairs* $\bigcup_{k=0}^{\infty}(V_k, W_k)$*, where* $0 = V_0 < V_1 < \ldots$ *is the set of all evil numbers, and* $R(W_k) = LR(V_k)$ *for all* k*. Then* $W_k - sV_k = tk$ *for all* k*.*

Proof. Induction on k. The assertion holds trivially for $k = 0$. Suppose that $W_k - sV_k = tk$ for arbitrary fixed k. Let $R(V_k) = \sum_{i=0}^{n} d_i u_i$. Then

$$R(W_k) - sR(V_k) = LR(V_k) - sR(V_k) = \sum_{i=0}^{n} d_i(u_{i+1} - su_i),$$

so

$$tk = \sum_{i=0}^{n} d_i(u_{i+1} - su_i). \tag{3}$$

We consider three cases.

I. The tail of $R(V_k)$ is as in case (i) of Lemma 1. Then $V_k + 1$ is evil, so $V_{k+1} = V_k + 1$ and $R(V_{k+1}) = (d_{2k} + 1)u_{2k} + \sum_{i=2k+1}^{n} d_i u_i$. Thus

$$LR(V_{k+1}) - sR(V_{k+1}) = (d_{2k} + 1)(u_{2k+1} - su_{2k}) + \sum_{i=2k+1}^{n} d_i(u_{i+1} - su_i) = u_{2k+1} - su_{2k} + \sum_{i=2k}^{n} d_i(u_{i+1} - su_i). \tag{4}$$

For case (i) of Lemma 1 we have by (3),

$$tk = q(u_1 - su_0) + r(u_2 - su_1) + q(u_3 - su_2) + r(u_4 - su_3)$$
$$+ \cdots + q(u_{2k-1} - su_{2k-2}) + r(u_{2k} - su_{2k-1}) + \sum_{i=2k}^{n} d_i(u_{i+1} - su_i).$$

We sum together the positive terms, adding and subtracting u_1. Then $ru_2 + su_1 = u_3$ is added to $(s-1)u_3$, and so on, leading to $u_{2k+1} - u_1$. We then sum all the negative terms, subtracting and adding su_0, leading to $-su_{2k} + su_0$. Thus

$$tk = u_{2k+1} - u_1 - su_{2k} + su_0 + \sum_{i=2k}^{n} d_i(u_{i+1} - su_i)$$
$$= u_{2k+1} - su_{2k} - t + \sum_{i=2k}^{n} d_i(u_{i+1} - su_i).$$

Hence by (4),

$$t(k+1) = u_{2k+1} - su_{2k} + \sum_{i=2k}^{n} d_i(u_{i+1} - su_i) = LR(V_{k+1}) - sR(V_{k+1}),$$

as was to be shown.

II. The tail of $R(V_k)$ is as in case (ii) of Lemma 1. Then $V_k + 1$ is old, but $V_k + 2$ is clearly evil, since $R(V_k + 2)$ ends in 1, so $V_{k+1} = V_k + 2$. Then $V_{k+1} = 1 + (d_{2k+1} + 1)u_{2k+1} + \sum_{i=2k+2}^{n} d_i u_i$, so $R(V_{k+1}) = u_0 + (d_{2k+1} + 1)u_{2k+1} + \sum_{i=2k+2}^{n} d_i u_i$. Hence

(5) $$LR(V_{k+1}) - sR(V_{k+1}) = (u_1 - su_0) + (d_{2k+1} + 1)(u_{2k+2} - su_{2k+1})$$
$$+ \sum_{i=2k+2}^{n} d_i(u_{i+1} - su_i) = t + u_{2k+2} - su_{2k+1} + \sum_{i=2k+1}^{n} d_i(u_{i+1} - su_i).$$

For case (ii) of Lemma 1 we have by (3),

$$tk = r(u_1 - su_0) + q(u_2 - su_1) + r(u_3 - su_2) + q(u_4 - su_3)$$
$$+ \cdots + q(u_{2k} - su_{2k-1}) + r(u_{2k+1} - su_{2k}) + \sum_{i=2k+1}^{n} d_i(u_{i+1} - su_i).$$

Summing the positive terms, adding and subtracting su_0, leads to $u_{2k+2} - s$. Summing the negative terms, subtracting and adding $s^2 q_{-1} = s$, gives $-su_{2k+1} + s$. Thus $tk = u_{2k+2} - su_{2k+1} + \sum_{i=2k+1}^{n} d_i(u_{i+1} - su_i)$.

Thus by (5), $LR(V_{k+1}) - sR(V_{k+1}) = t(k+1)$, as required.

III. The digit d_0 satisfies $q < d_0 < r$. Since clearly $d_1 < r$, we have $V_{k+1} = V_k + 1$, so by (3),

$$LR(V_{k+1}) - sR(V_{k+1}) = (d_0 + 1)(u_1 - su_0) + \sum_{i=1}^{n} d_i(u_{i+1} - su_i)$$
$$= t + \sum_{i=0}^{n} d_i(u_{i+1} - su_i) = t(k+1).$$

Every evil number P is of one of the three forms considered above. Note that if P ends in an even number of 0s, it is of the form **I**. ■

Lemma 2 enables us to prove our main result.

Theorem 3. *For all* $n \in \mathbb{Z}^0$, $(V_n, W_n) = (A_n, B_n)$.

Proof. Since $(V_0, W_0) = (A_0, B_0) = (0, 0)$, it suffices to show that for $n > 0$, the numbers V_n, W_n have the same inductive formation laws as the numbers A_n, B_n. By Lemma 2, $W_n = sV_n + tn$, the same as the formation rule for B_n given in (2). It remains only to show that $V_n = \text{mex}\, S$, where $S = \{V_i, W_i : i < n\}$. Suppose that $\text{mex}\, S = W_j$. Clearly $j \ge n$. But then $V_j \in S$, since $V_j < W_j$. Hence $j < n$, a contradiction.

Now $\bigcup_{i=1}^{\infty} V_i$ and $\bigcup_{i=1}^{\infty} W_i$ are complementary, since every positive integer has precisely one representation in $\mathcal{U}$, either ending in an even number of 0s, as the V_n do, or in an odd number, as the W_n do, since the W_n are a left shift of the V_n. Therefore if $\text{mex}\, S \ne W_j$, we must have $\text{mex}\, S = V_n$, so the formation laws are the same. ■

6. The End of the Games Story

Theorem 3 enables us to decide our main question, whether or not our heap games have a polynomial strategy. Given any game position (x, y) with $0 < x \le y$, compute $R(x)$ using the greedy algorithm mentioned in the first paragraph of §4. If $R(x)$ ends in an odd number of 0s, then $x = B_n$ for some $n > 0$. Then move $y \to A_n$, where $R(B_n) = LR(A_n)$. If $R(x)$ ends in an even number of 0s, then $x = A_n$ for some $n > 0$. We can also test the relative size of y and B_n, since $R(B_n) = LR(A_n)$. This information suffices for deciding the game, as indicated in the proof of Theorem 1 (and used again in §3). So the complexity of this computation is, up to a multiplicative constant, that of computing $R(x)$.

The recurrence $u_n = ru_{n-1} + su_{n-2}$ has characteristic polynomial $x^2 - rx - s = 0$, with roots $\alpha = (r + \sqrt{r^2 + 4s})/2$, $\beta = (r - \sqrt{r^2 + 4s})/2$. Since $s > 0$, we have $\alpha > 1$. Since $s = r - t + 1 \le r$, we have $0 < -\beta \le (\sqrt{(r+2)^2 - 4} - r)/2 < 1$, so $|\beta| < 1$. Therefore $u_n = E(c\alpha^n)$ for some constant $c > 0$, where $E(v)$ is the nearest integer to the real number v. It follows that $n = O(\log x)$ bases of $\mathcal{U}$ suffice for computing the strategy. Thus this strategy is in fact *linear* in the input size.

7. Yet Another Class of Sequences

In addition to the class of sequences $\{A_n\}$ and $\{B_n\}$ defined in (2), we now define another class of three sequences, $Q = \{Q_n\}$, $\{A'_n\}$, $\{B'_n\}$ $(n \in \mathbb{Z}^0)$, also depending on positive integer parameters s, t.

(a) $Q_n = Q_m$ if $n = tQ_m + sm$ and Q_m has already occurred precisely once; else

$$Q_n = \text{mex}\{Q_m : 0 \le m < n\}. \tag{6}$$

(b) $A'_n =$ smallest k such that $Q_k = n$.
(c) $B'_n =$ largest k such that $Q_k = n$.

Our main purpose here is to show that, despite the different definitions of the sequences, we actually have $A'_n = A_n$ and $B'_n = B_n$ for all $n \in \mathbb{Z}^0$.

Partition Q into subsequences $Q^1 = \{Q^1_n\}$, $Q^2 = \{Q^2_n\}$, where Q^1 consists of all the terms $Q_n = Q_m$ with smallest m, and Q^2 consists of the same terms, but with largest n.

Example. For $s = 2$, $t = 1$, Table 3 lists the first few terms of these sequences.

Table 3. The beginning terms of the five sequences for $s = 2$, $t = 1$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Q_n	0	1	2	1	3	4	2	5	6	7	8	3	9	10	4	11	12	13	14	5	15
Q_n^1	0	1	2		3	4		5	6	7	8		9	10		11	12	13	14		15
Q_n^2	0			1			2					3			4					5	
A'_n	0	1	2	4	5	7	8	9	10	12	13	15	16	17	18	20	21	23	24	26	
B'_n	0	3	6	11	14	19	22	25													

It is convenient to precede the proof with two Lemmas.

Lemma 3. *(i) Let $Q_i^2 = r$, $Q_j^2 = r+1$ be any two consecutive terms of Q^2. Then $j - i \geq 2$.*

(ii) Let $Q_i^1 = r$, $Q_j^1 = r+1$ be any two consecutive terms of Q^1. Then $j - i \leq 2$.

Proof. (i) We have $Q_i^2 = r$ if $i = tQ_m + sm$ for some $m < i$, and $Q_j^2 = r+1$ if $j = tQ_n + sn$ for some $m < n < i+1$, and Q_m, Q_n occurred precisely once before. Then $j - i = t(Q_n - Q_m) + s(n-m)$. We clearly have $Q_n > Q_m$. Therefore $j - i \geq t + s \geq 2$.

(ii) The mex property (6) implies $j = i+1$, unless $i+1 = tQ_m + sm$ for some $m < i+1$, where Q_m appeared precisely once before. In this case $Q_{i+1} \in Q^2$. Part (i) implies that in this latter case $Q_{i+2} \in Q^1$, so then $j = i+2$. ■

Lemma 4. *$A'_{r+1} - A'_r \in \{1, 2\}$ for all $r \in \mathbb{Z}^+$.*

Proof. A'_r is the smallest i such that $Q_i = r$, and A'_{r+1} is the smallest j such that $Q_j = r+1$. The minimality of i and j means that $Q_i = Q_i^1$, $Q_j = Q_j^1$, and $Q_i^1 = r$, $Q_j^1 = r+1$ are consecutive. The result now follows from Lemma 3 (ii). ■

We are now ready to prove

Theorem 4. *For every $s, t \in \mathbb{Z}^+$ we have $A'_n = A_n$, $B'_n = B_n$ for all $n \in \mathbb{Z}^+$, where A_n, B_n are defined in (2).*

Proof. Induction on n. By (2), $A_1 = 1$. Also $Q_1 = 1$ implies $A'_1 = 1$. Suppose we already showed that $A'_i = A_i$ for all $i \leq n$.

By Lemma 4, $A'_{n+1} - A'_n \in \{1, 2\}$. In the proof of Theorem 2 (§3), we showed that also $A_{n+1} - A_n \in \{1, 2\}$ for all $n \in \mathbb{Z}^+$.

Case (i). $A_{n+1} = A_n + 1$. Then by (2), $A_n + 1 = sA_m + tm$ for no $m \in \mathbb{Z}^+$. Suppose that $A'_{n+1} = A'_n + 2$. Let $A'_n = k$. Then $A'_{n+1} = k+2$, $Q_k = n$, $Q_{k+2} = n+1$; and $Q_{k+1} = Q_{A'_n+1}$ has the property that $A'_n + 1 = A_n + 1 = tQ_m + sm$, where Q_m has already occurred precisely once before. Thus if $Q_m = r$, then $A'_r = m$. Thus $A_n + 1 = tQ_{A'_n} + sA'_r = tr + sA'_r = tr + sA_r$ by the induction hypothesis, which is a contradiction.

Case (ii). $A_{n+1} = A_n + 2$. The argument is similar to that of Case (i), therefore it is omitted. Thus $A'_n = A_n$ for all $n \in \mathbb{Z}^+$.

Now $B'_n = k$ if $Q_k = n$ and Q_k occurred precisely once as some Q_j. It follows that for every $k \in \mathbb{Z}^+$ we have either $B'_n = k$ or $A'_n = k$, but not both. Hence A', B' are complementary sets, where $A' = \bigcup_{n=1}^{\infty} A'_n$, $B' = \bigcup_{n=1}^{\infty} B'_n$. The same was shown for $A = \bigcup_{n=1}^{\infty} A_n$, $B = \bigcup_{n=1}^{\infty} B_n$ at the beginning of the proof of Theorem 1 (§2). It follows that also $B'_n = B_n$ for all $n \in \mathbb{Z}^+$. ■

Notes.

1. The special case $s = 2$, $t = 1$ of the second class (without B'_n) is listed in Neil Sloane's database of sequences [Slo98], sequence numbers 26366 (Q_n), 26367 (A'_n), ascribed to Clark Kimberling. We have not found a reference to other sequences of these families in [Slo98]. In the definition of sequence 26366, "$a(n) = a(m)$ if m has already occurred exactly once…", m should presumably be replaced by $a(m)$.
2. We have $Q_0 = 0$, $Q_1 = 1$ for all $s, t \in \mathbb{Z}^0$; and $Q_2 = 1$ for $s = t = 1$, $Q_2 = 2$ for all s, t with $s + t > 2$.
3. The definition of the second class of sequences and the proof that both classes are identical, throws some light on the properties of both.

8. Epilogue

The heap games proposed and analysed here belong to the family of *succinct* games, so named because their input size is succinct: $O(\log n)$ rather than $O(n)$. Often an extra effort is required for showing that such games are polynomial, i.e., have a polynomial strategy, because not more than $O(\log n)$ computation steps can be used. Different families of succinct games seem to require different methods of strategy computations.

For example, in *octal* games, invented by Guy and Smith [GuSm56], a linearly ordered string of beads may be split and or reduced according to rules encoded in octal. See also [BCG82, Ch. 4], [Con76, Ch. 11]. The standard method for showing that an octal game is polynomial, is to demonstrate that its *Sprague-Grundy* function (the 0s of which constitute the set of P-positions) is periodic. Periodicity has been established for a number of octal games. Some of the periods and or preperiods may be very large; see [GaPl89]. Another way to establish polynomiality is to show that the Sprague-Grundy function values obey some other simple rule, such as forming an arithmetic sequence, as for Nim.

For the present class of heap games, polynomiality was established by a non-standard method. An arithmetic procedure, based on a class of special numeration systems, was the key to polynomiality. It appears that at this stage in the development of combinatorial game theory, there is no unified method for establishing polynomiality. But this malady seems to be common to most of discrete mathematics. Some might not even call it a malady, but consider it to be a feature inherent in the nature of mathematics.

In [YaYa67], the special case of the Zeckendorf numeration system [Zec72] was used to give one of the characterizations of the P-positions of Wythoff's game ($s = t = 1$). This method was extended in [Fra82] for the generalized Wythoff game introduced there ($s = 1$, $t \geq 1$). In both cases, the bases of the numeration system were the numerators of the simple continued expansion of α, were α is such that $A_n = \lfloor n\alpha \rfloor$ for all $n \geq 0$. The interesting aspect is that despite the fact that such α doesn't exist for $s > 1$ (Theorem 2, §3), the polynomial characterization based on special numeration systems nevertheless does exist. We also remark that it would be of interest to compute the Sprague-Grundy function for these heap games. For Wythoff's game this seems to be quite difficult, but this fact says nothing about the case $s > 1$.

In [BoFr81] it is shown that a sequence $\{A_n\}$ is spectral (defined in the proof of Theorem 2), if and only if $|(A_{n+i} - A_n) - (A_{m+i} - A_m)| \le 1$ for all $i, m, n \ge 1$. Another motivation for the present paper was to extend this condition, namely to create and characterize sequences satisfying $|(A_{n+i} - A_n) - (A_{m+i} - A_m)| \le 2$. Vera Sós told me that she has also been interested in this question. For the subfamily $s = t$ of the sequences $\{A_n\}$ defined in (2) (§2), we have perhaps $|(A_{n+i} - A_n) - (A_{m+i} - A_m)| \le s$. And if this is true, does also the converse hold, namely, does $|(A_{n+i} - A_n) - (A_{m+i} - A_m)| \le s$ imply (2) with $s = t$? Investigation of the full family of these sequences (any $s, t \in \mathbb{Z}^+$), is of independent interest. In §7 we defined a class of sequences, and demonstrated its equivalence with the class of sequences defined in (2).

Not always is a succinct game more difficult than "its nonsuccinct version"! We illustrate this with the game *vertex Kayles*. Given a finite (undirected) graph G. A move is to label an as yet unlabeled vertex not adjacent to any labeled vertex. The player first unable to play loses, and the opponent wins. A partizan variation is called *bigraph vertex Kayles*. Both versions have been proved Pspace-hard in [Sch78]. If G is a path, the resulting succinct game, known as *Kayles*, is actually polynomial! It is the octal game 0.137 — see [GuSm56], [BCG82], [Con76]. Incidentally, there is a large "no-man's-land" of games lying in between the polynomial 0.137 and the Pspace-hard vertex Kayles, and it would be of interest to reduce the boundary area.

Finally, the family of combinatorial games consists, roughly, of two-player games with perfect information (no hidden information as in some card games), no chance moves (no dice) and outcome restricted to (lose, win). These games are *completely determined*, so one of their main mathematical interests is in bounding the complexity of their strategies. This explains why we talked so much about efficiency of strategy computation in this paper.

References

1. [BCG82] E.R. Berlekamp, J.H. Conway and R.K. Guy [1982], *Winning Ways* (two volumes), Academic Press, London.

2. [BoFr81] M. Boshernitzan and A.S. Fraenkel [1981], Nonhomogeneous spectra of numbers, *Discr. Math.* **34**, 325–327

3. [BoFr84] M. Boshernitzan and A.S. Fraenkel [1984], A linear algorithm for nonhomogeneous spectra of numbers, *J. of Algorithms* **5**, 187–198.

4. [Con76] J.H. Conway [1976], *On Numbers and Games*, Academic Press, London.

5. [Cox53] H.S.M. Coxeter [1953], The golden section, phyllotaxis and Wythoff's game, *Scripta Math.* **19**, 135–143.

6. [Fra82] A.S. Fraenkel [1982], How to beat your Wythoff games' opponents on three fronts, *Amer. Math. Monthly* **89**, 353–361.

7. [Fra85] A.S. Fraenkel [1985], Systems of numeration, *Amer. Math. Monthly* **92**, 105–114.

8. [GaPl89] A. Gangolli and T. Plambeck [1989], A note on periodicity in some octal games, *Internat. J. Game Theory* **18**, 311–320.

9. [GLL78] R.L. Graham, S. Lin and C.-S. Lin [1978], Spectra of numbers, *Math. Mag.* **51**, 174–176.

10. [GuSm56] R.K. Guy and C.A.B. Smith [1956], The G-values of various games, *Proc. Camb. Phil. Soc.* **52**, 514–526.

11. [Sch78] T.J. Schaefer [1978], On the complexity of some two-person perfect-information games, *J. Comput. System Sci.* **16**, 185–225.

12. [Slo98] N.J.A. Sloane [1998], Sloane's On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/˜ njas/sequences/ .

13. [Wyt07] W.A. Wythoff [1907], A modification of the game of Nim, *Nieuw Arch. Wisk.* **7**, 199–202.

14. [YaYa67] A.M. Yaglom and I.M. Yaglom [1967], *Challenging Mathematical Problems with Elementary Solutions*, translated by J. McCawley, Jr., revised and edited by B. Gordon, Vol. II, Holden-Day, San Francisco.

15. [Zec72] E. Zeckendorf [1972], Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, *Bull. Soc. Roy. Sci. Liège* **41**, 179–182.