A change-of-coordinates from Geometry to Algebra, applied to Brick Tilings

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ABSTRACT. A proof is sketched of the Polynomial Conjecture of the author (circulated as preprint [Kin1], "Brick Tiling and Monotone Boolean Functions"), which says that the family of minimal tilable-boxes grows polynomially with dimension. An important ingredient of the argument is translating the problem from its finite-dimensional geometric framework to the algebraic setting of an in-finite-dimensional lattice.

§1   INGRESS

What connection could there possibly be between packing boxes by $\mathfrak{N}$ shapes of bricks, and the number of AND/OR logic circuits having $\mathfrak{N}$ Boolean inputs?

Several years ago I found an algorithmic solution to a tiling problem, aspects of which, it turned out, had been solved more than two decades earlier.[¥] A sudden gust of serendipity, in the guise of Neil Sloane's `superseeker` program, led from my quantitative results to numerical evidence for a "Polynomial Conjecture" on the growth of complexity (rank) of the tiling space as a function of dimension. In turn, the conjecture –henceforth abbreviated "PC"– led to algebraic questions involving the Dedekind sequence of integers (defined in §2, along with PC). During a sabbatical year at the University of Toronto, a fruitful collaboration with computer scientist Hugh Redelmeier led to additional numerical support for the conjecture, then to a computer-assisted proof for $\mathfrak{N} = 5$. In turn, this gave insight into the algebraic structure of the problem, eventually culminating in a (computerless) proof of the Polynomial Conjecture.

[¥] This is not uncommon in tiling theory, whose literature-of-record runs the gamut from technical research journals to puzzle books.

Packings, Tilings & Algebra

Packing-type problems are arguably among the most ancient of combinatorial co-nundrums. In recent times, several types of overtly algebraic methods have been used to study packings/tilings.

- Group theory, in the form of symmetry groups of tessellations and of crystals.
- Combinatorial group theory, e.g, [Co&La] [Thurst] [Propp].
- Commutative algebra, e.g, [Bar1,2].
- "Dehn invariant" and related Tensor Algebra methods, e.g, [Dehn] [Lac&Sze] [Fr&Ri] [Gal&G] [Ke&Ki1,2].

The tiling problem of the present paper is the first, to my knowledge, which seems to require a smidgeon of Lattice Theory. In this note, I will sketch the passage from the Geometry to the Algebra (from tilings to lattices) and show how PC reduces to a "finiteness certificate" which can be verified by computer.

The current §1 defines brick tilings and the "rank" of a set of protobricks. §2 states PC, illustrates how rank can be computed, and gives a brief introduction to distributive lattices and the Dedekind sequence. In §3, PC is restated in the lattice setting, and the finiteness certificate is described along with a sketch of ideas employed in the proof. The technical lattice-algebraic demonstration of PC will appear in [Kin5]. Geometric information on brick tilings/packings appears in preprints [Kin1] and [Kin2]. Lastly, §4 lists open questions.

Brick-Packings/Tilings. Published in innumerably many puzzle books is this chestnut: *Can the 8×8 chessboard, minus its North-East and South-West corner squares, be packed by (thirty-one) dominos?* (The dominos can be placed in both the 1×2 and 2×1 orientations.)

We are "born knowing" that there is no such packing: The two removed squares have the same color, yet each orientation of a domino must cover both a black and a white square. Even should we allow ourselves to place positive *and negative* copies of dominos (defined at (2)), this "coloring argument" still precludes a tiling. Depending on the shapes of the "proto-tiles", coloring ideas sometimes give an IFF-condition for whether a specified region is tilable.

Bricks. A $\mathfrak{D}$-dimensional ***brick*** B is a $\mathfrak{D}$-tuple

$$\mathsf{B} \;=\; b_1 \times \cdots \times b_d \times \cdots \times b_{\mathfrak{D}} \,,$$

where each ***sidelength*** b_d is a positive integer.[†] We will identify each $\mathfrak{D}$-brick B with a product of half-open intervals,

$$\mathsf{B} \;=\; [0, b_1) \times \cdots \times [0, b_{\mathfrak{D}}) \,,$$

[†]Brick-tiling questions which permit non-integral sidelengths are discussed in [Lac&Sze] [Fr&Ri] [Ke&Ki1,2].

a subset of Euclidean space $\mathbb{R}^{\mathfrak{D}}$. Translating our brick by a vector, $\mathsf{B} + \vec{w}$, gives the set of all sums $\vec{y} + \vec{w}$ for $\vec{y} \in \mathsf{B}$. Agree to use $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{T}$ to name bricks. A lowercase letter denotes the corresponding sidelengths, e.g,

$$\mathsf{A} = a_1 \times \cdots \times a_{\mathfrak{D}} \quad \text{and} \quad \mathsf{T} = t_1 \times \cdots \times t_{\mathfrak{D}}.$$

A **box** is another name for a brick; the latter are used to pack/tile the former. I am interested in when a specified box T (the target) can be packed or tiled by translates of copies of bricks in a specified finite set

$$\mathbf{P} = \left\{ \mathsf{B}^{\langle 1 \rangle}, \mathsf{B}^{\langle 2 \rangle}, \ldots, \mathsf{B}^{\langle \mathfrak{N} \rangle} \right\}$$

called the set of **"protobricks"**.

Definitions: Packing & Tiling. For a subset $S \subset \mathbb{R}^{\mathfrak{D}}$, the indicator function $\mathbf{1}_S$ is 1 at those points $\vec{y}$ in S, and $\mathbf{1}_S(\vec{y})$ is 0 on the complement $\mathbb{R}^{\mathfrak{D}} \smallsetminus S$. In order to show the connection between the problems considered in [Bar1,2] and [Ke&Ki1,2], I define "tilable" a touch more generally than is needed in the present paper.

Given a set $\mathbf{P}$ of protobricks in $\mathbb{R}^{\mathfrak{D}}$, a box T is **packable** if

$$(1) \qquad \mathbf{1}_\mathsf{T} = \sum_{\mathsf{H} \in \mathbf{G}} \mathbf{1}_\mathsf{H},$$

for some finite collection $\mathbf{G}$ of translates of the protobricks. Figure $1'$ exhibits a packing, by $\mathbf{P} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$.
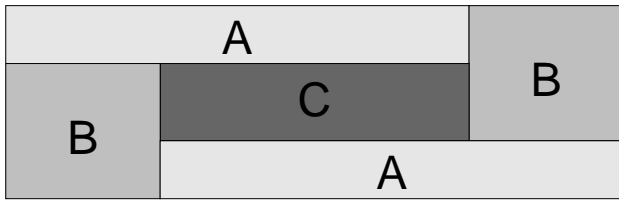

FIGURE $1'$. Two copies of rectangle A, two of B and a single C pack the 34×11 rectangle, where the protobricks are
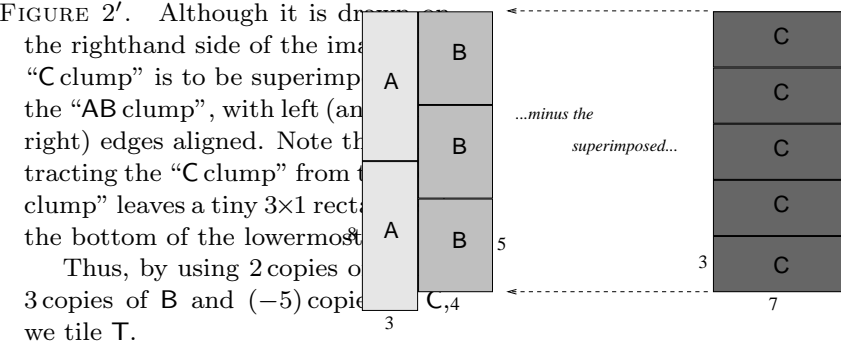
A:	25	×	3
B:	9	×	8
C:	16	×	5

For tiling, I allow weights from an arbitrary commutative monoid $(\Gamma, +, 0)$, with a distinguished non-zero element $1 \in \Gamma$. Say that a box T is Γ-**tilable**, by $\mathbf{P}$, if there exists a finite collection $\mathbf{G}$ of protobrick translates as well as coefficients[‡] $\gamma_\mathsf{H} \in \Gamma$, for H in $\mathbf{G}$, such that

$$(2) \qquad \mathbf{1}_\mathsf{T} = \sum_{\mathsf{H} \in \mathbf{G}} \gamma_\mathsf{H} \mathbf{1}_\mathsf{H}. \qquad \text{(The addition takes place in } \Gamma.)$$

[‡] For brick tiling, both [Bar2, thm 2.1] and [Kin1, Equality Thm] show that $\mathbb{C}$-tilability is equivalent to $\mathbb{Z}$-tilability. In contrast, [Bar2, P.14] has an example of a box which can be $\mathbb{Q}$-tiled by certain *polyominos*, but cannot be $\mathbb{Z}$-tiled by them.

When Γ is the additive group of integers, say simply that T is *tilable* by $\mathbf{P}$. As an illustration, consider the protobrick set consisting of three rectangles $\mathsf{A} = 3{\times}8$, $\mathsf{B} = 4{\times}5$ and $\mathsf{C} = 7{\times}3$. Our target is $\mathsf{T} := 3{\times}1$.

FIGURE $2'$.   Although it is dr[...]
   the righthand side of the ima[...]
   "C clump" is to be superimp[...]
   the "AB clump", with left (an[...]
   right) edges aligned. Note th[...]
   tracting the "C clump" from t[...]
   clump" leaves a tiny $3{\times}1$ rect[...]
   the bottom of the lowermost [...]
       Thus, by using 2 copies o[...]
   3 copies of B and (-5) copie[...]
   we tile T.

This figure depicts a way to tile $3{\times}1$ by the proto-set $\{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$. Certainly T cannot, however, be *packed* by these protobricks.

   Why Tiling? Trivially, there is an algorithm which is exhaustive –indeed, exhausting– for determining whether a given T is packable (by a fixed protoset $\mathbf{P}$). At first glance, one might think that tilability of T is more difficult to ascertain since, potentially, there are infinitely many collections $\mathbf{G}$, in (2), to consider. It transpires that the opposite is true. There is an analogy

$$\text{Packings} \hookrightarrow \text{Tilings}$$
$$\text{Semigroup} \hookrightarrow \text{Group}$$

between studying a semigroup by embedding it in a group, and studying packings by first understanding the (larger) space of tilings. And groups are easier to understand than semigroups, as everybody knows . . .

The partial order on Bricks. The goal of [Kin1] was to find a fast algorithm, with $\mathbf{P}$ fixed, for determining whether a target box T is packable or tilable.
   On the space of $\mathfrak{D}$-bricks there is a natural partial order "$\preccurlyeq$" of packability. For integers, use $a \lhd b$ for "a *divides* b" and $b \rhd a$ for "b *is a multiple of* a". Say that B *parallel-packs* T (or *divides* T), written $\mathsf{B} \preccurlyeq \mathsf{T}$, if translates of B can pack T. That is, $\mathsf{B} \preccurlyeq \mathsf{T}$ iff

$$\text{For each } d = 1, \ldots, \mathfrak{D} \text{ direction: sidelength } b_d \text{ divides } t_d.$$

Now consider a collection $\mathbf{G}$ of boxes which is an **"up-set"** in the partial order,

   Each box which is a multiple of some
   $\mathbf{G}$*-brick, is necessarily itself a* $\mathbf{G}$*-brick.*

Each up-set $\mathbf{G}$ is determined by its family of minimal elements (w.r.t. the $\preccurlyeq$ order). Writing this family as $\mathsf{Mml}(\mathbf{G})$, we have that $\mathsf{T} \in \mathbf{G}$ iff

$$(3) \qquad\qquad \exists \mathsf{B} \in \mathsf{Mml}(\mathbf{G}) \text{ with } \mathsf{B} \preccurlyeq \mathsf{T} .$$

Thus both $\mathsf{Pac}(\mathbf{P})$ –the set of $\mathbf{P}$-packable boxes– and $\mathsf{Til}(\mathbf{P})$ –the tilable boxes– are determined by their sets of minimal elements, respectively.

My purpose in [Kin1], partially successful, was to find finite descriptions of $\mathsf{Pac}(\mathbf{P})$ and $\mathsf{Til}(\mathbf{P})$ which allowed an efficient test for membership. Alas for packing, the minimal set doesn't work; typically $\mathsf{Mml}\big(\mathsf{Pac}(\mathbf{P})\big)$ is infinite.

Happily, several authors[¥] proved versions of the important result that the set $\mathbf{M}(\mathbf{P}) \coloneqq \mathsf{Mml}\big(\mathsf{Til}(\mathbf{P})\big)$ is finite, and that sufficiently large boxes are $\mathbf{P}$-packable if and only if they are $\mathbf{P}$-tilable. [See the end of §2 for an example computation of $\mathbf{M}(\mathbf{P})$.]

THEOREM 4. $\mathbf{M}(\mathbf{P})$ *is finite and is computable. Furthermore, there is a computable integer* $\mathcal{K} = \mathcal{K}(\mathbf{P})$ *so that, whenever* T *is a box whose sidelengths each exceed* $\mathcal{K}$, *then:* T *tilable* $\implies$ T *packable.*

This theorem yields an algorithm for testing whether a candidate box T is tilable: *Does* $\mathsf{B} \preccurlyeq \mathsf{T}$, *for some* B *in* $\mathbf{M}(\mathbf{P})$? Letting n denote the number of bits needed to describe T, this algorithm runs in linear time $O(n)$.

Computing rank

Is the algorithm practical? Well ... , this all depends on the magnitude of the constant in the $O(n)$ algorithm. [Kin1] called the cardinality of $\mathbf{M}(\mathbf{P})$ the **rank** of $\mathbf{P}$, showed it bounded by a pure function of $\mathfrak{N}$ and $\mathfrak{D}$ (the number of protobricks and their dimension), and produced two algorithms for computing it. Here is the first algorithm:

Given bricks $\mathsf{A}, \mathsf{B}, \ldots, \mathsf{C}$, we can use them to tile a box T built as follows. Let

$$g_1 \coloneqq \gcd\{a_1, b_1, \ldots, c_1\}, \quad \text{and for each other direction:}$$
$$\ell_e \coloneqq \mathrm{lcm}\{a_e, b_e, \ldots, c_e\}, \quad \text{for } e = 2, 3, \ldots, \mathfrak{D}.$$

Then $\mathsf{T} \coloneqq g_1 \times \ell_2 \times \cdots \times \ell_{\mathfrak{D}}$ is tilable by collection $\{\mathsf{A}, \mathsf{B}, \ldots, \mathsf{C}\}$. To see this, note that A parallel-packs the "slab"

$$\mathsf{A}' \coloneqq a_1 \times \ell_2 \times \cdots \times \ell_{\mathfrak{D}} .$$

[¥] Although they do not discuss the minimal set of $\mathsf{Til}(\mathbf{P})$, Katona & Szász (1971) give a criterion for a box being $\mathbf{P}$-tilable (and $\mathbf{P}$-packable, once the sidelengths are large enough) under the assumption that $\mathbf{P}$ comprises all $\mathfrak{D}!$ orientations of bricks in a brick-set.

Barnes, in two seminal papers (1982), uses ideals over polynomial rings to develop a general algebraic criterion for a polyomino to be tilable by other polyominos. Results (2.1) and (2.4) of [Bar2] imply our theorem 4, here.

These three papers do not address computability.

And T is tiled by slabs $\{\mathsf{A}', \mathsf{B}', \ldots, \mathsf{C}'\}$ in the same way that integer g_1 is an integral linear-combination of integers $\{a_1, b_1, \ldots, c_1\}$. We call this the ***combine*** operation, and write $\mathsf{T} = \mathsf{Comb}_1\big(\{\mathsf{A}, \mathsf{B}, \ldots, \mathsf{C}\}\big)$, the combine in direction 1.

More generally, given a brick-set $\mathbf{S}$ and direction d, let $\mathbf{S}_{\square \to d}$ denote the *set* $\{b_d \mid \mathsf{B} \in \mathbf{S}\}$ of dth sidelengths. Then $\mathsf{Comb}_d(\mathbf{S})$ is the brick $t_1 \times \cdots \times t_{\mathfrak{D}}$, where

$$t_d := \gcd\big(\mathbf{S}_{\square \to d}\big) \quad \text{and, for each direction } e \neq d:$$
$$t_e := \mathrm{lcm}\big(\mathbf{S}_{\square \to e}\big).$$

It turns out that iterating all possible Combines is powerful enough to generate all of $\mathbf{M}(\mathbf{P})$. Define the "dth extension of $\mathbf{P}$" to be the set of bricks

$$\mathsf{Ext}_d(\mathbf{P}) := \big\{ \mathsf{Comb}_d(\mathbf{S}) \mid \mathbf{S} \text{ is a non-void subset of } \mathbf{P} \big\}.$$

It is not difficult to see that each two of the Ext operators commute, and each is idempotent. So $(6a)$, below, is the set of all boxes that can be made by means of the $\mathsf{Combine}$ operation. Moreover,

THEOREM 5 [Kin1, Equality Thm]. *The brick-set $\mathbf{M}(\mathbf{P})$ equals the set of minimal bricks (w.r.t. divisibility) of*

$$(6a) \qquad\qquad \mathsf{Ext}_{\mathfrak{D}}\Big(\mathsf{Ext}_{\mathfrak{D}-1}\big(\ldots \mathsf{Ext}_2\big(\mathsf{Ext}_1(\mathbf{P})\big)\ldots\big)\Big).$$

As a corollary, we get this daunting bound on the rank of $\mathbf{P}$.

$$(6b) \qquad\qquad \mathrm{rank}(\mathbf{P}) \ \leq\ 2^{2^{2^{\cdot^{\cdot^{\cdot^{2^{\mathfrak{N}}}}}}}} \qquad \begin{array}{l} \text{(a tower of } \mathfrak{D} \text{ many} \\ \text{exponentiations)} \end{array}$$

On the one hand –perhaps unexpectedly– formula $(6a)$ gives a workable algorithm for computing the set $\mathbf{M}(\mathbf{P})$ of minimal tilable-boxes. For as the algorithm progressively generates bricks, we can discard bricks when they become divisible by a later-generated brick.

On the other hand, bound $(6b)$ is laughably too large. Other arguments in [Kin1] give a smaller bound of $\mathrm{rank}(\mathbf{P}) \leq \mathfrak{D}^{\mathfrak{D}^{\mathfrak{N}}}$. The corresponding algorithm, however, which this smaller bound engenders, typically runs more slowly than that from (6).

Two examples. The rank of $\mathbf{P}$ can be smaller or larger than cardinality $|\mathbf{P}|$.

Proto-set $\mathbf{P} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$ of Figure $1'$ tiles 34×11. What is the rank of $\mathbf{P}$? Evidently Comb_1 produces these bricks,

$$\mathsf{Comb}_1\{\mathsf{A}, \mathsf{B}\} \ = \ 1 \times (3 \cdot 8)$$
$$\mathsf{Comb}_1\{\mathsf{B}, \mathsf{C}\} \ = \ 1 \times (8 \cdot 5)$$
$$\mathsf{Comb}_1\{\mathsf{C}, \mathsf{A}\} \ = \ 1 \times (5 \cdot 3).$$

Applying Comb_2 to this family of three bricks yields the 1×1 brick. Thus $\mathsf{Til}(\mathbf{P})$ is the set of *all* boxes. So $\text{rank}(\mathbf{P})$ is 1.

As a second example, let A be 2×3×7 and let $\mathbf{P} := \{\mathsf{A}, \mathsf{A}', \mathsf{A}''\}$, where each stroke means to rotate the sides by one position; $\mathsf{A}' = 3×7×2$ and $\mathsf{A}'' = 7×2×3$. Necessarily, the set $\mathsf{Mml}\big(\mathsf{Til}(\mathbf{P})\big)$, the minimal tilable-boxes, is rotation invariant. It comprises these five bricks

$$
\begin{array}{llllll}
\mathsf{A} & & : & 2\times & 3 & \times & 7 \\
\mathsf{B} := \mathsf{Comb}_1\{\mathsf{A}, \mathsf{A}'\} & : & 1\times & (3\cdot 7) \times & (7\cdot 2) \\
\dot{\mathsf{B}} := \mathsf{Comb}_1\{\mathsf{A}', \mathsf{A}''\} & : & 1\times & (7\cdot 2) \times & (2\cdot 3) \\
\ddot{\mathsf{B}} := \mathsf{Comb}_1\{\mathsf{A}'', \mathsf{A}\} & : & 1\times & (2\cdot 3) \times & (3\cdot 7) \\
\mathsf{C} := \mathsf{Comb}_2\{\mathsf{B}, \dot{\mathsf{B}}, \ddot{\mathsf{B}}\} & : & 1\times & 1 & \times (2\cdot 3\cdot 7)
\end{array}
$$

and their rotates. Thus $\text{rank}\big(\{\mathsf{A}, \mathsf{A}', \mathsf{A}''\}\big) = 15$.

Simplifying Ext notation. Given a finite set S of directions (positive integers), let Ext_S mean

$$\mathsf{Ext}_{d_1} \circ \mathsf{Ext}_{d_2} \circ \cdots \circ \mathsf{Ext}_{d_\tau},$$

where $d_1, \ldots, d_\tau$ is some enumeration of S; this is well-defined since all the Ext operators commute. Henceforth, write $\mathsf{Ext}_{\mathfrak{D}}\big(\cdots \mathsf{Ext}_1(\mathbf{P}) \cdots\big)$ as $\mathsf{Ext}_{\{1,\ldots,\mathfrak{D}\}}(\mathbf{P})$, or just as $\mathsf{Ext}_{1..\mathfrak{D}}(\mathbf{P})$. When S is empty, $\mathsf{Ext}_{\varnothing}(\mathbf{P})$ is $\mathbf{P}$.

§2   POLYNOMIAL CONJECTURE

Since the tiling-rank of a set of $\mathfrak{N}$ many $\mathfrak{D}$-bricks is bounded by a function of $\mathfrak{N}$ and $\mathfrak{D}$, and since rank is essentially the constant in the linear-time algorithm tilability test, one naturally wishes to study the ***maxrank*** function μ:

> $\mu(\mathfrak{N}, \mathfrak{D})$ *is the maximum, as $\mathbf{P}$ ranges over all*
> $\mathfrak{N}$-*sets of $\mathfrak{D}$-dimensional bricks, of* $\text{rank}(\mathbf{P})$.

For each $\mathfrak{N}, \mathfrak{D}$ pair, there is a straightforward method to construct a "worst case" brick-set $\mathbf{P}$ whose rank is $\mu(\mathfrak{N}, \mathfrak{D})$. Letting $b^{\times \mathfrak{D}}$ denote the $\mathfrak{D}$-cube $b \times b \times \cdots \times b$, it turns out that a worst case $\mathbf{P}$ can be built from *cubes*,

$$(7a) \qquad \mathbf{P} = \left\{ (b^{\langle 1 \rangle})^{\times \mathfrak{D}}, (b^{\langle 2 \rangle})^{\times \mathfrak{D}}, \ldots, (b^{\langle \mathfrak{N} \rangle})^{\times \mathfrak{D}} \right\}.$$

Moreover, the collection of $\mathfrak{N}$ sidelengths $b^{\langle 1 \rangle}, b^{\langle 2 \rangle}, \ldots, b^{\langle \mathfrak{N} \rangle}$, can be chosen to depend only on $\mathfrak{N}$, and not on dimension. For $\mathfrak{N} = 3$, here is one such collection.

$$
\begin{aligned}
b^{\langle 1 \rangle} &:= 2^1 \cdot 3^1 \cdot 5^2 \cdot 7^2 \cdot 11^3 \cdot 13^3 \\
b^{\langle 2 \rangle} &:= 2^2 \cdot 3^3 \cdot 5^1 \cdot 7^3 \cdot 11^1 \cdot 13^2 \\
b^{\langle 3 \rangle} &:= 2^3 \cdot 3^2 \cdot 5^3 \cdot 7^1 \cdot 11^2 \cdot 13^1 .
\end{aligned}
$$

Reading the exponents down the columns, we see each of the 6 permutations of $\{1, 2, 3\}$.

More generally, let $\{p_\nu \mid \nu \in \text{Perms}\}$ be the first $\mathfrak{N}!$ prime numbers, indexed by the $\mathfrak{N}!$ permutations of $\{1, \ldots, \mathfrak{N}\}$.

PROPOSITION 7*b* [Kin1, Max-rank Proposition]. *For $n = 1, \ldots, \mathfrak{N}$, let*

$$b^{\langle n \rangle} := \prod_{\nu \in \text{Perms}} p_\nu^{\nu(n)}.$$

Then proto-set (7a) has rank equal to the maxrank value $\mu(\mathfrak{N}, \mathfrak{D})$.

So maxrank values can now be computed. A program I wrote in Common Lisp calculated the table below.

	$\mathfrak{D} \to$	*2*	*3*	*4*	*5*	*6*	*7*	*8*
	1	1	1	1	1	1	1	1
$\lvert \mathbf{P} \rvert = \mathfrak{N}$	**2**	4	5	6	7	8	9	10
$\downarrow$	**3**	18	36	61	93	132	178	231
	4	166	578	1372	2669	4590	?	
	5	7579	?					

TABLE 8. Maximal-rank values, $\mu(\mathfrak{N}, \mathfrak{D})$. The computer program omitted (serendipitously, as it later turned out) the $\mathfrak{D} = \mathit{1}$ column, which is trivially constant 1.

Enter Computer Serendipity. Neil Sloane's venerable "Handbook of Integer Sequences" was the standard reference for looking up mystery sequences of integers. As a wonderful service to the mathematical community, he has now made available an electronic version, called `superseeker`, which, upon receiving an email message comprising some terms of a sequence, mails back a list of journal citations where the sequence has been analysed.

The computed $\mathfrak{D} = \mathit{2}$ column, $1, 4, 18, 166, 7579$, were the only numbers in the table not entirely mysterious to me. The proof of (7*b*) showed that they were the *Dedekind numbers* (defined below), an explosively-growing sequence of integers well-known to combinatorists. In order to save a trip to the library, I used Sloane's program to get citations for Dedekind's sequence. On a lark, I subsequently emailed "`lookup 18 36 ... 231`", the $\mathfrak{N} = \mathbf{3}$ row, to `superseeker@research.att.com`. Knowing that the Dedekind sequence grew *doubly*-exponentially with $\mathfrak{N}$, then, I

was dumbfounded to receive

```
From:  superseq-reply@research.att.com
To:  squash@math.ufl.edu
Report on [ 18,36,61,93,132,178,231]:
Many tests are carried out, but only potentially useful information
(if any) is reported here.
TEST: IS THE k-TH TERM A POLYNOMIAL IN k?

        SUCCESS: k-th term is nontrivial polynomial in k of
degree   2
Polynomial is 18+29/2*k+7/2*k↑2
```

Completely floored, I hastily emailed off what little I had for the $\mathfrak{N} = \mathbf{4}$ row, only to see

```
Report on [ 166,578,1372,2669,4590]:
TEST: IS THE k-TH TERM A POLYNOMIAL IN k?

        SUCCESS: k-th term is nontrivial polynomial in k of
degree   3
Polynomial is 166+784/3*k+261/2*k↑2+121/6*k↑3
```

This was certainly food for contemplation ... –perhaps row $\mathfrak{N}$ was the output of a degree $\mathfrak{N} - 1$ polynomial?! Writing down the apparent polynomials for $\mathfrak{N} = \mathbf{1}, \mathbf{2}$ gave this list.

$$
\begin{aligned}
h_{\mathbf{1}}(k) &\coloneqq 1 \\
h_{\mathbf{2}}(k) &\coloneqq 4 + k \\
h_{\mathbf{3}}(k) &\coloneqq 18 + \tfrac{29}{2}k + \tfrac{7}{2}k^2 \, . \\
h_{\mathbf{4}}(k) &\coloneqq 166 + \tfrac{784}{3}k + \tfrac{261}{2}k^2 + \tfrac{121}{6}k^3 \, .
\end{aligned}
$$

The penny still had not dropped; I saw no pattern in this list. Moreover, plugging $k = -1$ (which corresponds to $\mathfrak{D} = \mathbf{1}$) into $h_{\mathbf{3}}$ did *not* give the correct value of $1 = \mu(\mathbf{3}, \mathbf{1})$, but rather gave 7.

But stay a moment -evaluating *all* the polynomials $h_{\mathbf{1}}, h_{\mathbf{2}}, h_{\mathbf{3}}, h_{\mathbf{4}}$ at $k = -1$, gave $1, 3, 7, 15$ –the diminished powers-of-two?! And plugging in $k = -2$ yielded $1, 2, 3, 4$. Hmm...

At last the penny dropped.

(9)   *Some phenomenon in Table 8 only kicked in at $\mathfrak{D} \geq \mathbf{2}$. However, the phenomenon was naturally indexed from $\mathfrak{D} = \mathbf{0}$.*

Shifting the polynomials back by 2, by letting $g_\mathfrak{N}(\mathfrak{D}) := h_\mathfrak{N}(\mathfrak{D} - 2)$, gave the first 4 lines of these next two tables. (The 5th lines were computed later, by Hugh Redelmeier.)

$$g_1(\mathfrak{D}) := 1$$
$$g_2(\mathfrak{D}) := 2 + \mathfrak{D}$$
$$g_3(\mathfrak{D}) := 3 + \tfrac{1}{2!}\big[\mathfrak{D} + 7\mathfrak{D}^2\big]$$
$$g_4(\mathfrak{D}) := 4 + \tfrac{1}{3!}\big[-112\mathfrak{D} + 57\mathfrak{D}^2 + 121\mathfrak{D}^3\big]$$
$$g_5(\mathfrak{D}) := 5 + \tfrac{1}{4!}\big[29898\mathfrak{D} - 81241\mathfrak{D}^2 + 48066\mathfrak{D}^3 + 3901\mathfrak{D}^4\big] \qquad \text{(Redelmeier)}$$

$\mathfrak{D} \to$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathfrak{N}$ **2**	2	3	4	5	6	7	8	9	10	11	12	13
$\downarrow$ **3**	3	7	18	36	61	93	132	178	231	291	358	432
4	4	15	166	578	1372	2669	4590	7256	10788	15307	20934	27790
5	5	31	7579	40517	120614	273540	(Redelmeier)					

TABLES 10a & 10b   Polynomials $g_\mathfrak{N}$ and their values $g_\mathfrak{N}(\mathfrak{D})$, for $\mathfrak{N} \le \mathbf{5}$. The zero-th column shows the naturals, the first exhibits diminished powers-of-two, and column *2* has Dedekind numbers.

With this much mathematical smoke in evidence, it was irresistable to conjecture that there was some mathematical fire underlying it.

POLYNOMIAL CONJECTURE. *As $\mathfrak{D}$ ranges over $[2..\infty)$, the mapping $\mathfrak{D} \mapsto \mu(\mathfrak{N}, \mathfrak{D})$ is a polynomial[†] of degree $\mathfrak{N} - 1$.*

In order to get a handle on this conjecture, we study the algebraic structure underlying (6a), which is that of a lattice.

Lattices & CIA operators

A **lattice** is a poset $(\mathcal{L}, \le)$ such that each pair $a, b \in \mathcal{L}$ has a *greatest lower bound*, written $a \wedge b$, and a *least upper bound*, $a \vee b$. Letting $c := a \wedge b$, then, $c \le a$ & $c \le b$ and, if c' is any other such element, then $c \ge c'$.

Automatically, $\wedge$ is a "CIA operator" –Commutative, Idempotent $(a \wedge a = a)$, Associative– and so is $\vee$. Moreover, the lattice operations fulfill the absorption laws

$$a \vee (a \wedge b) = a \qquad \& \qquad a \wedge (a \vee b) = a\,.$$

[†]Every degree-$(\mathfrak{N} - 1)$ polynomial which takes on integer values at integers necessarily has coefficients of the form $q \,/\, (\mathfrak{N} - 1)!$ where q is integral. This form of the coefficients will arise naturally, in (13), from the proof of PC in §3.

Now consider the brick-set $\mathbf{S} := \mathsf{Ext}_{1..\mathfrak{D}}(\mathbf{P})$, for a fixed $\mathfrak{D} \geq 2$. Observe that the collection $\mathbf{S}_{\square \to 1}$ of first sidelengths is a lattice with respect to divisibility; here "$\wedge$" is gcd and "$\vee$" is lcm. Indeed, in each direction d,

$$\left(\mathbf{S}_{\square \to d} ,\; \lessdot \right) \text{ is a distributive lattice,}$$

since $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, and $a \wedge (b \vee c)$ equals $(a \wedge b) \vee (a \wedge c)$.

Each lattice $\mathbf{S}_{\square \to d}$ is generated by the (at most) $\mathfrak{N}$ numbers $\mathbf{P}_{\square \to d}$. Thus each of these lattices is a homomorphic image of $\mathcal{L}[\mathfrak{N}]$, the ***Dedekind lattice***, which is the *free* distributive lattice on $\mathfrak{N}$ generators.

Picturing $\mathcal{L}[\mathfrak{N}]$. It is convenient to describe $\mathcal{L} = \mathcal{L}[\mathfrak{N}]$ as the lattice of non-decreasing Boolean functions of $\mathfrak{N}$ Boolean variables. To this end, we will write "$\vee$" as addition, logical OR, and write "$\wedge$" as multiplication, logical AND.

FIGURE 11.   The Dedekind lattice $\mathcal{L}[3]$, here generated by the three symbol alphabet $\{\mathtt{w}, \mathtt{x}, \mathtt{y}\}$, has 18 members. Each node is labeled by a phrase—a sum of products. The dotted equal-sign connects two semicircles, which represent two instances of the *same* node, $\mathtt{wx} + \mathtt{wy} + \mathtt{xy}$. This node is self-dual.

The lattice exhibits a mirror symmetry across a line passing through nodes $\mathtt{w}$, $\mathtt{x}$ and $\mathtt{y}$. Each node in the lattice is mirror-symmetric with its dual. [The *dual* of a phrase is obtained by replacing addition by multiplication, and vice versa. So the dual of $\mathtt{w} + (\mathtt{xy})$ is $\mathtt{w}(\mathtt{x} + \mathtt{y})$, which equals $\mathtt{wx} + \mathtt{wy}$.]
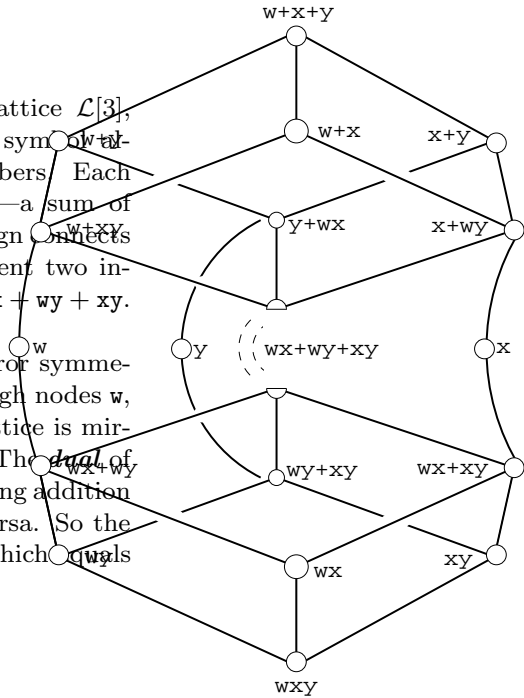
Fix an alphabet $\{\mathtt{w}, \mathtt{x}, \mathtt{y}, \ldots, \mathtt{z}\}$ comprising $\mathfrak{N}$ letters. A ***word*** is a non-empty product of letters, e.g $\mathtt{wyz}$. Each expression built from AND/OR, e.g,

$$\alpha \;:=\; \Big(\mathtt{x} + \mathtt{w}\big((\mathtt{wz} + \mathtt{xz})\mathtt{y} + \mathtt{x}\big) \Big)\big(\mathtt{w} + \mathtt{y} + \mathtt{z}\big) \;+\; \mathtt{y}(\mathtt{x} + \mathtt{y})$$

can be rewritten –courtesy of the distributive laws– as a non-empty sum of words. Moreover:

- By idempotency and commutativity, each word has no repeated letters.
- By absorption, no word is a subword of another word in the sum.

Such a reduced sum will be called a ***phrase***. The above expression reduces to the phrase $\alpha = \mathtt{wx} + \mathtt{xz} + \mathtt{y}$.

Assigning 0 (=**false**) and 1 (=**true**) to each of the $\mathfrak{N}$ symbols $\mathtt{w}, \ldots, \mathtt{z}$, gives a phrase α the value 0 or 1. So the phrase, thus viewed, is a non-decreasing Boolean function of its variables. Conversely, each non-decreasing Boolean function reduces to a unique phrase. Consequently: *The free distributive lattice can be written as the lattice of phrases, or of non-decreasing Boolean functions over $\mathfrak{N}$ variables.*

Dedekind Numbers. Dede$[\mathfrak{N}]$ is the cardinality of $\mathcal{L}[\mathfrak{N}]$. Some known values[‡] are $1, 4, 18, 166, 7579, 7828352, 2414682040996, 56130437228687557907786$.

A lower bound on the sequence comes from words using half the alphabet. Let $H := \lfloor \mathfrak{N}/2 \rfloor$ and consider those words which use exactly H of the $\mathfrak{N}$ letters. Evidently every sum of such words is a phrase, and there are $-1 + \binom{\mathfrak{N}}{H}$ such non-empty sums. Stirling's approximation to the binomial coefficient $\binom{\mathfrak{N}}{H}$ gives

$$\mathrm{Dede}(\mathfrak{N}) \quad \geq \quad \tfrac{1}{2} \cdot 2 {\uparrow} \binom{\mathfrak{N}}{\lfloor \mathfrak{N}/2 \rfloor} \quad \approx \quad 2^{\left[2^{\mathfrak{N}} / \sqrt{\pi \mathfrak{N}/2} \right]}.$$

The upshot is that (8) is a naturally occuring table of numbers which grows doubly-exponentially in one direction, and apparently polynomially in the other.

Lifting $\mathsf{Mml}\big(\mathsf{Til}(\mathbf{P})\big)$**.** There is a lattice homomorphism φ_1 from $\mathcal{L} = \mathcal{L}[\mathfrak{N}]$ onto $\mathbf{S}_{\square \to 1}$. Simply specify a bijection φ_1 from the $\mathfrak{N}$ generators $\{\mathtt{w}, \ldots, \mathtt{z}\}$ onto the multiset $\mathbf{P}_{\square \to 1}$, then extend φ_1 by the two lattice operations. Similarly, let φ_d be a homomorphism from $\mathcal{L}$ onto $\mathbf{S}_{\square \to d}$.

In consequence, the Cartesian product $\overline{\varphi} := \varphi_1 \times \cdots \times \varphi_{\mathfrak{D}}$ is a lattice homomorphism which, for each direction d, makes the following diagram commute.

$$
\begin{array}{ccc}
\mathcal{L} \times \cdots \times \mathcal{L} & \xrightarrow{\;\mathsf{Comb}_d\;} & \mathcal{L} \times \cdots \times \mathcal{L} \\[2pt]
\overline{\varphi} \downarrow & & \overline{\varphi} \downarrow \\[2pt]
\mathbf{S}_{\square \to 1} \times \cdots \times \mathbf{S}_{\square \to \mathfrak{D}} & \xrightarrow{\;\mathsf{Comb}_d\;} & \mathbf{S}_{\square \to 1} \times \cdots \times \mathbf{S}_{\square \to \mathfrak{D}}
\end{array}
$$

The set of cubes $\mathsf{W} := \mathtt{w}^{\times \mathfrak{D}}, \ldots, \mathsf{Z} := \mathtt{z}^{\times \mathfrak{D}}$ in $\mathcal{L}^{\times \mathfrak{D}}$, upstairs, corresponds to the proto-set $\mathbf{P}$, downstairs. So the homomorphism $\overline{\varphi}$ provides an order-preserving surjection

$$\mathsf{Ext}_{1..\mathfrak{D}}\big(\{\mathsf{W}, \ldots, \mathsf{Z}\}\big) \; \longrightarrow \; \mathsf{Ext}_{1..\mathfrak{D}}\big(\mathbf{P}\big),$$

[‡]See [Comtet] or `superseeker`. In my definition of $\mathcal{L}[\mathfrak{N}]$ I omitted two phrases: the constant 0 function (the empty sum) and the constant 1 function (the sum whose only term is the empty word). Some authors include these phrases, and so their Dedekind numbers are two higher than those listed here.

and consequently each minimal brick in $\mathsf{Ext}_{1..\mathfrak{D}}(\mathbf{P})$ comes from some minimal brick upstairs.

The upshot is this: Suppose we once-and-for-all compute $\mathbf{F} \coloneqq \mathbf{M}(\{\mathsf{W}, \ldots, \mathsf{Z}\})$, the minimal bricks in $\mathsf{Ext}_{1..\mathfrak{D}}(\{\mathsf{W}, \ldots, \mathsf{Z}\})$. Then we know $\mathbf{M}(\mathbf{P})$ for *every* proto-set $\mathbf{P}$ of $\mathfrak{N}$ many $\mathfrak{D}$-bricks: $\mathbf{M}(\mathbf{P})$ *comprises the minimal members of the homomorphic image* $\overline{\varphi}(\mathbf{F})$.

This observation intimates that we might profitably lift our regard to the product lattice $\mathcal{L}^{\times \mathfrak{D}}$.

§3   THE PRODUCT LATTICE

We now recast the PC in a lattice setting, stating five Facts, F1–F5, used for its proof, but leaving their technical demonstration to the purely algebraic paper [Kin5].

Let $\Lambda = \Lambda[\mathfrak{N}, \mathfrak{D}]$ denote the $\mathfrak{D}$-fold product lattice $\mathcal{L}[\mathfrak{N}] \times \cdots \times \mathcal{L}[\mathfrak{N}]$. Given bricks $\mathsf{A}, \mathsf{B} \in \Lambda$, write "$\mathsf{A} \vee_d \mathsf{B}$" for $\mathsf{Comb}_d\{\mathsf{A}, \mathsf{B}\}$. Evidently $\vee_1, \ldots, \vee_{\mathfrak{D}}$ are CIA operators, each distributing over every other. However, once $\mathfrak{D}$ exceeds 2, the $\vee_d$ operators *no longer fulfill absorption*. Here is a $\mathfrak{D} = 3$ counterexample.

$$\mathsf{W} \vee_1 \left(\mathsf{W} \vee_2 \mathsf{Y} \right) \;=\; \mathtt{w(w+y)} \;\times\; \mathtt{w+(wy)} \;\times\; \mathtt{w+(w+y)}$$
$$= \;\; \mathtt{w} \;\times\; \mathtt{w} \;\times\; \mathtt{(w+y)} \;\neq\; \mathsf{W}\,.$$

Such operators $\{\vee_d\}_d$ are called **"semilattice operators"**, [Kn&Ro], and the algebraic structure $\left(\Lambda, (\vee_1, \ldots, \vee_{\mathfrak{D}}) \right)$ is a **"multi-semilattice"**.

DEFINITIONS. The **alphabet** of a phrase is the set of letters it uses. So the alphabet of the product $\alpha \coloneqq (\mathtt{w} + \mathtt{yz})(\mathtt{w} + \mathtt{x} + \mathtt{yz})$ is just $\{\mathtt{w}, \mathtt{y}, \mathtt{z}\}$, since α reduces to $\mathtt{w} + \mathtt{yz}$. The alphabet of a brick B, written $\mathrm{Alf}(\mathsf{B})$, is the union of the alphabets of all his sidelengths. Say that B is **balanced** if all his sidelengths have the same alphabet.

The sidelengths of B live in a Dedekind sublattice of $\mathcal{L}[\mathfrak{N}]$; the sublattice generated by $\mathrm{Alf}(\mathsf{B})$. The maximum element of this sublattice, which is the sum of the letters of $\mathrm{Alf}(\mathsf{B})$, will be called the **envelope** of B. We write it as e^{B}. For example, if B is the 2-brick $(\mathtt{xy} + \mathtt{xz}) \times (\mathtt{zw} + \mathtt{y})$, then $e^{\mathsf{B}} = \mathtt{w} + \mathtt{x} + \mathtt{y} + \mathtt{z}$.

A Comb expression which only uses those bricks that are cubes over the given alphabet, e.g,

$$(12) \qquad\qquad (\mathsf{Y} \vee_{19} \mathsf{W}) \vee_{57} \left(\mathsf{Z} \vee_{18} (\mathsf{W} \vee_{19} \mathsf{X}) \right)$$

will be called a **good** expression. A *brick* B is **good** if it is the value of some good expression. So the set of $\Lambda[\mathfrak{N}, \mathfrak{D}]$-good bricks is precisely

$$\mathbf{G}_{\mathfrak{D}} \coloneqq \mathsf{Ext}_{1..\mathfrak{D}}\left(\{\mathsf{W}, \ldots, \mathsf{Z}\} \right).$$

If B is a *minimal* member of $\mathbf{G}_\mathfrak{D}$, say that B is $\Lambda[\mathfrak{N}, \mathfrak{D}]$-**minimal**. Alternatively, "B is minimal for $\Lambda[\mathfrak{N}, \mathfrak{D}]$".

FACT F1 [Kin5, Full-Alphabet/Decomp Lemmata]. *In* $\Lambda[\mathfrak{N}, \mathfrak{D}]$ *suppose brick* B *is good. Then* B *can be built by some good expression which only uses cubes over the alphabet of* B.

Moreover, if B *is minimal then this expression can be chosen to employ operations* $\vee_d$ *only in those directions* d *where the sidelength,* b_d, *is* **not** *the envelope* e^B.

FACT F2 [Kin5, Equal-Alphabet Lemma]. *If* B *is* $\Lambda[\mathfrak{N}, \mathfrak{D}]$-*minimal, then* B *is balanced.*

Overall Strategy

An impediment to discussing the function $\mathfrak{D} \mapsto \mu(\mathfrak{N}, \mathfrak{D})$ is that the ambient lattice $\Lambda[\mathfrak{N}, \mathfrak{D}]$ changes, alas, with $\mathfrak{D}$. Further, a natural avenue towards proving PC is by inducting on $\mathfrak{N}$, and here too this inconvenience arises. In order to bypass this hindrance, a straightforward approach is to create a big lattice, $\Lambda[\infty, \infty]$, to serve as a common setting for all values of $\mathfrak{N}$ and $\mathfrak{D}$.

Let $\mathtt{w}^{\langle 1 \rangle}, \mathtt{w}^{\langle 2 \rangle}, \ldots$ be an infinite list of letters and let $\mathcal{L}[\infty]$ be the free distributive lattice that they generate; $\mathcal{L}[\infty]$ comprises all finite sums of finite words. Identifying $\mathcal{L}[\mathfrak{N}]$ with the lattice generated by $\mathtt{w}^{\langle 1 \rangle}, \ldots, \mathtt{w}^{\langle \mathfrak{N} \rangle}$ shows that $\mathcal{L}[\infty]$ is the direct limit $\mathcal{L}[1] \hookrightarrow \mathcal{L}[2] \hookrightarrow \ldots$ of lattices. Lastly, for $\mathfrak{N}$ any value in $\{1, 2, \ldots, \infty\}$, let $\Lambda[\mathfrak{N}, \infty]$ represent the infinite-product lattice

$$\Lambda[\mathfrak{N}, \infty] \;:=\; \mathcal{L}[\mathfrak{N}] \times \mathcal{L}[\mathfrak{N}] \times \cdots .$$

Minimality and Alphabet Size. As a first step to implementing the strategy we ask: *If brick* B *is* $\Lambda[\mathfrak{N}, \mathfrak{D}]$-*minimal, must he be minimal for* $\Lambda[\mathfrak{N} + 1, \mathfrak{D}]$?

"Yes", except for the following type of triviality when $\mathfrak{D} = 1$: The 1-dimensional brick $\mathsf{W} \vee_1 \mathsf{X} = \mathtt{wx}$ is minimal with respect to alphabet $\{\mathtt{w}, \mathtt{x}\}$; it is $\Lambda[2, 1]$-minimal. But $\mathsf{W} \vee_1 \mathsf{X}$ is not $\Lambda[3, 1]$-minimal, since $\mathsf{W} \vee_1 \mathsf{X} \vee_1 \mathsf{Y} = \mathtt{wxy}$ is a proper divisor.

Once $\mathfrak{D} = 2$, however, this triviality evaporates. Now $\mathsf{W} \vee_1 \mathsf{X}$ equals $(\mathtt{wx}) \times (\mathtt{w} + \mathtt{x})$, which is minimal even for $\Lambda[\infty, 2]$.

These observations can be interpreted as explaining why the $\mathfrak{D} = 1$ values of rank $\mu(\mathfrak{N}, 1)$ do not fit the polynomial pattern observed in Table 8.

Minimality and Dimension. Given a good $\Lambda[\mathfrak{N}, \mathfrak{D}]$-brick B, there exists some good expression which fabricates B, using only cubes over $\mathrm{Alf}(\mathsf{B})$. This same expression, when interpreted in $\Lambda[\mathfrak{N}, \infty]$, yields a brick, B^*, which is infinite dimensional. Evidently

$$\mathsf{B}^* \;=\; b_1 \times b_2 \times \cdots \times b_\mathfrak{D} \times e^\mathsf{B} \times e^\mathsf{B} \times \cdots, \textit{ where the}$$
envelope e^B *is the sum of the letters in* $\mathrm{Alf}(\mathsf{B})$.

It turns out that if B is $\Lambda[\mathfrak{N}, \mathfrak{D}]$-minimal, then B^* is $\Lambda[\mathfrak{N}, \infty]$-minimal. The converse holds[¥] once $\mathfrak{D} \geq 2$.

We can summarize these facts as follows.

FACT F3 [Kin5, Universally-minimal Lemma]. *For each $\mathfrak{D} \geq 2$ and each $\mathfrak{N}$: Brick B is $\Lambda[\mathfrak{N}, \mathfrak{D}]$-minimal IFF B^* is $\Lambda[\infty, \infty]$-minimal.*

Courtesy of F3, we can henceforth work entirely in the infinite lattice $\Lambda :=\Lambda[\infty, \infty]$, and so we should adjust our notation accordingly. Let $\mathsf{W}^{\langle n \rangle}$ be the ∞-dimensional $\mathbf{w}^{\langle n \rangle} \times \mathbf{w}^{\langle n \rangle} \times \cdots$ cube. A brick $\mathsf{B} \in \Lambda$ is "$\mathfrak{N}, \mathfrak{D}$-**good**" if

$$\mathsf{B} \in \mathsf{Ext}_{1..\mathfrak{D}}\left(\left\{\mathsf{W}^{\langle 1 \rangle}, \mathsf{W}^{\langle 2 \rangle}, \ldots, \mathsf{W}^{\langle \mathfrak{N} \rangle}\right\}\right).$$

Further, B is "$\mathfrak{N}, \mathfrak{D}$-**minimal**" if it is $\mathfrak{N}, \mathfrak{D}$-good *and* is minimal for Λ. Redefine[†] $\mu(\mathfrak{N}, \mathfrak{D})$ to now mean the number of $\mathfrak{N}, \mathfrak{D}$-minimal bricks. For $\mathfrak{N} = 1, 2, \ldots$, the Polynomial Conjecture now becomes

PC[$\mathfrak{N}$] : *As $\mathfrak{D}$ takes on the values $0, 1, 2, \ldots$, the resulting $\mu(\mathfrak{N}, \cdot)$ function,*

$$\mathfrak{D} \mapsto [\text{Number of } \mathfrak{N}, \mathfrak{D}\text{-minimal bricks}],$$

is a degree-$(\mathfrak{N} - 1)$ polynomial.

An Interpretation. For $\mathfrak{D} = 0, 1, 2$, the maxrank number $\mu(\mathfrak{N}, \mathfrak{D})$ can be regarded as the cardinality of particular subsets of the Dedekind lattice $\mathcal{L}[\mathfrak{N}]$.

- $\mu(\mathfrak{N}, 0) = \mathfrak{N}$ is the cardinality of the set of *generators* of $\mathcal{L}(\mathfrak{N})$.
- Taking the closure of the generating set, under $\cdot$ (product), gives the set of *words*. Consequently $\mu(\mathfrak{N}, 1) = 2^{\mathfrak{N}} - 1$.
- Closing the set of words under $+$ (sum), yields $\mathcal{L}(\mathfrak{N})$, the set of *phrases*. It is straightforward to check that

$$\alpha \mapsto \alpha \times \alpha^{\divideontimes}$$

is a bijection from $\mathcal{L}(\mathfrak{N})$ onto $\Lambda[\mathfrak{N}, 2]$, where $\alpha^{\divideontimes}$ denotes the dual of α. Thus $\mu(\mathfrak{N}, 2) = \big|\mathcal{L}(\mathfrak{N})\big| = \mathrm{Dede}(\mathfrak{N})$.

[¥]For each good brick A in $\Lambda[\mathfrak{N}, \infty]$, there is some integer $\mathfrak{D}$ such that $a_{\mathfrak{D}+1} = a_{\mathfrak{D}+2} = \ldots$, all being the envelope of A. Thus each in-finite-dimensional good A is of the form B^*, for some finite-dimensional brick B.

[†]This changes the value of $\mu(\mathfrak{N}, \mathfrak{D})$ only for $\mathfrak{D} \leq 1$.

A finite certificate for polynomialness

Now consider a $\Lambda[\infty, \infty]$-minimal brick B; suppose he can be built with $\vee_1, \ldots, \vee_{\mathfrak{D}}$. Let $b_{d_1}, \ldots, b_{d_\tau}$ be an enumeration of the non-envelope sidelengths of B; so $d_\tau \leq \mathfrak{D}$. A non-envelope sidelength b_d can be recognized immediately: Since B is balanced, $\mathrm{Alf}(b_d) = \mathrm{Alf}(e^{\mathsf{B}})$, yet $b_d \neq e^{\mathsf{B}}$. Thus b_d is not a "pure sum" of letters —it must contain a word of length at least 2.

Because all the Ext operators mutually commute, the permuted brick

$$\mathsf{C} := b_{d_1} \times b_{d_2} \times \cdots \times b_{d_\tau} \times e^{\mathsf{B}} \times e^{\mathsf{B}} \times \cdots$$

is also $\Lambda[\infty, \infty]$-minimal. Furthermore, courtesy of Fact F1, brick C can be constructed only using $\vee_1, \ldots, \vee_\tau$. This number τ is what we will call the ***true dimension*** of B. For example, the true dimension of (12) is three. Each cube $\mathsf{W}^{\langle n \rangle}$ has true dimension zero, and these are the only minimal bricks with true-dim zero.

Both bricks B and C are built from the pure sum e^{B} and the multiset $\{b_{d_1}, \ldots, b_{d_\tau}\}$ of sidelengths. In order to systematically count the bricks thus-buildable, we write this data in a canonical way, by fixing some strict total-order $\lessdot$ on $\mathcal{L} = \mathcal{L}[\infty]$.

Letting K be the number of *distinct* sidelengths in $b_{d_1}, \ldots, b_{d_\tau}$, we may rewrite this multiset as

$$a_1, .^{r_1}., a_1, \, a_2, .^{r_2}., a_2, \, \cdots, a_K, .^{r_K}, a_K \ ,$$

where a_i is repeated r_i times, the sum $r_1 + \cdots + r_K$ equals τ, and where $a_1 \lessdot a_2 \lessdot \ldots \lessdot a_K$. Thus the expression $\left[e \, ; \, a_1^{\times r_1}, a_2^{\times r_2}, \ldots, a_K^{\times r_K} \right]$ tells us what bricks can be build from the multiset.

Conversely, an expression $\mathsf{A} := \left[e \, ; \, a_1^{\times r_1}, a_2^{\times r_2}, \ldots, a_K^{\times r_K} \right]$, formed from members $e, a_1, \ldots, a_K \in \mathcal{L}$, is a ***archetype*** if

- Sidelength e is a pure sum.
- Sidelength $a_1 \lessdot \ldots \lessdot a_K$, and none is a pure sum.
- $a_1^{\times r_1} \times \cdots \times a_K^{\times r_K} \times e^{\times \infty}$ is a minimal brick.

Naturally, we call $r_1 + \cdots + r_K$ the true dimension of A and write it $\tau(\mathsf{A})$.

Counting. Now consider a dimension $\mathfrak{D}$ greater-equal the true-dim $\tau = \tau(\mathsf{A})$. The number of ways –let's call it $\#\mathrm{Bricks}^{\mathsf{A}}(\mathfrak{D})$– of placing τ sidelengths together with $\mathfrak{D} - \tau$ copies of e, into $\mathfrak{D}$ positions, is expressible by the multinomial coefficient

$$\#\mathrm{Bricks}^{\mathsf{A}}(\mathfrak{D}) \; = \; \binom{\mathfrak{D}}{r_1, \ldots, r_K, \mathfrak{D} - \tau} \; \overset{\text{note}}{=\!=\!=} \; \frac{\mathfrak{D}!}{r_1! \cdots r_K! \cdot (\mathfrak{D} - \tau)!} \; .$$

In consequence, $\#\mathrm{Bricks}^{\mathsf{A}}(\mathfrak{D})$ is a polynomial in $\mathfrak{D}$,

$$
\begin{aligned}
(13) \qquad \#\mathrm{Bricks}^{\mathsf{A}}(\mathfrak{D}) \; &= \; \binom{\tau}{r_1, \ldots, r_K} \cdot \binom{\mathfrak{D}}{\tau} \\
&= \; \frac{q}{\tau!} \; \cdot \; \mathfrak{D} \cdot [\mathfrak{D} - 1] \cdots [\mathfrak{D} - (\tau - 1)] \; ,
\end{aligned}
$$

where q is the *integer* $\binom{\tau}{r_1,\dots,r_K}$. Remark that this polynomial has degree τ, and gives the correct value of $\#\text{Bricks}^{\mathsf{A}}(\mathfrak{D})$ –namely zero– for each $0 \le \mathfrak{D} < \tau$.

Say that A is an "$\mathfrak{N}, \tau$-archetype", where $\tau = \tau(\mathsf{A})$, if $\mathfrak{N}$ is large enough that $\text{Alf}(\mathsf{A})$ is a subset of $\{\mathtt{w}^{\langle 1 \rangle}, \dots, \mathtt{w}^{\langle \mathfrak{N} \rangle}\}$. Then, with $\mathfrak{N}$ held fixed,

$$\mu(\mathfrak{N}, \mathfrak{D}) = \sum_{\tau=0}^{\infty} \sum_{\mathsf{A}} \#\text{Bricks}^{\mathsf{A}}(\mathfrak{D}),$$

where A ranges over the (finite) set of $\mathfrak{N}, \tau$-archetypes. Therefore, the inner sum $\sum_{\mathsf{A}} \#\text{Bricks}^{\mathsf{A}}(\cdot)$ is a polynomial of degree τ. We obtain the following.

THEOREM 14. *The function* $\mathfrak{D} \mapsto \mu(\mathfrak{N}, \mathfrak{D})$ *is a polynomial IFF the set of* $\mathfrak{N}$-*archetypes is finite. In that instance, letting* M *denote the maximum true dimension taken over the* $\mathfrak{N}$-*archetypes, the polynomial* $\mu(\mathfrak{N}, \cdot)$ *has degree* M.

This theorem permits a computer proof of $\text{PC}[\mathfrak{N}]$, for small values of $\mathfrak{N}$. As $\mathfrak{D} = 1, 2, \dots$, perform the following:

(*i*) *Having computed* $\mathbf{M}_{\mathfrak{D}-1}$, *the set of* $\Lambda[\mathfrak{N}, \mathfrak{D}-1]$-*minimal bricks, "extend" each such brick* B *to the* $\mathfrak{D}$-*brick* $\mathsf{B}' \coloneqq \mathsf{B} \times e^{\mathsf{B}}$.

(*ii*) *Compute* $\mathbf{S}$, *the set of bricks* $\text{Comb}_{\mathfrak{D}}(\{\mathsf{A}', \dots, \mathsf{C}'\})$ *as* $\{\mathsf{A}', \dots, \mathsf{C}'\}$ *ranges over all non-void sets of extended bricks. Thus* $\mathbf{M}_{\mathfrak{D}}$ *equals* $\text{Mml}(\mathbf{S})$. *If some brick in* $\mathbf{M}_{\mathfrak{D}}$ *has true-dim* $\mathfrak{D}$, *then GOTO step* (*i*). *Otherwise STOP; the maximum true-dimension of an* $\mathfrak{N}$-*archetype is* $\mathfrak{D}-1$, *and collection* $\mathbf{M}_{\mathfrak{D}}$ *is a certificate of this.*

My friend Hugh Redelmeier wrote an intricate computer program to compute archetypes. After running for more than a week on the $\mathfrak{N} = 5$ case, his program constructed all the archetypes and discovered that the maximum true-dim is 4, thus establishing $\text{PC}[5]$. The certificate $\mathbf{M}_5$ has 273540 members.

The last ingredient. Courtesy of the theorem, the Polynomial Conjecture follows from these two facts.

FACT F4. *The brick*

$$\left(\dots \left(\left(\mathsf{W}^{\langle 1 \rangle} \vee_1 \mathsf{W}^{\langle 2 \rangle}\right) \vee_2 \mathsf{W}^{\langle 3 \rangle}\right) \vee_3 \dots\right) \vee_{\mathfrak{N}-1} \mathsf{W}^{\langle \mathfrak{N} \rangle}$$

is $\mathfrak{N}$-*minimal. Furthermore, none of its first* $\mathfrak{N}-1$ *sidelengths is a pure sum, so* $\mathfrak{N}-1$ *is indeed the true dimension of the brick.*

FACT F5. *Each* $\mathfrak{N}$-*minimal brick has true dimension at most* $\mathfrak{N}-1$.

This latter result follows from a simultaneous induction on $\mathfrak{N}$ and $\mathfrak{D}$ within the $\Lambda[\infty, \infty]$ lattice.

§4 Egress

Professors T. Hamachi and Y. Tomita recently sent me a preprint, [Ha&To], which develops a new technique to extend the computations done in [Kin1] for the maxrank numbers in Table 8. And –happily– their results agree with Redelmeier's.

I warmly thank George Bergman, Kevin Keating, Eric Mendelsohn and Hugh Redelmeier, as well as the University of Toronto for its hospitality during a sabbatical visit.

Questions. Here are some algebraic questions suggested by the argument.

Is there a reasonably simple recurrence relation among the $\mu(\mathfrak{N}, \cdot)$ polynomials? If so, this would likely lead to a new method to compute the Dedekind numbers, a sequence which has been the object of considerable study.

An even more likely place to find a recurrence relation is in the 2-parameter table of values $\mathrm{Arch}(\mathfrak{N}, \mathfrak{D})$, for $\mathfrak{D} < \mathfrak{N}$, whose entry is the number of $\mathfrak{N}, \mathfrak{D}$-archetypes.

Affirmative answers to the following would speed up the computation of archetypes: *If minimal bricks* B *and* C *have disjoint alphabets, must* B$\vee_1$C *be minimal? Can each minimal brick* T *be obtained* (*from the given cubes*) *by a succession of* $\vee_d$ *operations, so that at every stage the two operand bricks are minimal?*

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