

# Enumerations of plane meanders

Iwan Jensen

*Department of Mathematics & Statistics, The University of Melbourne,  
Parkville, Victoria 3052, Australia  
e-mail: i.jensen@ms.unimelb.edu.au*

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## Abstract

A closed plane meander of order  $n$  is a closed self-avoiding loop intersecting an infinite line  $2n$  times. Meanders are considered distinct up to any smooth deformation leaving the line fixed. We have developed an improved algorithm, based on transfer matrix methods, for the enumeration of plane meanders. This allows us to calculate the number of closed meanders up to  $n = 24$ . The algorithm is easily modified to enumerate various systems of closed meanders, semi-meanders or open meanders.

*Key words:* meanders; polymers; folding; exact enumerations.

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## 1 Introduction

Meanders form a set of combinatorial problems concerned with the enumeration of self-avoiding loops crossing a line through a given number of points. Meanders are considered distinct up to any smooth deformation leaving the line fixed. This problem seems to date back at least to the work of Poincaré on differential geometry. More recently it has been related to enumerations of ovals in planar algebraic curves [1] and the classification of 3-manifolds [2]. During the last decade or so it has received considerable attention in other areas of science. In computer science meanders are related to the sorting of Jordan sequences [3]. In physics meanders are relevant to the study of compact foldings of polymers [4,5], properties of the Temperley-Lieb algebra [6,7], and defects in liquid crystals and  $2 + 1$  dimensional gravity [8].

The difficulty in the enumeration of most interesting combinatorial problems is that, computationally, they are of exponential complexity. Initial efforts at computer enumeration of meanders have been based on direct counting. Lando and Zvonkin [9] studied closed meanders, open meanders and systems of closed meanders, while Di Francesco *et al.* [5] studied semi-meanders. In this

paper we describe a new and improved algorithm, based on transfer matrix methods, for the enumeration of closed plane meanders. While the algorithm still has exponential complexity, the growth in computer time is much slower than that experienced with direct counting, and consequently the calculation can be carried much further. The algorithm is easily modified to enumerate systems of closed meanders, semi-meanders or open meanders.

## 2 Definitions of meanders

A *closed meander* of order  $n$  is a closed self-avoiding curve crossing an infinite line  $2n$  times. Fig. 1 shows some meanders. The meandric number  $M_n$  is simply the number of such meanders distinct up to smooth transformations. We define the generating function

$$M(x) = \sum_{n=1}^{\infty} M_n x^n. \quad (1)$$

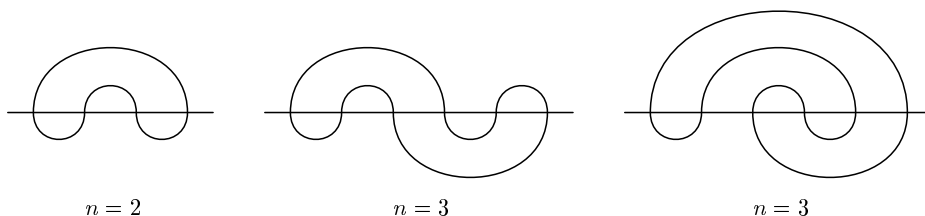


Fig. 1. A few examples of closed meanders of order 2 and 3, respectively.

We can extend the definition to *systems of closed meanders*, by allowing configurations with disconnected closed meanders. The meandric numbers  $M_n^{(k)}$  are the number of meanders with  $2n$  crossings and  $k$  components. An *open meander* of order  $n$  is a self-avoiding curve running from west to east while crossing an infinite line  $n$  times. The number of such curves is  $m_n$ . It should be noted [9] that  $M_n = m_{2n-1}$ . Finally, we could consider a semi-infinite line and allow the curve to wind around the end-point of the line. A *semi-meander* of order  $n$  is a closed self-avoiding loop crossing the semi-infinite line  $2n$  times. The number of semi-meanders of order  $n$  is denoted by  $\overline{M}_n$ . In this case a further interesting generalisation is to study the number of semi-meanders  $\overline{M}_n(w)$ , which wind around the end-point of the line  $w$  times.

## 3 Enumeration of meanders

The method used to enumerate meanders is similar to the transfer matrix technique devised by Enting [10] in his pioneering work on the enumeration

of self-avoiding polygons. The first terms in the series for the meander generating function can be calculated using transfer matrix techniques. This involves drawing a boundary perpendicular to the infinite line. Meanders are enumerated by successive moves of the boundary, so that one crossing at a time is added to the meanders as illustrated in Fig. 2. At each position of the boundary we have a configuration of loop-ends closed to the left, and for each configuration we count all the possible meanders that could give rise to that particular configuration of loop-ends. Since the curve making up a meander is self-avoiding each configuration can be represented by an ordered set of edge states  $\{x_i\}$  = 0 (1) indicates the lower (upper) part of loop closed to the left of the boundary. In addition we need to know where the infinite line is situated within the loop-ends. This can be done simply by specifying how many loop-ends lie beneath the infinite line. Configurations are read from the bottom to the top. As an example we note that the configuration along the boundary of the meander in Fig. 2 at position 4 is  $\{2; 001011\}$ .

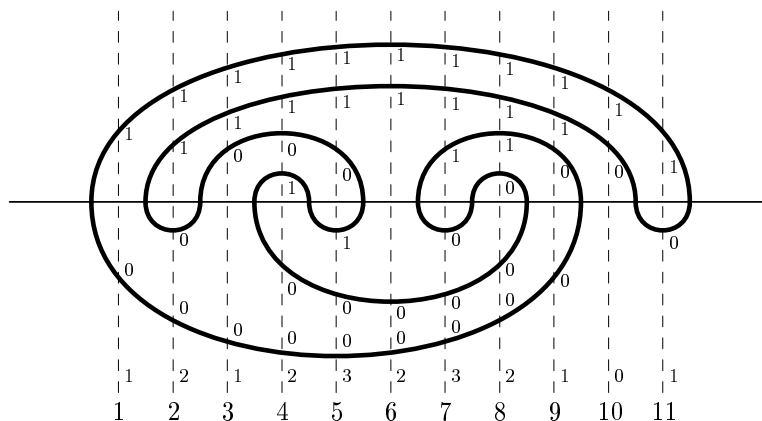


Fig. 2. Positions of the boundaries (dashed lines) during the transfer matrix calculation. Numbers along the boundaries give the encoding of the loop structure in the partially completed meander to the left of the boundary.

We start with the configuration  $\{1;01\}$  with a count of 1, that is one loop crossing the infinite line. Next we move the boundary one step ahead and add a new crossing. So we either put in an additional loop or we take an existing loop-end immediately above or below the infinite line and drag it across the line. The first possibility is illustrated in Fig. 2 in going to position 2 where we get the configuration  $\{2;0011\}$ . Additional loops are also inserted while going to positions 4 and 7. As we cross the infinite line with an existing loop-end we may be allowed to connect it to the loop-end on the other side. In going to position 6 we connect a '1' below the line to a '0' above the line and no further processing is required. In going to position 8 or 9 a '0' below the line is connected to a '0' above the line. This requires further processing because in connecting two lower loop-ends an upper loop-end elsewhere in the old configuration becomes a lower loop-end in the new configuration. In going to position 8 we see that the configuration  $\{2;000111\}$  before the step forward

becomes the configuration  $\{1;0011\}$  after the step. That is the upper end of the third loop before the step becomes the lower end of the second loop after the step. We refer the reader to [10] for the detailed rules for relabeling of configurations. Finally, note that connecting a ‘0’ below the line to a ‘1’ above the line results in a closed loop. So this is only allowed if there are no other loops cut by the boundary and the result is a valid closed meander. As we move along and generate a new ‘target’ configuration its count is calculated by adding up the count for the various ‘source’ configurations which could generate that target. For example the ‘target’  $\{2;0011\}$  could be generated from the ‘sources’  $\{1;01\}$ ,  $\{1;0011\}$ ,  $\{3;0011\}$  and  $\{3;001011\}$ , by, respectively, putting in an additional loop, moving a loop-end below the line, moving a loop-end above the line and connecting two loop-ends across the line.

The number of configurations, which need be generated in a calculation of  $M_n$ , is restricted by the fact that at each step we change the number of loop-ends above and/or below the infinite line by at most one. So if we have already taken  $k$  steps then there can be at most  $2n - k$  loop-ends above or below the line. Any configurations violating this criterion can be discarded. Furthermore we can reduce the number of distinct configurations by a factor of two by using the symmetry with respect to reflection in the infinite line.

As we noted above connecting a ‘0’ below the line to a ‘1’ above the line results in a closed loop. Failure to observe the restriction on this closure would result in graphs with disconnected components, either one closed meander over another or one closed meander within another. Obviously these are just the types of graphs required in order to enumerate systems of closed meanders. So by noting that each such closure adds one more component it is straightforward to generalise the algorithm to enumerate *systems of closed meanders*. *Open meanders* are a little more complicated. Suffice to say at this stage that the main part of the necessary generalisations consists in adding an extra piece of information. We have to add a free end and specify its position within the configuration of loop-ends. In order to enumerate *semi-meanders* all we just change the initial configuration, and start in a position just before the first crossing of the semi-infinite line with  $w$  loops nested within one another. By running the algorithm for each  $w$  from 0 to  $n$  we count semi-meanders with up to  $n$  crossings.

Using the new algorithm we have calculated  $M_n$  up to  $n = 24$  as compared to the previous best of  $n = 17$  obtained by V. R. Pratt [11]. To fully appreciate the advance it should be noted that the computational complexity grows exponentially, that is the time required to obtain  $n$  term grows asymptotically as  $\lambda^n$ . For direct enumerations time is simply proportional to  $M_n$  and thus  $\lambda = \lim_{n \rightarrow \infty} M_{n+1}/M_n \approx 12.26$ . The transfer matrix method employed in this paper is far more efficient and the numerical evidence suggests that the computational complexity is such that  $\lambda \approx 2.5$ . Another way of gauging the improved

efficiency is to note that the calculations for semi-meanders carried out in [5] were “done on the parallel Cray-T3D (128 processors) of the CEA-Grenoble, with approximately 7500 hours  $\times$  processors.” Or in total about 100 years of CPU time. The equivalent calculations can be done with the transfer matrix algorithm in about 15 minutes on a single processor DEC-Alpha workstation! The price we have to pay is that unlike for direct enumeration memory use grows exponentially with growth factor  $\lambda$ .

## 4 Results and analysis

Table 1

The number,  $M_n$ , of connected closed meanders with  $2n$  crossings.

$n$	$M_n$	$n$	$M_n$	$n$	$M_n$
1	1	9	933458	17	59923200729046
2	2	10	8152860	18	608188709574124
3	8	11	73424650	19	6234277838531806
4	42	12	678390116	20	64477712119584604
5	262	13	6405031050	21	672265814872772972
6	1828	14	61606881612	22	7060941974458061392
7	13820	15	602188541928	23	74661728661167809752
8	110954	16	5969806669034	24	794337831754564188184

The enumerations undertaken thus far are too numerous to detail here. We thus only give the results for  $M_n$  which are listed in Table 1. The series for the meander generating function is characterised by coefficients which grow exponentially, with sub-dominant term given by a critical exponent. The generating function behaviour is  $M(x) = \sum_n M_n x^n \sim A(x)(x_c - x)^\xi$ , and hence the coefficients of the generating function  $M_n = [x^n]M(x) \sim \sigma^n/n^{\xi+1} \sum_i c_i/n^{f(i)}$ , where  $\sigma = 1/x_c$  is the connective constant. We analyzed the series by the numerical method of differential approximants [12], and obtained the very accurate estimates  $x_c = 0.08154695(10)$  and  $\xi = 2.4206(4)$ , and we thus find that the connective constant  $\sigma = 12.262874(15)$ . Having obtained these accurate estimates we used them to fit the asymptotic form of the coefficient to the formula above. The results were fully consistent with  $f(i) = i$ . There were no signs of half-integer or other powers, showing that there does not appear to be any non-analytic correction terms to the generating function.

## 5 Conclusion

We have presented an improved algorithm for the enumeration of closed meanders. The computational complexity of the algorithm is estimated to be  $2.5^n$ , much better than direct counting algorithms which have complexity  $12.26^n$ . Implementing this algorithm enabled us to obtain closed meanders up to order 24. From our extended series we obtained precise estimates for the connective constant and critical exponent. An alternative analysis provides very strong evidence for the absence of any non-analytic correction terms to the proposed asymptotic form for the generating function.

## Acknowledgments

This work was supported by a grant from the Australian Research Council.

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