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Combinatoric explosion of renormalization tamed by Hopf algebra: 30-loop Padé-Borel resummation

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Abstract It is easy to sum chain-free self-energy rainbows, to obtain contributions to anomalous dimensions. It is also easy to resum rainbow-free self-energy chains. Taming the combinatoric explosion of all possible nestings and chainings of a primitive self-energy divergence is a much more demanding problem. We solve it in terms of the coproduct Δ, antipode S, and grading operator Y of the Hopf algebra of undecorated rooted trees. The vital operator is $S\star Y$, with a star product effected by Δ. We perform 30-loop Padé-Borel resummation of 463 020 146 037 416 130 934 BPHZ subtractions in Yukawa theory, at spacetime dimension $d=4$, and in a trivalent scalar theory, at $d=6$, encountering residues of $S\star Y$ that involve primes with up to 60 digits. Even with a very large Yukawa coupling, $g=30$, the precision of resummation is remarkable; a 31-loop calculation suggests that it is of order 10^{-8}.

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1 Introduction

In this work we develop the Hopf algebra of renormalization [1, 2, 3, 4, 5] to progress beyond the rainbow [6, 7] and chain [8, 9] approximations for anomalous dimensions.

Summing rainbows: In d dimensions, the massless scalar one-loop integral with propagators to the powers α, β is

$$G(\alpha,\beta;d) := g(\alpha)g(\beta)g(d-\alpha-\beta); \quad g(\alpha) := \Gamma(d/2-\alpha)/\Gamma(\alpha) \tag{1}$$

Now consider the interaction $g\phi^\dagger\sigma\phi$, with a neutral scalar particle σ coupled to a charged scalar ϕ, in the critical dimension, $d_c = 6$. To find the anomalous field dimension γ of ϕ, in the rainbow approximation of [7], one solves the consistency condition

$$1 = aG(1, 1+\gamma; 6) = \frac{a}{\gamma(\gamma-1)(\gamma-2)(\gamma-3)} \tag{2}$$

which ensures that the coupling $a := g^2/(4\pi)^{d_c/2}$ cancels the insertion of the anomalous self energy. The perturbative solution of the resulting quartic is easily found:

$$\gamma_{\mathrm{rainbow}} = \frac{3-\sqrt{5+4\sqrt{1+a}}}{2} = -\frac{a}{6} + 11\frac{a^2}{6^3} - 206\frac{a^3}{6^5} + \cdots \tag{3}$$

Resumming chains: At the other extreme, one may easily perform the Borel resummation of chains of self-energy insertions, within a single rainbow. Suppose that the self energy $p^2\overline{\Sigma}(a, p^2/\mu^2)$ is renormalized in the momentum scheme, and hence vanishes at $p^2 = \mu^2$. The renormalized massless propagator is $\overline{D} = 1/(p^2 - p^2\overline{\Sigma})$. Then (3) is the rainbow approximation for $\partial\overline{\Sigma}/\partial\log(\mu^2)$ at $p^2 = \mu^2$. Following the methods of [8], one finds that the corresponding asymptotic series for chains is Borel resummable:

$$\gamma_{\mathrm{chain}} = -6\int_0^\infty \frac{\exp(-6x/a)dx}{(x+1)(x+2)(x+3)} \simeq -\frac{a}{6} + 11\frac{a^2}{6^3} - 170\frac{a^3}{6^5} + \cdots \tag{4}$$

which differs from the rainbow approximation at 3 loops, with 206 in (3) coming from the triple rainbow, while 170 in (4) comes from a chain of two self energies inside a third.

Hopf algebra: We shall progress beyond the rainbow and chain approximations by including all possible nestings and chainings of the one-loop self-energy divergence. In other words, we consider the full Hopf algebra of undecorated rooted trees, established in [1] and implemented in [4]. Two figures suffice to exhibit the class of diagrams considered, and their divergence structure. The first exhibits a 12-loop example, the second exhibits its divergence structure. Due to the fact that we combine chains and rainbows, we have a full tree structure [1]: the depth of the tree is larger than one, and there can be more than one edge attached to a vertex.

There are 4 notable features of this analysis.

1. We use the coproduct Δ to combine the antipode S and grading operator Y in a star product $S \star Y$ whose residue delivers the contribution of each rooted tree.

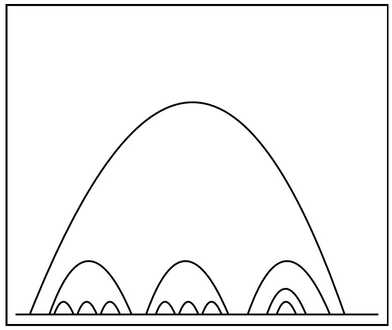

Figure 1: A 12-loop diagram based on a one-loop skeleton.

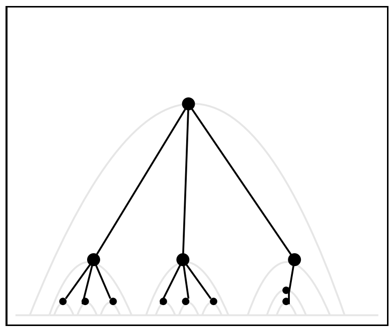

Figure 2: The divergence structure of the previous figure.

2. We show that the rationality of rainbows [10] extends to the contribution of every undecorated rooted tree, as had been inferred from examples in [11].

3. We confirm that a recent analysis [12] of dimensional regularization applies at both $d_c = 4$ and $d_c = 6$, detecting poles of Γ functions that occur in even dimensions.

4. We obtain, to 30 loops, highly non-trivial alternating asymptotic series, which we resum, to high precision, by combining Padé [13] and Borel [8, 9] methods.

2 Hopf-algebra method

Let t be an undecorated rooted tree, denoting the divergence structure of a Feynman diagram. Then its coproduct is defined, recursively, by

$$\Delta(t) = t \otimes e + id \otimes B_+(\Delta(B_-(t))) \tag{5}$$

where e is the empty tree, evaluating to unity, id is the identity map, B_- removes the root, giving a product of trees in general, and B_+ is the inverse of B_-, combining products by restoring a common root. The recursion terminates with $\Delta(e) = e \otimes e$ and develops a highly non-trivial structure by the operation of the coproduct on products of trees

$$\Delta\left(\textstyle\prod_k t_k\right) = \textstyle\prod_k \Delta(t_k) \tag{6}$$

between each removal and restoration of a root. In Sweedler notation, it takes the form

$$\Delta(t) = \sum_k a_k^{(1)} \otimes a_k^{(2)} = t \otimes e + e \otimes t + \sum_k' a_k^{(1)} \otimes a_k^{(2)} \tag{7}$$

with single trees on the right and, in general, products on the left. The prime in the second summation indicates the absence of the empty tree. The field-theoretic role of the coproduct is clear: on the left products of subdivergences are identified; on the right these shrink to points. Subtractions are effected by the antipode, defined by the recursion

$$S(t) = -t - \sum_k' S(a_k^{(1)}) a_k^{(2)} \tag{8}$$

for a non-empty tree, with $S(\prod_k t_k) = \prod_k S(t_k)$ for products and $S(e) = e$.

Renormalization involves a twisted antipode, S_R. Let ϕ denote the Feynman map that assigns a dimensionally regularized bare value $\phi(t)$ to the diagram whose divergence structure is labelled by the tree t. Then we apply the recursive definition [4]

$$S_R(t) = -R\left(\phi(t) + \sum_k' S_R(a_k^{(1)}) \phi(a_k^{(2)})\right) \tag{9}$$

with a renormalization operator R that sets $p^2 = \mu^2$, in both the momentum and MS schemes, and in the MS scheme selects only the poles in $\varepsilon := (d_c - d)/2$.

We can use the coproduct to combine operators. Suppose that O_1 and O_2 operate on trees and their products. Then we define the star product $O_1 \star O_2$ by

$$O_1 \star O_2(t) = \sum_k O_1(a_k^{(1)}) O_2(a_k^{(2)}) \tag{10}$$

with ordinary multiplication performed after O_1 operates on the left and O_2 on the right of each term in the coproduct. By construction, $S \star id$ annihilates everything except the empty tree, e. The presence of R makes $S_R \star \phi$ finite and non-trivial. In particular, the renormalized Green function is simply

$$\Gamma_R(t) = \lim_{\varepsilon \to 0} S_R \star \phi(t) \tag{11}$$

whose evaluation was efficiently encoded in [4], using a few lines of computer algebra.

Here we present a new – and vital – formula for efficiently computing the contribution of an undecorated tree to the anomalous dimension. It is simply

$$\gamma(t) = \lim_{\varepsilon \to 0} \varepsilon \phi(S \star Y(t)) \tag{12}$$

where Y is the grading operator, with $Y(t) = nt$, for a tree with n nodes. In general, Y multiplies a product of trees by its total number of nodes. To see that this works, consider the terms in (11), in the momentum scheme, before taking the limit $\varepsilon \to 0$. Each term has a momentum dependence $(p^2)^{n(d-d_c)/2}$, where n is the number of loops (and hence nodes) of the tree on the right of the term in the Sweedler sum. If we multiply by $n\varepsilon$, and then let $\varepsilon \to 0$, we clearly obtain the derivative w.r.t. $\log(\mu^2/p^2)$. Setting $p^2 = \mu^2$ we obtain the contribution to the anomalous dimension. Thus R plays no role and we may replace $S_R(t)$ by $\lim_{R \to id} S_R(t) = \phi(S(t))$, where S is the canonical antipode. Multiplication by $n\varepsilon$ is achieved by $\varepsilon\phi(Y(t)) = n\varepsilon\phi(t)$ on the right of the coproduct, where Y acts only on single trees. Hence the abstract operator $S \star Y$ delivers the precise combination of products of trees whose bare evaluation as Feynman diagrams is guaranteed to have merely a $1/\varepsilon$ singularity, with residue equal to the contribution to the anomalous dimension. Thus we entirely separate the combinatorics from the analysis.

3 Example

By way of example, we show how the 3-loop expansions of (3,4) result from (12). The combinatorics are now clear. The analysis, at first sight, seems to entail the detailed properties of Γ functions. However, appearances can be misleading.

In general, a dimensionally regularized bare value for a n-loop diagram, corresponding to the undecorated rooted tree t, is evaluated by the recursion [4]

$$\phi(t) = \frac{L(\varepsilon, n\varepsilon)}{n\varepsilon} \prod_k \phi(b_k) \tag{13}$$

where b_k are the branches originating from the root of t. It terminates with $\phi(e) = 1$. For the scalar theory with $d_c = 6$, the master function is

$$L(\varepsilon, \delta) = \frac{a\delta}{(p^2)^\varepsilon} G(1, 1+\delta-\varepsilon; 6-2\varepsilon) = -\frac{a}{(p^2)^\varepsilon} \frac{\Gamma(1-\delta)\Gamma(1+\delta)\Gamma(2-\varepsilon)}{\Gamma(4-\delta-\varepsilon)\Gamma(1+\delta-\varepsilon)} \tag{14}$$

Now the wonderful feature of (12) is that it depends only on the derivatives of $L(\varepsilon, \delta)$ w.r.t. δ at $\varepsilon = 0$. This reflects the fact that the anomalous dimension, unlike the Green function, is insensitive to the details of the regularization method. Thus we may, with huge savings in computation time, replace the master function by

$$L(0, \delta) = \frac{a}{(\delta-1)(\delta-2)(\delta-3)} = \sum_{n\geq 0} g_n \delta^n = -\frac{a}{6} + 11\frac{a\delta}{6^2} - 85\frac{a\delta^2}{6^3} + O(\delta^3) \tag{15}$$

which establishes that the contribution of each rooted tree is rational. The residue of the anomalous dimension operator $S \star Y$ feels only the rational residues of Γ functions; it is blind to the zeta-valued derivatives that contribute to the renormalized Green function.

Now that the analysis has been drastically simplified, we return to the combinatorics. The double rainbow, t_2, has coproduct $\Delta(t_2) = t_2 \otimes e + e \otimes t_2 + t_1 \otimes t_1$ where t_1 is the single rainbow, with $\Delta(t_1) = t_1 \otimes e + e \otimes t_1$. The antipodes are $S(t_1) = -t_1$ and $S(t_2) = -t_2 + t_1^2$. The star products are $S \star Y(t_1) = t_1$ and $S \star Y(t_2) = 2t_2 - t_1^2$. Hence the contributions to the anomalous dimensions are the residues of $L(0, \varepsilon)/\varepsilon$ and $(L(0, 2\varepsilon) - L(0, \varepsilon))L(0, \varepsilon)/\varepsilon^2$, namely $g_0 = -a/6$ and $g_1 g_0 = 11a^2/6^3$.

Following this simple example, the reader should find it easy to determine the anomalous dimension contributions of the two rooted trees at 3 loops. For t_3, the triple rainbow graph, $S \star Y$ delivers $3t_3 - 3t_1t_2 + t_1^3$, with residue $g_2 g_0^2 + g_1^2 g_0 = -(85 + 11^2)a^3/6^5$, in agreement with (3). For the other diagram, t_3', with a double chain in a single rainbow, it delivers $3t_3' - 4t_1t_2 + t_1^3$ with residue $2g_2g_0^2 = -2 \times 85a^3/6^5$, in agreement with (4). The Borel resummation (4) of chains corresponds to the result $n! g_n g_0^n$ for a chain of n self energies, inside a single rainbow. Writing the anomalous dimension contribution of the full Hopf algebra as the asymptotic series

$$\gamma_{\rm hopf} \simeq \sum_{n>0} G_n \frac{(-a)^n}{6^{2n-1}} \tag{16}$$

we find that $G_3 = 3 \times 85 + 11^2 = 376$.

In this paper, we undertake Padé-Borel resummation of the full Hopf series (16), to 30 loops. We also resum

$$\widetilde{\gamma}_{\rm hopf} \simeq \sum_{n>0} \widetilde{G}_n \frac{(-a)^n}{2^{2n-1}} \tag{17}$$

for the anomalous dimension of a fermion field with a Yukawa interaction $g\overline{\psi}\sigma\psi$, at $d_c = 4$, whose rainbow approximation

$$\widetilde{\gamma}_{\rm rainbow} = 1 - \sqrt{1+a} \tag{18}$$

was obtained in [6]. At the other extreme, the Borel-resummed chain approximation

$$\widetilde{\gamma}_{\rm chain} = -2 \int_0^\infty \frac{\exp(-2x/a)dx}{x+2} \tag{19}$$

is easily obtained from the Yukawa generating function, $\widetilde{L}(0,\delta) = a/(\delta - 2)$.

4 Results to 30 loops

At 4 loops, there are 5 undecorated Wick contractions, corresponding to 4 rooted trees, one of which has weight 2. For the scalar theory, at $d_c = 6$, the tally is

$$G_4 = 4890 + 4711 + 3595 + 3595 + 3450 = 20241 = 3^2 \times 13 \times 173 \tag{20}$$

Already this becomes tedious to compute by hand. Fortunately, the recursions (5,8) of the coproduct and antipode make it sublimely easy to automate the procedure (12).

At n loops, the number of relevant Wick contractions is the Catalan number C_{n-1}, where $C_n := \frac{1}{n+1}\binom{2n}{n}$. At 30 loops, there are $C_{29} = 1\,002\,242\,216\,651\,368$ contractions. Symmetries reduce these to rooted trees, with weights determined recursively by $W(t) = w(t)\prod_k W(b_k)$ where b_k are the branches obtained by removing the root of t. The symmetry factor of the root is $w(t) = (\sum_j n_j)!/\prod_j n_j!$ where n_j is the number of branches of type j. The generating formula for R_n, the number of rooted trees with n nodes, is [14] $\sum_{n>0} R_n x^n = x\prod_{n>0}(1-x^n)^{-R_n}$ which expresses the fact that removal of roots from all trees with n nodes produces all products of trees with a total of $n-1$ nodes. This gives $R_{30} = 354\,426\,847\,597$. The number of terms produced by applying the BPHZ procedure [15] to a single tree with n nodes is 2^n.

From these enumerations, one finds – with some trepidation – that computation to 30 loops entails $\sum_{n\le 30} 2^n R_n = 463\,020\,146\,037\,416\,130\,934$ subtractions, each requiring 30 terms in its Laurent expansion, with coefficients involving integers of $O(10^{60})$. Brute force would require processing of $O(10^{24})$ bits of data, which is far beyond anything contemplated by current computer science. The remedy is clear: recursion of coproduct and antipode, to compute the residues of the anomalous dimension operator $S \star Y$.

Each new coproduct or antipode refers to others with fewer loops. By storing these we easily progressed to 13 loops, extending the sequence G_n to

1, 11, 376, 20241, 1427156, 121639250, 12007003824, 1337583507153,
165328009728652, 22404009743110566, 3299256277254713760,
524366465815117346250, 89448728780073829991976

For $\widetilde{G}_n$, in the Yukawa case, we obtained the 13-loop sequence

1, 1, 4, 27, 248, 2830, 38232, 593859, 10401712, 202601898
4342263000, 101551822350, 2573779506192

At this point, recursion of individual trees hit a ceiling imposed by memory limitations.

Beyond 13 loops, we stored only the unique combination of terms that is needed at higher loops, namely the momentum-scheme renormalized self energy. Allocating 750 megabytes of main memory to Reduce 3.7 [16], the time to reach 30 loops was 8 hours. Of these, more than 2 hours were spent on garbage collection, indicating the combinatoric complexity. Results for the scalar and Yukawa theories are in Tables 1 and 2. They are highly non-trivial. Factorization of $G_{27} = 2^6 \times 5 \times 103 \times 18489245764504883 6717 \times 6994310485062168126832946962 4581$ needed significant use of Richard Crandall's elliptic curve routine [17], while $G_{29}/240$ is a 60-digit integer that is most probably prime.

5 Padé-Borel resummation

We combine Padé-approximant [13] and Borel-transformation [8, 9] methods. From (4) we obtain the pure chain contribution $G^{\rm chain}_{n+1} = (2^n + (2^n - 1)3^{n+1})n!$ with, for example, $G^{\rm chain}_4 = (8 + 7 \times 81) \times 6 = 3450$ appearing in (20) as the smallest contribution of the 5 Wick contractions at 4 loops, while the pure rainbow contribution, 4711, is next to largest. This is far removed from the situation at large n, where the pure rainbow term is factorially smaller than the pure chain term. At large n, we combine $C_{n-1} \approx 4^{n-1}/\sqrt{n^3\pi}$ Wick contractions, some of which are of order $G^{\rm chain}_n$, while some are far smaller. It is thus difficult to anticipate the large-n behaviour of G_n. We adopted an empirical approach, finding that $S_n := 12^{1-n} G_n/\Gamma(n + 2)$ varies little for $n \in [14, 30]$, as shown in the final column of Table 1. In the Yukawa case of Table 2, we found little variation in $\widetilde{S}_n := 2^{1-n}\widetilde{G}_n/\Gamma(n + 1/2)$.

In the scalar case, at $d_c = 6$, Padé-Borel resummation may be achieved by the Ansatz

$$\gamma_{\rm hopf} \approx -\frac{a}{12}\int_0^\infty P(ax/3)e^{-x}x^2\,dx \tag{21}$$

where $P(y) = 1 + O(y)$ is a $[M\backslash N]$ Padé approximant, with numerator $1 + \sum_{m=1}^M c_m y^m$ and denominator $1+\sum_{n=1}^N d_n y^n$, chosen so as to reproduce the first $M+N+1$ terms in the asymptotic series (16). We expect $P(y)$ to have singularities only in the left half-plane. In particular, a pole near $y = -1$ is expected, corresponding to the approximate constancy of S_n in Table 1. We fitted the first 29 values of G_n with a $[14\backslash 14]$ Padé approximant $P(y)$, finding a pole at $y \approx -0.994$. The other 13 poles have $\Re y < -1$. Moreover there is no zero with $\Re y > 0$. The test-value G_{30} is reproduced to a precision of 5×10^{-16}.

In the Yukawa case, at $d_c = 4$, we made the Ansatz

$$\widetilde{\gamma}_{\rm hopf} \approx -\frac{a}{\sqrt{\pi}} \int_0^\infty Q(ax/2) e^{-x} x^{1/2}\, dx; \quad Q(y) := \frac{\widetilde{P}(y)}{1+y} \qquad (22)$$

suggested by Table 2. Here we put in by hand the suspected pole at $y = -1$. The [14\14] approximant to $\widetilde{P}(y) = 1 + O(y)$ then has all its 14 poles at $\Re y < -1$ and no zero with $\Re y > 0$. The test-value $\widetilde{G}_{30}$ is reproduced to a reassuring precision of 4×10^{-17}.

Table 3 compares resummation of the full Hopf results (16,17) with those from the far more restrictive chain and rainbow subsets. To test the precision of resummations (21,22), we used the star product (12) to perform the 2.6×10^{21} BPHZ subtractions that yield the exact 31-loop coefficients

$$\begin{aligned} G_{31} &= 2^6 \times 3^3 \times 5 \times 139 \times 2957 \times 22279 \times 69318820356301 \times 9602299922477621 \\ &\quad \times 144927172127490232568467 \qquad (23) \\ \widetilde{G}_{31} &= 2^5 \times 3^4 \times 5 \times 71 \times 109 \times 13224049649 \times 473202021103152647613521 \qquad (24) \end{aligned}$$

No change in the final digits of Table 3 results from using these. At the prodigious Yukawa coupling $g = 30$, corresponding to $a = (30/4\pi)^2 \approx 5.7$, a [15\15] Padé approximant gives $\widetilde{\gamma}_{\rm hopf} \approx -1.85202761$, differing by less than 1 part in 10^8 from the [14\14] result $\widetilde{\gamma}_{\rm hopf} \approx -1.85202762$. It appears that resummation of undecorated rooted trees is under very good control, notwithstanding the combinatoric explosion apparent in (23,24).

6 Conclusions

As stated in the introduction, we achieved 4 goals. First, we found the Hopf-algebra construct (12) that delivers undecorated contributions to anomalous dimensions. Then we found that these are rational, with the Γ functions of (14) contributing only their residues, via (15). Next, we exemplified the analysis of dimensional regularization in [12], at two different critical dimensions, $d_c = 6$ and $d_c = 4$. The residues of a common set (1) of Γ functions determine both results. Finally, we obtained highly non-trivial results, from all combinations of rainbows and chains, to 30 loops. *A priori*, we had no idea how these would compare with the easily determined pure chain contributions. Tables 1 and 2 suggest that at large n the full Hopf-algebra results exceed pure chains by factors that scale like $n^2 2^n$ and $n^{1/2} 2^n$, respectively. Padé approximation gave 15-digit agreement with exact 30-loop results. In Table 3, we compare the Borel resummations (21,22) of the full Hopf algebra with the vastly simpler rainbow approximations (3,18) and the still rather trivial chain approximations (4,19). Even at the very large Yukawa coupling $g = 30$ we claim 8-digit precision. Apart from large-N_f approximations [13], we know of no other large-coupling analysis of anomalous dimension contributions, at spacetime dimensions $d \geq 4$, that progresses beyond pure rainbows [6, 7] or pure chains [8, 9].

In conclusion: Hopf algebra tames the combinatorics of renormalization, by disentangling the iterative subtraction of primitive subdivergences from the analytical challenge of evaluating dimensionally regularized bare values for Feynman diagrams. Progress with the analytic challenge shall require the expansion of skeleton graphs in the regularization parameter $D-4$. After that, the Hopf algebra of decorated rooted trees provides the tool to take care of the combinatorial challenge of renormalization in general. Generalizations of the methods here to cases where decorations are different, but still analytically trivial, are conceivable. The results in [18] are of this form. In the present case, where the combinatoric explosion is ferocious, while the analysis is routine, the automation of renormalization by Hopf algebra is a joy. How else might one resum 2.6×10^{21} BPHZ subtractions at 31 loops and achieve 8-digit precision at very strong coupling?

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Table 1: Scalar coefficients in (16), with $S_n := 12^{1-n} G_n / \Gamma(n+2)$

n	G_n	S_n
14	16301356287284530869810308	0.1165
15	3161258841758986060906197536	0.1177
16	650090787950164885954804021185	0.1186
17	141326399508139818539694443090940	0.1194
18	32389192708918228594003667471390750	0.1200
19	7805642594117634874205145727265669184	0.1205
20	1973552096478862083584247237907087008846	0.1209
21	522399387732959889862436502331522596697560	0.1212
22	144486332652501966354908665093390779463113660	0.1215
23	41681362292986022786933211385817840822702468640	0.1217
24	12520661507532542738174037622803485508817145773050	0.1218
25	3910338928202486568787314743084879349561179264255736	0.1220
26	1267891158800355844456289086726128521948839015617187260	0.1221
27	426237156086127437403654947366849019736474802601497417920	0.1221
28	148382376919675149120919349602375065827367635238832722748020	0.1222
29	53428133467243180546330391126922442419952183999220340144106320	0.1222
30	19876558632009586773182109989526780486481329823560105761256963720	0.1222

Table 2: Yukawa coefficients in (17), with $\widetilde{S}_n := 2^{1-n}\widetilde{G}_n/\Gamma(n+1/2)$

n	$\widetilde{G}_n$	$\widetilde{S}_n$
14	70282204726396	0.3715
15	2057490936366320	0.3750
16	64291032462761955	0.3780
17	2136017303903513184	0.3806
18	75197869250518812754	0.3828
19	2796475872605709079512	0.3848
20	109549714522464120960474	0.3865
21	4509302910783496963256400	0.3880
22	194584224274515194731540740	0.3894
23	8784041120771057847338352720	0.3906
24	414032133398397494698579333710	0.3917
25	20340342746544244143487152873888	0.3928
26	1039819967521866936447997028508900	0.3937
27	55230362672853506023203822058592752	0.3946
28	3043750896574866226650924152479935036	0.3953
29	173814476864493583374050720641310171808	0.3961
30	10272611586206353744425870217572111879288	0.3968

Table 3: Comparison of chain, rainbow and full Hopf contributions

a	$-\gamma_{\text{chain}}$	$-\gamma_{\text{rainbow}}$	$-\gamma_{\text{hopf}}$	$-\widetilde{\gamma}_{\text{chain}}$	$-\widetilde{\gamma}_{\text{rainbow}}$	$-\widetilde{\gamma}_{\text{hopf}}$
0.5	0.0727579	0.0731322	0.0742476	0.2245593	0.2247449	0.2278233
1.0	0.1301409	0.1322419	0.1373080	0.4126913	0.4142136	0.4281423
1.5	0.1773375	0.1825988	0.1937609	0.5765641	0.5811388	0.6118625
2.0	0.2172313	0.2268615	0.2455916	0.7226572	0.7320508	0.7837372
2.5	0.2516214	0.2665867	0.2939133	0.8549759	0.8708287	0.9464649
3.0	0.2817148	0.3027756	0.3394353	0.9762193	1.0000000	1.1017856
3.5	0.3083635	0.3361156	0.3826462	1.0883141	1.1213203	1.2509126
4.0	0.3321923	0.3671015	0.4239016	1.1926947	1.2360680	1.3947383
4.5	0.3536734	0.3961033	0.4634712	1.2904639	1.3452079	1.5339452
5.0	0.3731724	0.4234058	0.5015652	1.3824908	1.4494897	1.6690711
5.5	0.3909778	0.4492331	0.5383523	1.4694751	1.5495098	1.8005504
6.0	0.4073216	0.4737658	0.5739698	1.5519895	1.6457513	1.9287404