A BIJECTION BETWEEN THE d-DIMENSIONAL SIMPLICES WITH ALL DISTANCES IN $\{1, 2\}$ AND THE PARTITIONS OF $d + 1$

Sascha Kurz
University of Bayreuth
95440 Bayreuth
Germany
sascha.kurz@stud.uni-bayreuth.de

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Abstract

We give a construction for the d-dimensional simplices with all distances in $\{1, 2\}$ from the set of partitions of $d + 1$.

1 Introduction

Because there is some interest in integral point sets, i.e. sets of n points in the euclidean $\mathbb{E}^d$ with integral distances between vertices, we examined such point sets for $n = d + 1$. We use the term simplex for a point set of $d+1$ points in the euclidean $\mathbb{E}^d$ not all points lying in a hyperplane of the $\mathbb{E}^d$. For $d = 2$ this is a triangle and for $d = 3$ this is a tetrahedron. Similar to integral point sets we define integral simplices as simplices with integral distances between the vertices. The largest distance of a point set is called its diameter. Calculations, done by a computer, gave a short table of the number of nonisomorphic integral simplices by diameter and dimension.

dimension diameter	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1
2	4	6	10	14	21	29	41
3	16	56	197	656	2127	6548	19130
4	45	336	3133	31771	329859	3336597	32815796

Table 1. Number of integral simplices by diameter and dimension.

There is clearly an unique integral simplex with diameter 1 in any dimension. By testing the sequence of the number of simplices with diameter 2 with N.J.A. Sloane's marvellous 'Online-Encyclopedia of Integer Sequences' [4] we learned that the first calculated terms are equal to numbers of partitions of a natural number fewer 1.

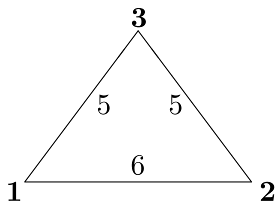
A partition of a natural number n is an r-tuple of natural numbers $(i_1, \ldots, i_r)$ with $i_1 \geq i_2 \geq \ldots \geq i_r > 0$ and $i_1 + i_2 + \ldots + i_r = n$.

In the next section we give an algorithm which constructs an integral simplex from a partition. And in Section 3 we proof that there is indeed a bijection.

Theorem. The number of integral d-dimensional simplices with diameter at most 2 is the number of partitions of $d+1$, and all simplices can be constructed by the algorithm of Section 2.

2 Construction

We would like to give an algorithm which constructs an integral simplex from of a given partition. Therefore we represent an integral simplex by its distance matrix $A = (d_{ij})$ with point i and point j of the integral simplex having distance d_{ij}. See Figure 1 for an example.



$$\begin{pmatrix} 0 & 6 & 5 \\ 6 & 0 & 5 \\ 5 & 5 & 0 \end{pmatrix}$$

Figure 1. An integral triangle with its distance matrix.

Algorithm.
Input: A partition $(i_1, \dots, i_r)$
Output: A distance matrix A, corresponding to an integral simplex

We construct recursively.
(i) The partition (n) yields the n-dimensional matrix

$$\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}.$$

(ii) For $r > 1$ we construct the matrix B from the partition $(i_1, \dots, i_{r-1})$. And set

$$A = \begin{pmatrix} 0 & 2 & \cdots & 2 & \mathbf{1} & \cdots & \cdots & \mathbf{1} \\ 2 & & & & 2 & \cdots & \cdots & 2 \\ \vdots & & B & & \vdots & \ddots & \ddots & \vdots \\ 2 & & & & 2 & \cdots & \cdots & 2 \\ 1 & 2 & \cdots & 2 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 2 & \cdots & 2 & 1 & \cdots & 1 & 0 \end{pmatrix}$$

with the right upper block of 1's (bold printed) of width $i_r - 1$.

We would like to illustrate the bijection by the first few examples.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$$(2) \qquad\qquad (1,1)$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

$$(3) \qquad\qquad (2,1) \qquad\qquad (1,1,1)$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 2 & 1 \\ 2 & 0 & 1 & 2 \\ 2 & 1 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

$$(4) \qquad\qquad (2,2) \qquad\qquad (3,1)$$

$$\begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{pmatrix}$$

$$(2,1,1) \qquad\qquad (1,1,1,1)$$

Figure 2. Distance matrices and corresponding partitions.

In the next section we will show that this algorithm gives all integral simplices of diameter at most 2 without repetitions.

3 Proof for the bijection

At first we want to illustrate the bijection between partitions and integral simplices with diameter at most 2 by an example. Let A be the distances matrix constructed of the partition $(4,3,3,2,2)$. Thus

$$A = \begin{pmatrix}
0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\
2 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 \\
2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 \\
2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 1 & 1 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 0 & 1 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 0 & 2 & 2 & 2 & 2 \\
2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 1 & 2 & 2 \\
2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 & 2 & 2 \\
2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0
\end{pmatrix}.$$

This yields the upper right triangle matrix of A.

$$\overline{A} = \begin{pmatrix}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \mathbf{1} \\
 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \mathbf{1} & 2 \\
 & & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \mathbf{1} & \mathbf{1} & 2 & 2 \\
 & & & 2 & 2 & 2 & 2 & \mathbf{1} & \mathbf{1} & 2 & 2 & 2 & 2 \\
 & & & & \mathbf{1} & \mathbf{1} & \mathbf{1} & 2 & 2 & 2 & 2 & 2 & 2 \\
 & & & & & \mathit{1} & \mathit{1} & 2 & 2 & 2 & 2 & 2 & 2 \\
 & & & & & & \mathit{1} & 2 & 2 & 2 & 2 & 2 & 2 \\
 & & & & & & & 2 & 2 & 2 & 2 & 2 & 2 \\
 & & & & & & & & \mathit{1} & 2 & 2 & 2 & 2 \\
 & & & & & & & & & 2 & 2 & 2 & 2 \\
 & & & & & & & & & & \mathit{1} & 2 & 2 \\
 & & & & & & & & & & & 2 & 2 \\
 & & & & & & & & & & & & 2
\end{pmatrix}$$

Figure 3. Boundary of ones.

The shape of $\overline{A}$ can be described as follows. The length of a bold printed block of ones is one less than the corresponding summand of the partition. Each block of such ones is completed to a upper right triangular matrix consisting only of ones (printed in italics) at the bottom of the corresponding columns. The remaining places are filled with twos. It is not difficult to show that every partition yields a matrix with such a shape.

There are several things to prove the proposed bijection. We have to show that the matrices, constructed by the algorithm of Section 2, are integral simplices. Not every symmetric matrix can be realized as a distance matrix in the euclidean space. There is for example no triangle with side length 4,2, and 1. At this point we can use a theorem by Menger [2], which reduces the problem to calculate certain determinants. With Definition 1 we can state the used part of Menger's theorem as follows. If M is a set of $d+1$ points with distance matrix $D=(d_{ij})$ and squared distance matrix $A=(d_{i,j}^2)$, then M is realizable in the euclidean d-dimensional space, iff $(-1)^{d+1}det(\hat{A})\ge 0$ and each subset of M is realizable in the $(d-1)$-dimensional space.

Definition 1. For a matrix A we define $\hat{A}$ by

$$\hat{A} := \begin{pmatrix} & & & 1 \\ & A & & \vdots \\ & & & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}.$$

We denote the determinant of a matrix A by $det(A)$, and its number of rows by $dim(A)$.

Lemma 1. If $D=(d_{i,j})_{1\le i,j\le dim(A)}$ is a matrix, which is constructed by the algorithm in Section 2, and $A=(d_{i,j}^2)$ then it holds

$$(-1)^{dim(A)}(4\,det(\hat{A})+det(A))>0$$

and

$$(-1)^{dim(A)}det(\hat{A})>0.$$

Proof.
We prove by induction on $dim(A)$.

Induction start $dim(A) = 2$.
Here there are only the two cases corresponding to lines of length 1 and 2.
For $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we get

$$det(A) = -1,\ det(\hat{A}) = 2$$

and so

$$(-1)^2(4 \cdot 2 - 1) > 0,\ (-1)^2 2 > 0.$$

For $A = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$ we get

$$det(A) = -16,\ det(\hat{A}) = 8$$

and so

$$(-1)^2(4 \cdot 8 - 16) > 0,\ (-1)^2 8 > 0.$$

Induction step.
(i) The corresponding simplex to A has only unit distances. (This is case (i) of the algorithm in Section 2.) A little calculation leads to

$$det(A) = (-1)^{dim(A)-1}(dim(A) - 1),\ det(\hat{A}) = (-1)^{dim(A)} dim(A).$$

Thus

$$\begin{gathered} (-1)^{dim(A)}(4 \cdot (-1)^{dim(A)} dim(A) + (-1)^{dim(A)-1}(dim(A) - 1)) \\ = 3\, dim(A) + 1 > 0, \\ (-1)^{dim(A)}(-1)^{dim(A)} dim(A) = dim(A) > 0 \end{gathered}$$

holds.

(ii) The matrix A has the shape

$$\begin{pmatrix} 0 & 4 & \cdots & 4 \\ 4 & & & \\ \vdots & & B & \\ 4 & & & \end{pmatrix}$$

with a Matrix B corresponding to one from the construction algorithm. Dividing the first row and the first column by 4, and reordering yields $det(A) = 16\, det(\hat{B})$.

$$det(\hat{A}) = \begin{vmatrix} 0 & 4 & \cdots & 4 & 1 \\ 4 & & & & 1 \\ \vdots & & B & & \vdots \\ 4 & & & & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix}$$

Subtracting 3 times the last row from the first row,4 times the last column from the first column yields

$$det(\hat{A}) = \begin{vmatrix} -7 & 1 & \cdots & 1 & 1 \\ 0 & & & & 1 \\ \vdots & & B & & \vdots \\ 0 & & & & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix}.$$

After developing the first column and reordering we get

$$det(\hat{A}) = -7\, det(\hat{B}) - det(\hat{B}) - det(B).$$

Inserting into the two conditions yields

$$\begin{gathered}(-1)^{dim(A)}(4(-8\, det(\hat{B}) - det(B)) + 16\, det(\hat{B})) \\ = 4((-1)^{dim(B)}(4\, det(\hat{B}) + det(B))) > 0,\end{gathered}$$

$$\begin{gathered}(-1)^{dim(A)}(-8\, det(\hat{B}) - det(B)) \\ > 4(-1)^{dim(B)}\, det(\hat{B}) > 0.\end{gathered}$$

(iii) The matrix $A = C_j$ has the shape

$$\begin{pmatrix} 0 & 4 & \cdots & 4 & 1 & \cdots & \cdots & 1 \\ 4 & & & & 4 & \cdots & \cdots & 4 \\ \vdots & & B & & \vdots & \ddots & \ddots & \vdots \\ 4 & & & & 4 & \cdots & \cdots & 4 \\ 1 & 4 & \cdots & 4 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 4 & \cdots & 4 & 1 & \cdots & 1 & 0 \end{pmatrix}$$

with the upper right block of width j. With this notation we have shown the lemma for C_0 in (ii).

Subtracting the first column of C_j from the last column yields

$$det(C_j) = \begin{vmatrix} 0 & 4 & \cdots & 4 & 1 & \cdots & \cdots & 1 & 1 \\ 4 & & & & 4 & \cdots & \cdots & 4 & 0 \\ \vdots & & B & & \vdots & \ddots & \ddots & \vdots & \vdots \\ 4 & & & & 4 & \cdots & \cdots & 4 & 0 \\ 1 & 4 & \cdots & 4 & 0 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & 1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots \\ 1 & 4 & \cdots & 4 & 1 & \cdots & 1 & 0 & 0 \\ 1 & 4 & \cdots & 4 & 1 & \cdots & \cdots & 1 & -1 \end{vmatrix}.$$

Developing the last column, and moving, in the first submatrix, the last row to the first, row yields

$$= -det(C_{j-1}) - \begin{vmatrix} 0 & 4 & \cdots & 4 & 1 & \cdots & \cdots & 1 \\ 4 & & & & 4 & \cdots & \cdots & 4 \\ \vdots & & B & & \vdots & \ddots & \ddots & \vdots \\ 4 & & & & 4 & \cdots & \cdots & 4 \\ 1 & 4 & \cdots & 4 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & 0 & 1 \\ 1 & 4 & \cdots & 4 & 1 & \cdots & 1 & 1 \end{vmatrix}.$$

If $j > 1$ we get by extracting the bottom right 1 of the second summand

$$det(C_j) = -det(C_{j-1}) - det(C_{j-1}) - det(C_{j-2}) = -2\,det(C_{j-1}) - det(C_{j-2}).$$

For $j = 1$ we get by the same operation

$$det(C_1) = -det(C_0) - det(C_0) - det(B) = -32\,det(\hat{B}) - det(B).$$

We define $C_{-1} = B$ to avoid the distinction. Now we use the $\hat{}$ operator on the above calculation to get

$$det(\hat{C}_j) = -2\,det(\hat{C}_{j-1}) - det(\hat{C}_{j-2}).$$

By induction we get

$$4\,det(\hat{C}_j) + det(C_j) = (-1)^{j+1}(4j + 1)(4\,det(\hat{B}) + det(B)).$$

With $dim(C_j) = 1 + dim(B) + j$ and the induction hypothesis, we see that the first condition holds. We can also derive by induction

$$det(\hat{C}_j) = (-1)^{j+1}((7j + 8)det(\hat{B}) + (j + 1)det(B))$$

and conclude

$$(-1)^{dim(C_j)}det(\hat{C}_j) = (j + 1)(-1)^{dim(B)}(4\,det(\hat{B}) + det(B)) +$$
$$(3j + 4)(-1)^{dim(B)}det(\hat{B}) > 0.$$

□

Lemma 2.
If we chooses a matrix A with $dim(A) \geq 2$ as in Lemma 1 then for every sub-matrix A_i, obtained by deleting row and column i of A, $(-1)^{dim(A_i)}det(\hat{A}_i) > 0$ holds.

Proof. Suppose we are given such a matrix

$$A = \begin{pmatrix} 0 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{1} \\ 4 & 0 & 4 & 4 & 4 & 4 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 4 \\ 4 & 4 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 4 & 4 & 4 & 4 \\ 4 & 4 & 1 & 0 & 1 & 1 & 4 & 4 & 4 & 4 \\ 4 & 4 & 1 & 1 & 0 & 1 & 4 & 4 & 4 & 4 \\ 4 & 4 & 1 & 1 & 1 & 0 & 4 & 4 & 4 & 4 \\ 4 & 1 & 4 & 4 & 4 & 4 & 0 & 1 & 1 & 4 \\ 4 & 1 & 4 & 4 & 4 & 4 & 1 & 0 & 1 & 4 \\ 4 & 1 & 4 & 4 & 4 & 4 & 1 & 1 & 0 & 4 \\ 1 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 0 \end{pmatrix}$$

and the corresponding partition $(4, 4, 2)$.

If we delete a row, and the corresponding column, below the "boundary of ones", e.g. row 4 or 5 we get a matrix corresponding to $(3, 4, 2)$. In the other case we delete above the "boundary of ones",and so delete a block of, bold printed, ones. Deleting row and column 2 yields the matrix

$$\begin{pmatrix} 0 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{1} \\ 4 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 4 & 4 & 4 & 4 \\ 4 & 1 & 0 & 1 & 1 & 4 & 4 & 4 & 4 \\ 4 & 1 & 1 & 0 & 1 & 4 & 4 & 4 & 4 \\ 4 & 1 & 1 & 1 & 0 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 0 & \mathbf{1} & \mathbf{1} & 4 \\ 4 & 4 & 4 & 4 & 4 & 1 & 0 & 1 & 4 \\ 4 & 4 & 4 & 4 & 4 & 1 & 1 & 0 & 4 \\ 1 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 0 \end{pmatrix}.$$

Here we marked also the ones in row 6 which correspond to the block of the deleted bold printed ones. If we move row and column 6 between row respectively column 1 and 2, we get a matrix corresponding to $(4, 3, 2)$. The switch of rows and columns can be described by a permutation τ, in this case $\tau = (3\ 4\ 5\ 6)$.

What might happen, and happened in the first case is that the description as a tuple is not a partition anymore. In $(3, 4, 2)$ we must interchange the 3 and the 4 to gain a partition again. Translated as a operation on the matrix this is again a suitable permutation τ.

Since a permutation acting simultaneous on the rows and columns of a matrix does not change its determinant the lemma is proven. $\square$

Lemma 3.
The matrices constructed by the algorithm of section 2 are simplices.
Proof. Distance matrices that are realizable in the euclidean d-dimensional space have been characterized by Menger [2]. If M is a set of $d+1$ points with distance matrix D, and squared distance matrix A, then M is realizable in the euclidean d-dimensional space, iff $(-1)^{d+1}det(\hat{A}) \geq 0$ and each subset of M is realizable in $(d-1)$-dimensional space.
$(-1)^{d+1}det(\hat{A}) = 0$ is equivalent to the realizability in the $d-1$-dimensional space. Since we do not want degenerate simplices we request $(-1)^{d+1}det(\hat{A}) > 0$. The use of Lemma 1 and Lemma 2 completes the proof of Lemma 3. $\square$

Now we know that the algorithm of Section 2 converts a partition into an integral simplex. To prove that the algorithm yields an injection we define an order on the partitions and a term named value of a matrix.

Definition 2. The value $val(A,b)$ in base b of a symmetric matrix A is defined by

$$val(A,b) = \sum_{i=2}^{dim(A)} \sum j = 1^{i-1} a_{i,j} b^{\frac{dim(A)(dim(A)-1)-i(i-1)}{2}+i-1-j}.$$

The matrix A is called maximal in base b if for every permutation τ

$$val(A,b) \geq val((a_{\tau(i),\tau(j)})_{1\leq i,j\leq dim(A)}, b)$$

holds.

Let us give a little more insight in this definition of the value $val(A,b)$ of a symmetric matrix A. As the diagonal elements are not involved in the definition of $val(A,b)$ and A is symmetric we can consider the upper right triangle matrix of A

$$\overline{A} = \begin{pmatrix} a_{1,2} & a_{1,3} & \cdots & a_{1,dim(A)} \\ & a_{2,3} & \ddots & a_{2,dim(A)} \\ & & \ddots & \vdots \\ & & & a_{dim(A)-1,dim(A)} \end{pmatrix}.$$

If we read this triangle matrix columnwise, and regard the sequence of coefficients as a number in base b notation we have the definition of $val(A, b)$. A simple fact is, that the matrix consisting of the first columns and rows of a maximal in base b matrix is maximal in base b.

Lemma 4.
The matrices constructed by the algorithm in section 2 are maximal in base 3.
Proof. We will prove by induction on the dimension of the matrices. For shortness the coefficients of the matrix A will be denoted by $a_{s,t}$ and similar for the other matrices.

Since the two matrices of dimension 2 are maximal in base 3 the induction start is made.

Induction step.
Let A be such a matrix.

The corresponding partition has the shape $p = (p_1, p_2, \ldots, , p_r)$. Now we consider the set S of partitions where one summand of p is decreased by one. Since we know the maximality of the corresponding matrices by induction hypothesis we can conclude, by a look at the "boundary of ones", that $(p_1, p_2, \ldots, p_{r-1}, p_r - 1)$ leads to a matrix with the maximum value. If $p_r = 1$ then A clearly is maximal in base 3 because the entries of the last column of A do not increase. In the other case there must be at most a one in the last column. Consider the maximal in base 3 representation R of A. We know that the first $dim(A) - 1$ columns of A equal those of the maximal representation. The row number of the first one in the last column of R can not be bigger than the row number of the first one in the last but one column of A, because interchanging the last two columns would yield a contradiction to the maximality of the submatrix. If $p_r \neq 2$ the row number of the first one in the last column of A equals the row number of the first one in the last but one column because the last block of ones has a length at most 2 due to $p_r > 2$. Since possible other ones are at the bottom of the last column of A, thus A is maximal in base 3 in this case. In the other case, $p_r = 2$, consider the number of rows which contain a single one. Since this number is equal for R and A, and R,A equal in the first columns, the row with the first one in the last column of R must be a row with a single one. So we can conclude that A is maximal in base 3. $\Box$.

Definition 3. Given two partitions $i = (i_1, \ldots, i_r)$, $j = (j_1, \ldots, j_s)$ of a natural number n. If $r < s$ then $i < j$, similar if $r > s$ then $i > j$. In the case $r = s$ we choose lexicographical order.

Lemma 5.
For two partitions $p_1 < p_2$, the algorithm of Section 2 produces two matrices A_1, A_2 satisfying $val(A_1, 3) < val(A_2, 3)$
Proof. A look at Figure 3 and a little thought shows the fact. $\square$

We can conclude from Lemma 3, Lemma 4, and Lemma 5 that there are at least as many d-dimensional integral simplices of diameter at most 2 as the number of partitions of $d + 1$.

To construct all integral d-dimensional simplices of diameter at most 2 in order, we let the algorithm of Section 2 work on a complete increasing sequence of the partitions of $d + 1$.

The last thing to prove is that every integral simplex with distances in $\{1, 2\}$ is isomorph to a simplex constructed by the algorithm of Section 2. Due to isomorphism we only must consider integral simplices which are maximal in base 3.

Lemma 6.
Every integral maximal (in base 3) simplex with diameter at most 2 is constructed by the algorithm of Section 2.
Proof. Let A be the distance matrix of the simplex. We will use induction on $dim(A)$. As a induction start we check that the examples in Section 2 are all integral simplices with diameter at most 2 and dimension at most 3 which are maximal in base 3.

Induction step.
By B we denote the matrix consisting of the first $dim(A) - 1$ columns and rows of A. With A maximal in base 3 B is also maximal in base 3. So B was constructed by the algorithm of Section 2 due to induction hypothesis and there is a partition $i = (i_1, \ldots, i_r)$ which corresponds to B. Thus we must only consider the last column of A. We distinguish three cases for the permutation $\tau = (dim(A) - 1, dim(A))$.

(i) τ is an automorphism on A.
If the last column of B contains only twos, then A is constructible from one of the two partitions $(1, \ldots, 1)$ or $(2, 1, \ldots, 1)$. Else both the last and the last but one columns of A contain a one at a row number

say l. Applying the triangle inequality yields $a_{dim(A)-1,dim(A)} = 1$, so A corresponds to the sequence $\bar{i} = (i_1, \ldots, i_r + 1)$. If $\bar{i}$ is a partition then A is constructible by the algorithm, else A is not maximal in base 3.

(ii) $val(\tau(A), 3) > val(A, 3)$.
A contradiction to the maximality of A.

(iii) $val(\tau(A), 3) < val(A, 3)$.
(iii) a) The last but one column of A contains no one, then by induction hypothesis all non diagonal entries of B must equal 2. The last column of A can not contain more than 1 one, because of the triangle inequality and the absence of another 1 in B. If the last column of A contains no one, than A corresponds to the partition $(1, 1, \ldots, 1)$. In the other case the last column of A contains a single one. Due to the maximality of A the single one must be located in the lower right corner of A, and so A is constructible from the partition $(2, 1, \ldots, 1)$.

Define h_1 as the row number of the first one of the last but one column of A. Because $val(\tau(A), 3) < val(A, 3)$ there must exist the row number h_2 of the first one of the last column of A fulfilling $h_1 \geq h_2$. We distinguish the two cases of equality and strict inequality.

(iii) b) $h_1 = h_2$.
Let $l \neq h_2$ be a row number with a one in the last column of A. From $a_{h_2,dim(A)} = a_{l,dim(A)} = 1$ we conclude with the help of the triangle inequality that $a_{h_2,l} = 1$ must hold. Again by using the triangle inequality and $a_{h_2,dim(A)-1} = 1$ we conclude $a_{l,dim(A)-1} = 1$. Together with $val(\tau(A), 3) < val(A, 3)$ we get a contradiction.

(iii) c) $h_1 > h_2$.
From the induction hypothesis we know that the first $h_1 - 1$ rows of B contain only twos, so due to the maximality of A it holds $a_{h_1-1,dim(A)} = 1$. Now assume that there exists a row number $l \neq h_1 - 1$ with a one in the l's row of the last column of A. Applying the triangle inequality yields $a_{h_1-1,l} = 1$, a contradiction. So A is constructible from the partition $(i_1, \ldots, i_{r-h_1+1}, 2, 1, \ldots, 1)$. $\square$

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