Integrability, Exact Solvability, and Algebraic Combinatorics: A Three-Way Bridge?

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I. INTRODUCTION

Some integrable systems

Continuous Painlevé II:

$$y''(x) = 2y^3 + xy + c$$

Discrete version:

$$nx_n + t(1 - x_n^2)(x_{n+1} + x_{n-1}) = 0$$

Autonomous discrete version:

$$x_n + t(1 - x_n^2)(x_{n+1} + x_{n-1}) = 0$$

The dimer model

```
o       o       o------o
|       |
|       |
o       o       o      o
                |      |
                |      |
o------o        o      o



o------o        o------o
```

Fisher and Temperley (1961) and Kasteleyn (1961) showed that the number of dimer configurations on the m-by-n rectangle with mn even is asymptotic to C^{mn}, where $C = e^{G/\pi} \approx 1.34$, and G is Catalan's constant $1-1/9+1/25-1/49+1/81-\ldots$

The Discrete Hirota equation (see Zabrodin):

Let u, v, w be fixed vectors with the vector x varying over some fixed coset of $\mathbf{Z}u + \mathbf{Z}v + \mathbf{Z}w$.

Symmetric form:

$$aF(x+u)F(x-u)+bF(x+v)F(x-v)$$
$$+cF(x+w)F(x-w)=0$$

Asymmetric form:

$$F(x+w)F(x-w) =$$
$$aF(x+u)F(x-u)+bF(x+v)F(x-v)$$

Note:

$$(x+u)+(x-u)$$
$$=(x+v)+(x-v)$$
$$=(x+w)+(x-w).$$

Rhombus tilings of hexagons

Let $H(a,b,c)$ be the number of rhombus tilings of an a,b,c,a,b,c semiregular hexagon (opposite sides of equal length, all internal angles equal to 120 degrees). Then $H(a,b,c)$ satisfies the discrete Hirota relations

$$H(a,b,c)H(a,b-1,c-1)$$
$$= H(a+1,b-1,c-1)H(a-1,b,c)$$
$$+H(a,b-1,c)H(a,b,c-1)$$

and

$$H(a,b,c)H(a,b,c-2)$$
$$= H(a,b,c-1)H(a,b,c-1)$$
$$-H(a-1,b+1,c-1)H(a+1,b-1,c-1)$$

Plücker relations for Schur functions

Let $s_{m,n}$ be the Schur function whose shape is the rectangular Young diagram with m rows and n columns.
Then we have the formula (observed by Kirillov and later generalized by Kleber):

$$s_{m,n}\, s_{m,n} =$$

$$s_{m-1,n}\, s_{m+1,n} + s_{m,n-1}\, s_{m,n+1} \ .$$

Symmetry check:

$$(m,n) + (m,n)$$

$$= (m-1,n) + (m+1,n)$$
$$= (m,n-1) + (m,n+1).$$

Number friezes ("frieze patterns")

(Conway and Coxeter;
Conway and Guy)

```
                           a
Rule: for the pattern  b     c , we have ad=bc-1.
                           d


1   1   1   1   1   1   1   1   1   1   1   1   1   1

  1   3   1   2   2   1   3   1   2   2   1   3   1

    2   2   1   3   1   2   2   1   3   1   2   2

      1   1   1   1   1   1   1   1   1   1   1
```

Number walls

(Conway and Guy; Plouffe and Sloane; Lunnon)

```
                              a
Rule: For the pattern  b c d , we have ae = c^2 - bd
                              e
(with extra provisos we use when a = c^2 - bd = 0)


1    1    1    1    1    1    1    1    1    1    1

1    1    2    3    5    8   13   21   34   55   89

    -1    1   -1    1   -1    1   -1    1   -1

          0    0    0    0    0    0    0
```

If we start with a row of 1's, then we eventually get a row of 0's if and only if the sequence in the second row satisfies a linear recurrence with constant coefficients.

Somos-4 and Somos-5 sequences

The Somos-4 sequence:

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, ...$$

$$S_n S_{n-4} = S_{n-1} S_{n-3} + S_{n-2} S_{n-2}$$

The Somos-5 sequence:

$$1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, 6161, ...$$

$$S_n S_{n-5} = S_{n-1} S_{n-4} + S_{n-2} S_{n-3}$$

II. CONDENSATION OF DETERMINANTS

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,n-1} & m_{1,n} \\ m_{2,1} & m_{2,2} & \dots & m_{2,n-1} & m_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{n-1,1} & m_{n-1,2} & \dots & m_{n-1,n-1} & m_{n-1,n} \\ m_{n,1} & m_{n,2} & \dots & m_{n,n-1} & m_{n,n} \end{pmatrix}$$

$$M_C = \begin{pmatrix} m_{2,2} & \dots & m_{2,n-1} \\ \vdots & \ddots & \vdots \\ m_{n-1,2} & \dots & m_{n-1,n-1} \end{pmatrix} \quad (\text{“center”})$$

$$M_{TL} = \begin{pmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,n-1} \\ m_{2,1} & m_{2,2} & \dots & m_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1,1} & m_{n-1,2} & \dots & m_{n-1,n-1} \end{pmatrix} \quad (\text{“top left”})$$

with the $n - 1$ by $n - 1$ minors M_{TR} (“top right”), M_{BL} (“bottom left”), and M_{BR} (“bottom right”) defined similarly.

The Desnanot-Jacobi identity

$$\det(M)\ \det(M_C) =$$

$$\det(M_{TL})\ \det(M_{BR}) - \det(M_{TR})\ \det(M_{BL})$$

Dodgson condensation

To compute the determinant of an n-by-n matrix, iteratively use this identity to compute the determinants of the connected minors of orders 1, 2, ..., n. (A "connected" k-by-k minor is formed by taking k consecutive rows and k consecutive columns.)
Note: a minor of order 0 has determinant 1.
Example:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 \\ 1 & 6 & 36 \\ 1 & 12 & 144 \end{pmatrix} \to$$

$$\begin{pmatrix} 2 & 12 \\ 2 & 48 \end{pmatrix} \to \begin{pmatrix} 12 \end{pmatrix}$$

Dodgson pyramids

Stacking the matrices in layers, we get a relation of discrete Hirota type, relating the values associated with the vertices of an octahedron:

```
                                a
                             b------c
Rule: For the 3D pattern    /      /   ,
                           d------e
                                f
      we have af = be - cd.
```

Condensation applied to tridiagonal matrices (number friezes)

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 2 & 1 & \mathbf{0} \\ 1 & 2 & 1 \\ \mathbf{0} & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \end{pmatrix}$$

Condensation applied to Töplitz matrices (number walls)

$$\begin{pmatrix} 13 & 21 & 34 & 55 \\ 8 & 13 & 21 & 34 \\ 5 & 8 & 13 & 21 \\ 3 & 5 & 8 & 13 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \end{pmatrix}$$

$$\begin{pmatrix} 12 & 18 & 27 & 41 \\ 8 & 12 & 18 & 27 \\ 5 & 8 & 12 & 18 \\ 3 & 5 & 8 & 12 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & -9 \\ 4 & 0 & 0 \\ 1 & 4 & 0 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{-1} \end{pmatrix}$$

III. ALTERNATING-SIGN MATRICES

For $1 \leq m \leq n-1$, the determinant of the n-by-n matrix M can be expressed as a rational function of the determinants of the $(n-m)$-by-$(n-m)$ and $(n-m-1)$-by-$(n-m-1)$ connected minors of M.

(We can arrange these determinants in the form of an $(m+1)$-by-$(m+1)$ and an $(m+2)$-by-$(m+2)$ matrix; we can superimpose these two matrices, obtaining a "bimatrix".)

Theorem (Robbins and Rumsey, 1986): This rational function (the "determinant" of the bimatrix) is formally a Laurent polynomial in the determinants of the connected minors of order $n-m$ and $n-m-1$.

$m = 1$:

$$\ldots \to \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \to \begin{pmatrix} j & k \\ l & m \end{pmatrix} \to \begin{pmatrix} X \end{pmatrix} \ ,$$

$$\begin{aligned} X &= (jm - kl)/e \\ &= j^1 m^1 e^{-1} - k^1 l^1 e^{-1} \\ &= e^{-1} j^1 k^0 l^0 m^1 - e^{-1} j^0 k^1 l^1 m^0 \\ &= \left(\begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 1 \end{pmatrix}\right) - \left(\begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}\right) \ . \end{aligned}$$

$m = 2$:

$$\ldots \to \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \to \begin{pmatrix} q & r & s \\ t & u & v \\ w & x & y \end{pmatrix} \to$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} X \end{pmatrix} \ ,$$

$m = 2$ (continued):

$X \;=\;$ an alternating sum of eight Laurent monomials

$$
\begin{aligned}
= \; & \left(\left(\begin{array}{ccccc} 1 & & 0 & & 0 \\ & -1 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & -1 & \\ 0 & & 0 & & 1 \end{array}\right)\right) - \left(\left(\begin{array}{ccccc} 0 & & 1 & & 0 \\ & -1 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & -1 & \\ 0 & & 0 & & 1 \end{array}\right)\right) \\
& - \left(\left(\begin{array}{ccccc} 1 & & 0 & & 0 \\ & -1 & & 0 & \\ 0 & & 0 & & 1 \\ & 0 & & -1 & \\ 0 & & 1 & & 0 \end{array}\right)\right) - \left(\left(\begin{array}{ccccc} 0 & & 0 & & 1 \\ & 0 & & -1 & \\ 0 & & 1 & & 0 \\ & -1 & & 0 & \\ 1 & & 0 & & 0 \end{array}\right)\right) \\
& + \left(\left(\begin{array}{ccccc} 0 & & 1 & & 0 \\ & 0 & & -1 & \\ 0 & & 0 & & 1 \\ & -1 & & 0 & \\ 1 & & 0 & & 0 \end{array}\right)\right) + \left(\left(\begin{array}{ccccc} 0 & & 0 & & 1 \\ & 0 & & -1 & \\ 1 & & 0 & & 0 \\ & -1 & & 0 & \\ 0 & & 1 & & 0 \end{array}\right)\right) \\
& + \left(\left(\begin{array}{ccccc} 0 & & 1 & & 0 \\ & -1 & & 0 & \\ 1 & & -1 & & 1 \\ & 0 & & -1 & \\ 0 & & 1 & & 0 \end{array}\right)\right) - \left(\left(\begin{array}{ccccc} 0 & & 1 & & 0 \\ & 0 & & -1 & \\ 1 & & -1 & & 1 \\ & -1 & & 0 & \\ 0 & & 1 & & 0 \end{array}\right)\right).
\end{aligned}
$$

Theorem (Robbins and Rumsey, 1986, continued):
Moreover, "Laurentness" continues to hold if the Dodgson recurrence

$$af = be - cd$$

is replaced by the recurrence

$$af = be + cd$$

or the more general

$$af = be + \lambda cd.$$

This is called the λ-determinant of the bimatrix.

Theorem (Robbins and Rumsey, 1986, continued):
The λ-determinant of a bimatrix is a sum of

$$2^{m(m+1)/2}$$

monomials, in which each coefficient is plus or minus a power of λ, and the exponents of all the variables equal $+1$, 0, or -1. The terms correspond to the compatible pairs of ASMs (Alternating-Sign Matrices) of order m and $m+1$.

Compatibility

The ASMs of order 3, and the ASMs of order 2 they are respectively compatible with:

```
(1 0 0) (0 1 0) (1 0 0) (0  1 0) (0 0 1) (0 1 0) (0 0 1)
(0 1 0) (1 0 0) (0 0 1) (1 -1 1) (1 0 0) (0 0 1) (0 1 0)
(0 0 1) (0 0 1) (0 1 0) (0  1 0) (0 1 0) (1 0 0) (1 0 0)
      \       \     |       / \       |    /       /
         \      \   |     /     \     |   /     /
             \   \  |   /         \   |  /   /
                \ \ | /             \ | / /
                (1 0)                (0 1)
                (0 1)                (1 0)
```

Aztec diamond graphs

Grensing, Carlsen, and Zapp (1980) conjectured that for graphs of the form

```
            o-----o
            |     |
            |     |
      o-----o-----o-----o
      |     |     |     |
      |     |     |     |
o-----o-----o-----o-----o-----o
|     |     |     |     |     |
|     |     |     |     |     |
o-----o-----o-----o-----o-----o
      |     |     |     |
      |     |     |     |
      o-----o-----o-----o
            |     |
            |     |
            o-----o
```

the number of perfect matchings (aka dimer configurations) is $2^{n(n+1)/2}$, where the rows and columns have respective lengths $2, 4, 6, \ldots, 2n, 2n, \ldots, 6, 4, 2$.

This was proved by Elkies, Kuperberg, Larsen, and Propp (1992), using the language of tilings instead of dimer covers:

```
         -------
        |       |                             o---o
     ---------------
    |       |       |                     o---o   o---o
 ---|-------|-------|---
|   |   |   |       |   |             o   o   o   o---o   o
|   |   |   |-------|   |   <--->     |   |   |           |
|   |   |   |   |   |   |             o   o   o   o   o   o
 ---|-------|   |   |---                          |   |
    |       |   |   |                     o---o   o   o
     -----------|---
        |       |                             o---o
         -------

    "domino tiling of                 "perfect matching of an
    an Aztec diamond"                   Aztec diamond graph"
```

But in a sense the result was proved by Robbins and Rumsey (1986) using the language of compatible pairs of ASMs.

From matchings to pairs of ASMs

(Elkies, Kuperberg, Larsen, Propp)

In each hole, write 1 minus the number of neighboring edges included in the matching.

```
0   o   0   o   1   o   0
  /       /           \
o   0   o  -1   o   0   o
              /
0   o   1   o   0   o   0
  /                   \
o   0   o   0   o  -1   o
          \       \
1   o  -1   o   0   o   1
      \
o  -1   o   0   o   0   o
  \           /       /
0   o   1   o   0   o   0
```

Decompose the bimatrix into two matrices, and negate the entries in the smaller matrix.

From pairs of ASMs to matchings (Carroll)

Insert 0's into the holes in the bimatrix, and at each location, record the (possibly empty) sum of all the entries above and/or to the left of it, obtaining the "corner-sum matrix" of the bimatrix.

```
  0    0    0    0    0    0    0    0
         o         o         o
  0    0    0    0    0    1    1    1
    o         o         o         o
  0    0    0    0   -1    0    0    0
         o         o         o
  0    0    0    1    0    1    1    1
    o         o         o         o
  0    0    0    1    0    1    0    0
         o         o         o
  0    1    1    1    0    1    0    1
    o         o         o         o
  0    1    0    0   -1    0   -1    0
         o         o         o
  0    1    0    1    0    1    0    1
```

Then replace each corner-sum by a mark, as prescribed by the following table, based on the dimensions of the sub-array and the value of the sum.

Even-by-even block:
corner sum is -1: edge (/)
corner sum is 0: no edge
corner sum is 1: **incompatible**
Even-by-odd or odd-by-even block:
corner sum is -1: **incompatible**
corner sum is 0: no edge
corner sum is 1: edge (\)
Odd-by-odd block:
corner sum is -1: **incompatible**
corner sum is 0: edge (/)
corner sum is 1: no edge

The generic Dodgson pyramid with formal indeterminates at levels -1 and 0:

Level 1: Laurent polynomials with 2 terms
Level 2: Laurent polynomials with 8 terms
Level 3: Laurent polynomials with 64 terms
... etc.

Specialize!

(i.e., turn the Laurent polynomials into numbers)

The numbers 1, 1, 2, 8, 64, ... (of the form $2^{n(n+1)/2}$) yield a solution to the discrete Hirota equation, with initial conditions in a slab made of two planes of 1's:

```
1   1   1   1   1
                       1   1   1   1
1   1   1   1   1                       2   2   2
                       1   1   1   1                  8   8
1   1   1   1   1                       2   2   2                 64
                       1   1   1   1                  8   8
1   1   1   1   1                       2   2   2
                       1   1   1   1
1   1   1   1   1
```

If we let $M(n)$ be the number of perfect matchings of the Aztec diamond graph of order n, then we obtain

$$M(n+1)M(n-1) =$$

$$M(n)M(n) + M(n)M(n) \ ,$$

a 1-dimensional discrete Hirota equation that gives us an inductive proof of the formula $M(n) = 2^{n(n+1)/2}$.

In 1997 Eric Kuo found a direct combinatorial proof of this formula (via “graphical condensation”).

These same methods apply to weighted enumeration of perfect matchings, where the weight of a particular perfect matching is product of the weights of its edges under some fixed weighting scheme.

The number of perfect matchings of a $2n$-by-$2n$ square equals the sum of the weights of the perfect matchings of the Aztec diamond of order $2n-1$, where edges are assigned weight 0 or weight 1 as shown below for the case $n = 3$:

```
  o   o   o   o   o
 1 0 1 0 1 1 0 1 0 1
o   o   o   o   o   o
 0 0 0 1 1 1 1 0 0 0
  o   o   o   o   o
 1 0 1 1 1 1 1 1 0 1
o   o   o   o   o   o
 0 1 1 1 1 1 1 1 1 0
  o   o   o   o   o
 1 1 1 1 1 1 1 1 1 1
o   o   o   o   o   o
 1 1 1 1 1 1 1 1 1 1
  o   o   o   o   o
 0 1 1 1 1 1 1 1 1 0
o   o   o   o   o   o
 1 0 1 1 1 1 1 1 0 1
  o   o   o   o   o
 0 0 0 1 1 1 1 0 0 0
o   o   o   o   o   o
 1 0 1 0 1 1 0 1 0 1
  o   o   o   o   o
```

The number of perfect matchings of a $2n$-by-$2n$ square equals the ($\lambda = 1$)-determinant of the $(2n+1)$-by-$(2n+1)$ bimatrix shown below for the case $2n = 6$:

$$\left(\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}\right)$$

(The intermeshed 6-by-6 matrix has all entries equal to 1.)

E.g., for $2n = 4$,

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 4 & 1 \\ 4 & 8 & 4 \\ 1 & 4 & 1 \end{pmatrix} \to \begin{pmatrix} 12 & 12 \\ 12 & 12 \end{pmatrix} \to \begin{pmatrix} 36 \end{pmatrix}$$

IV. SOMOS SEQUENCES

Theorem (Somos; Fomin and Zelevinsky): If we put $S(1) = w$, $S(2) = x$, $S(3) = y$, and $S(4) = z$, and

$$S(n) = \frac{S(n-1)S(n-3) + S(n-2)^2}{S(n-4)}$$

for $n > 4$, then $S(n)$ is a Laurent polynomial in w, x, y, z (i.e., a polynomial in $w, w^{-1}, x, x^{-1}, y, y^{-1}, z, z^{-1}$).

Theorem (Somos; Fomin and Zelevinsky): If we put $S(1) = 1$, $S(2) = 1$, $S(3) = 1$, and $S(4) = 1$, and

$$S(n) = \frac{uS(n-1)S(n-3) + vS(n-2)^2}{S(n-4)}$$

for $n > 4$, then $S(n)$ is a polynomial in u, v.

Theorem (Fomin and Zelevinsky): Fix positive integers a, b, d with $a + b < d$. Put $S(i) = x_i$ for $i = 1, 2, ..., d$ and put

$$S(n) = \frac{uS(n-a)S(n-d+a) + vS(n-b)S(n-d+b)}{S(n-d)}$$

for $n > d$. Then $S(n)$ is polynomial in $x_1, x_1^{-1}, ..., x_d, x_d^{-1}, u, v$.

Special cases:

$d = 4, a = 1, b = 2$: Somos-4
$d = 5, a = 1, b = 2$: Somos-5

Quasi-theorem (Propp, Bousquet-Mélou and West, 2002):

All the coefficients in this polynomial are positive integers.

The values of Somos-4 also yield an intrinsically one-dimensional solution to the discrete Hirota equation, with initial conditions in a tilted slab made of four tilted planes of 1's:

1	2	3	7						
				3	7	23			
1	1	2	3				23	59	
				2	3	7			314
1	1	1	2				7	23	
				1	2	3			
1	1	1	1						

Un-specialize! (???)

(i.e., turn the numbers back into Laurent polynomials)

Implementing recurrences in MAPLE

```
f := proc(n,i,j) option remember;
    if n<0 then undefined
    elif n<4 then x(n,i,j);
    else simplify( ( f(n-1,i-1,j)*f(n-3,i+1,j)
                   + f(n-2,i,j-1)*f(n-2,i,j+1))
                  /  f(n-4,i,j)); fi; end;
```

Empirically, one finds that `f(n,i,j)` is a Laurent polynomial in the x-variables in which every coefficient equals $+1$, so that the number of terms in `f(n,i,j)` is equal to the result of specializing all the x's to equal 1, which is equal to the nth term of the Somos-4 sequence.

The Somos-4 graphs

```
                                             o--o
                                             |  |
                                          o--o--o--o
                                          |  |  |  |
                           o--o        o--o  o--o  o--o
                           |  |        |  |  |  |  |  |
                        o--o  o--o     o  o--o  o--o  o
                        |  |  |  |        |  |  |  |
            o--o        o--o--o--o        o--o--o--o     . . .
            |  |        |  |  |  |        |  |  |  |
         o--o  o--o     o--o  o--o        o--o  o--o
         |  |  |  |     |  |  |  |        |  |  |  |
o--o     o  o--o  o     o  o--o  o        o  o--o  o
|  |        |  |           |  |              |  |
o--o        o--o           o--o              o--o
|  |        |  |           |  |              |  |
o--o        o--o           o--o              o--o
|  |        |  |           |  |              |  |
o  o        o  o           o  o              o  o
```

V. GROVES

Theorem (Fomin and Zelevinsky): Fix positive integers a, b, c and let $a' = b + c$, $b' = a + c$, and $c' = a + b$. Put $S(i) = x_i$ for $i = 1, 2, ..., a + b + c$ and put

$$S(n) = \frac{uS(n-a)S(n-a') + vS(n-b)S(n-b') + wS(n-c)S(n-c')}{S(n-a-b-c)}$$

for $n > a+b+c$. Then $S(n)$ is polynomial in $x_1, x_1^{-1}, ..., x_{a+b+c}, x_{a+b+c}^{-1}$, u, v, w.

Special cases:

$a = 1, b = 2, c = 3$: Somos-6

$$S(n)S(n-6) = S(n-1)S(n-5)$$
$$+S(n-2)S(n-4) + S(n-3)^2$$

$a = 1, b = 2, c = 4$: Somos-7

$$S(n)S(n-7) = S(n-1)S(n-6)$$
$$+S(n-2)S(n-5) + S(n-3)S(n-4)$$

(Somos-6 and Somos-7 were first proved to give integers by Dean Hickerson.)

The cube recurrence (or Miwa equation)

```
                                  a
                                / | \
                              /   |   \
                            /     |     \
                          b       c       d
                          | \   / | \   / |
Rule: For the 3D pattern  |   X       X   | ,
                          | /   \ | /   \ |
                          e       f       g
                            \     |     /
                              \   |   /
                                \ | /
                                  h

      we have ah = bg + cf + de.
```

Conjecture (Propp, 1998):
For $i+j+k = -1, 0$, or 1, let $S(i, j, k) = x_{i,j,k}$, and for $i + j + k > 1$ inductively define $S(i, j, k)$ by the cube recurrence

$$\begin{aligned} & S(i,j,k)S(i-1,j-1,k-1) \\ = \ & S(i-1,j,k)S(i,j-1,k-1) \\ & +S(i,j-1,k)S(i-1,j,k-1) \\ & +S(i,j,k-1)S(i-1,j-1,k). \end{aligned}$$

Then for all i, j, k with $i + j + k \geq -1$, $S(i, j, k)$ is a Laurent polynomial in the x-variables, with all coefficients equal to 1 and all exponents bounded between -1 and 4 (inclusive).

The Laurentness part of the claim was proved by Fomin and Zelevinsky. The rest was proved by my students Gabriel Carroll and David Speyer (with some input from me) in Spring 2002, when they discovered that these Laurent polynomials are encoding a hitherto unstudied kind of combinatorial object, which they call a *grove*.

A grove of order n is a special kind of forest on the triangle graph with $n+1$ vertices on a side. It is required that:

- every vertex is joined to a point on the boundary by a path; and
- two boundary-vertices v, w are joined by a path iff they are related by a reflection through a median that passes through a corner-point P of the triangle with the property that no corner of the triangle is closer to v or w than P.

(It follows that the corner vertices are isolated, and that every vertex other than the three corner vertices belongs to at least one edge.)

A grove of order 5

```
          o

        o---o

      o   o---o
       \ /
    o   o   o   o
     \     /   /
  o   o---o   o   o
   \     /   /   /
o   o   o   o   o   o
```

A grove of order 6

```
            o

          o---o
           \
        o   o   o
         \     /
      o   o---o   o
       \         /
    o   o---o---o   o
     \       \     /
  o   o---o   o   o   o
   \     /   /   /   /
o   o   o   o   o   o   o
```

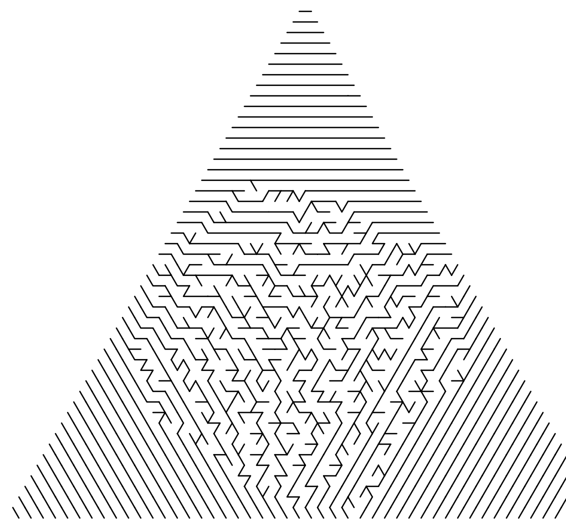
Theorem (Carroll and Speyer, 2002): The number of groves of order m is

$$3^{\lfloor m^2/4 \rfloor},$$

and these correspond to the monomials in $S(i,j,k)$, where $m = i + j + k$.

We knew the RHS ahead of time; it was the LHS that was mysterious!

In[19]:= **Grove[50]**



Out[19]= - Graphics -

VI. AN INTEGRABLE SYSTEM

Bilinear version of autonomous discrete Painlevé II: Replace x_n by ix_n, so that the recurrence becomes

$$x_n + t(1 + x_n^2)(x_{n+1} + x_{n-1}) = 0.$$

Let $u = x_{-1}$ and $v = x_0$. Then the sequence can also be generated by the recurrence

$$x_{n+1}x_{n-1} = (x_n^2 + d)/(x_n^2 + 1)$$

where

$$d = -u^2 - v^2 - u^2v^2 - uv/t.$$

If we treat d as a formal variable, x_n (with $n \geq 1$) can be written as

$$P_n(u, v, d)/Q_n(u, v, d)$$

where P_n, Q_n are Laurent polynomials whose degrees grow subexponentially (quadratically, in fact).

P_n and Q_n satisfy a pair of joint (1D) discrete Hirota equations:

$$P_{n+1}P_{n-1} = P_nP_n + dQ_nQ_n$$

$$Q_{n+1}Q_{n-1} = P_nP_n + Q_nQ_n$$

with initial conditions $P_{-1} = u$, $P_0 = v$, $Q_{-1} = 1$, $Q_0 = v$.

The one-dimensional family of polynomials P_n, Q_n can be lifted to a three-dimensional family of Laurent polynomials $P(n, i, j)$, $Q(n, i, j)$ satisfying a pair of joint 3D discrete Hirota equations:

$$P(n+1, i, j)P(n-1, i, j) =$$
$$P(n, i-1, j)P(n, i+1, j)+$$
$$dQ(n, i, j-1)Q(n, i, j+1) \ ,$$

$$Q(n+1, i, j)Q(n-1, i, j) =$$
$$P(n, i-1, j)P(n, i+1, j)+$$
$$Q(n, i, j-1)Q(n, i, j+1)$$

The Laurent polynomials $P(n, i, j)$, $Q(n, i, j)$ enumerate weighted perfect matchings of the order n Aztec diamond graph.

VII. CONCLUSION

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See also links accessible from the following web-pages:
`www.math.wisc.edu/~propp/somos.html`
`www.math.wisc.edu/~propp/bilinear.html`
`www.math.harvard.edu/~propp/reach`