COMPOSITIONS OF n WITH PARTS IN A SET

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Abstract

Let A be any set of positive integers and $n \in \mathbb{N}$. A *composition of* n *with parts in* A is an ordered collection of one or more elements in A whose sum is n. A *palindromic composition of* n *with parts in* A is a composition of n with parts in A in which the summands are the same in the given or in reverse order. A *Carlitz (palindromic) composition of* n *with parts in* A is a (palindromic) composition of n with parts in A in which no adjacent parts are the same. In this paper, we study the generating functions for several counting problems for compositions, palindromic compositions, Carlitz compositions, and Carlitz palindromic compositions with parts in A, respectively. Our theorems contain many previously known results as special cases, as well as many new results. We also prove some of our results using direct counting arguments.

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1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. A *composition* of $n \in \mathbb{N}$ is an ordered collection of one or more positive integers whose sum is n. The number of summands is called the number of *parts* of the composition. For $A = \{a_1, a_2, \dots\} \subseteq \mathbb{N}$, we denote the number of compositions of n with parts in A (respectively with j parts in A) by $C_A(n)$ (respectively $C_A(j; n)$). The corresponding generating functions are given by

$$C_A(x) = \sum_{n\geq 0} C_A(n)x^n \quad \text{and} \quad C_A(j; x) = \sum_{n\geq 0} C_A(j; n)x^n,$$

where we define $C_A(0) = C_A(0; 0) = 1$, $C_A(j; 0) = 0$ for $j \geq 1$, and $C_A(j; n) = 0$ for $n < 0$.

A *palindromic composition* of n is a composition of n in which the summands are the same in the given or in reverse order. We denote the number of palindromic compositions of n with parts in

A (respectively with j parts in A) by $P_A(n)$ (respectively $P_A(j;n)$). The corresponding generating functions are given by

$$P_A(x) = \sum_{n\geq 0} P_A(n)x^n \quad \text{and} \quad P_A(j;x) = \sum_{n\geq 0} P_A(j;n)x^n,$$

where we define $P_A(0) = P_A(0;0) = 1$,$P_A(j;0) = 0$ for $j \geq 1$, and $P_A(j;n) = 0$ for $n < 0$.

In addition, we define $\hat{F}_n = F_n$ for $n \geq 1$ and $\hat{F}_0 = 1$, where F_n is the n^{th} Fibonacci number. Then, using the generating function for the Fibonacci sequence, we can compute the generating function for $\hat{F}_n$ as

$$g_{\hat{F}}(x) = \sum_{n\geq 0} \hat{F}_n x^n = \frac{1-x^2}{1-x-x^2}. \tag{1.1}$$

2. The number of compositions with parts in a given set

In the following theorem we will present the generating function for the number of compositions of n with j parts in A.

Theorem 2.1. *Let $A \subseteq \mathbb{N}$ and $j \geq 0$. Then the generating function for the number of compositions of n with j parts in A is given by*

$$C_A(j;x) = \left(\sum_{a\in A} x^a\right)^j.$$

Proof. To derive the generating function we create the compositions of n with j parts in A recursively by the following process: to any composition of $n-a$ with $j-1$ parts in A, add an $a \in A$ at the right end. Thus, for $n \geq 1$,

$$C_A(j;n) = \sum_{a\in A} C_A(j-1;n-a).$$

Multiplying by x^n and summing over $n \geq 1$, then changing the order of summation and simplifying results in

$$C_A(j;x) = C_A(j-1;x)\sum_{a\in A} x^a.$$

Iterating this equation and using the fact that $C_A(0;x) = 1$ we get the desired result. □

Example 2.2. *Theorem 2.1 gives an easy way to derive the generating function for the number of compositions of n without k's that have j parts. Chinn and Heubach gave recursions for this quantity in* [CH1, Theorem 5]*, but did not derive the generating function. Using $A = \mathbb{N}_k = \mathbb{N}\backslash\{k\}$ yields*

$$C_{\mathbb{N}_k}(j;x) = \left(\frac{x}{1-x} - x^k\right)^j = x^j\left(\frac{1}{1-x} - x^{k-1}\right)^j.$$

Example 2.3. *Theorem 2.1 gives $C_{\{1,3,5,\dots\}}(j;x) = \left(\frac{x}{1-x^2}\right)^j$ and $C_{\{2,4,6,\dots\}}(j;x) = \left(\frac{x^2}{1-x^2}\right)^j$. Since $\frac{1}{(1-x)^{k+1}} = \sum_{n\geq 0}\binom{n+k}{n}x^n$ (see for example* [W]*, Eq. (2.5.7), p. 53), we get that $\frac{x^k}{(1-x)^{k+1}} = \sum_{n\geq 0}\binom{n}{k}x^n$ after appropriate re-indexing. Since*

$$\frac{x^j}{(1-x^2)^j} = \frac{x^{-j+2}\cdot(x^2)^{j-1}}{(1-x^2)^j} = x^{2-j}\sum_{n\geq 0}\binom{n}{j-1}x^{2n} = \sum_{n\geq 0}\binom{n}{j-1}x^{2n+2-j},$$

the number of compositions of $2n-j$ with j parts in $\{1,3,5,\dots\}$ is given by $\binom{n-1}{j-1}$. Likewise,

$$\frac{x^{2j}}{(1-x^2)^j} = \frac{x^2 \cdot x^{2j-2}}{(1-x^2)^j} = \sum_{n\geq 0} \binom{n}{j-1} x^{2n+2},$$

and therefore, the number of compositions of $2n$ with j parts in $\{2,4,6,\dots\}$ is also given by $\binom{n-1}{j-1}$. The fact that these two counts are identical can be easily seen using a direct combinatorial argument: In any composition of $2n-j$ with j parts in $\{1, 3, 5, \dots\}$, increase each part by 1. This yields a composition of n with j parts in $\{2, 4, 6, \dots\}$.

Using Theorem 2.1 we get an alternative derivation of Theorem 1.1 in [HB].

Theorem 2.4. *Let A be any set of natural numbers. Then the generating function for the number of compositions of n with parts in A is given by*

$$C_A(x) = \frac{1}{1-\sum_{a\in A} x^a}.$$

Proof. Since $C_A(n) = \sum_{j\geq 0} C_A(j;n)$ we get that $C_A(x) = \sum_{j\geq 0} C_A(j;x)$. Using Theorem 2.1 we get the desired result. □

Example 2.5. (see [CH1, Theorem 1]) *Let $\mathbb{N}_k = \mathbb{N}\backslash\{k\}$ be the set of all natural numbers without k. Then Theorem 2.4 yields*

$$C_{\mathbb{N}_k}(x) = \frac{1}{1-\frac{x}{1-x}+x^k} = \frac{1-x}{1-2x+x^k-x^{k+1}}.$$

Example 2.6. (see [CH2, Lemma 1, Part 1]) *Let $A = \{1,k\}$. Then Theorem 2.4 yields*

$$C_A(x) = \frac{1}{1-x-x^k}.$$

Example 2.7. *Applying Theorem 2.4 to the set $A = \{1,2,...,\ell\}$, we obtain that*

$$C_A(x) = \frac{1}{1-x-x^2-\cdots-x^\ell} = \frac{1}{x^{\ell-1}} \cdot \frac{x^{\ell-1}}{1-x-x^2-\cdots-x^\ell}.$$

Thus, the number of compositions of n with parts that are less than or equal to ℓ is given by the shifted ℓ-generalized Fibonacci number. Specifically, $C_{\{1,2,\dots,\ell\}}(n) = F(\ell;n+\ell-1)$ for $n\geq 0$, where $F(\ell;n) = \sum_{i=1}^{\ell} F(\ell;n-i)$ with $F(\ell;0) = \cdots = F(\ell;\ell-2) = 0, F(\ell;\ell-1) = 1$ [F]. Combinatorially, this can be seen using the following recursive method to create these compositions: To create the compositions of n, add i to the right end of the compositions of $n-i$, for $i = 1,2,\dots,\ell$. The ℓ-generalized Fibonacci sequences occur in [S] as sequence A000073 for $\ell = 3$, A000078 for $\ell = 4$, A001591 for $\ell = 5$, and A001592 for $\ell = 6$.

Example 2.8. *Applying Theorem 2.4 to the set of prime numbers $A = \{2,3,5,7,11,13,...\}$,we get that the sequence $\{C_A(n)\}_{n\geq 0}$ is given by $\{1,0,1,1,1,3,2,6,6,10,16,20,35,46,72,105,152,232,332,501,$ $732,1081,1604,2352,3493,5136,7595,11212,16534,24442,...\}$, which occurs in* [S] *as sequence A023360.*

Example 2.9. *If A consists of the set of odd numbers, then Theorem 2.4 gives*

$$C_{\{1,3,5,\dots\}}(x) = \frac{1}{1-\frac{x}{1-x^2}} = \frac{1-x^2}{1-x-x^2} = g_{\hat{F}}(x).$$

Thus, the number of compositions of n with parts in $\{1, 3, 5, \dots\}$ is given by F_n for $n \geq 1$, as was also shown by [G] using a recursive equation. More generally, let $\mathbb{N}_{d,e}$ be the set of all numbers of the form $k \cdot d + e$ where $k \geq 0$, $d, e \in \mathbb{N}$. Then Theorem 2.4 yields

$$C_{\mathbb{N}_{d,e}}(x) = \frac{1}{1 - \frac{x^e}{1-x^d}} = \frac{1 - x^d}{1 - x^e - x^d}.$$

If $d = 2e$, then the set A consists of all odd multiples of e, and

$$C_{\mathbb{N}_{2e,e}}(x) = C_{\{e,3e,5e,\dots\}}(x) = \frac{1 - x^{2e}}{1 - x^e - x^{2e}} = g_{\hat{F}}(x^e).$$

Thus, the number of compositions of $n = k \cdot e$ with parts in $\{e, 3e, 5e, \dots\}$ is given by F_k for $k \geq 1$. This result can be explained combinatorially as follows: Multiply each part of a composition of n with parts in $\{1, 3, 5, \dots\}$ by e to create a composition of $n \cdot e$ with parts in $\{e, 3e, 5e, \dots\}$.

The above theorems can be generalized as follows. Let $A = \{a_1, a_2, \dots\}$, and let $D_A(n, r_1, r_2, \dots)$ (respectively $D_A(j; n, r_1, r_2, \dots)$) be the number of compositions of n (respectively with j parts) such that the part $a_i \in A$ occurs exactly r_i times in the composition (with $\sum_i r_i = j$) and $D_A(0; 0, 0, 0 \dots) = 1$ and $D_A(0; n, 0, 0, \dots) = 0$ for $n > 0$. We define

$$D_A(j; x, t_{a_1}, t_{a_2}, \dots) = \sum_{n \geq 0} \sum_{r_1 + r_2 + \ldots = j} D_A(j; n, r_1, r_2, \dots) x^n \prod_{i \geq 0} t_{a_i}^{r_i},$$

and

$$D_A(x, t_{a_1}, t_{a_2}, \dots) = \sum_{n \geq 0} \sum_{r_1 + r_2 + \ldots \geq 0} D_A(n, r_1, r_2, \dots) x^n \prod_{i \geq 0} t_{a_i}^{r_i},$$

where x and t_{a_i} are indeterminate variables. Theorem 2.10 gives formulas for these generating functions.

Theorem 2.10. *For any $j \geq 0$,*

$$D_A(j; x, t_{a_1}, t_{a_2}, \dots) = \left(\sum_{a \in A} t_a x^a \right)^j.$$

Moreover,

$$D_A(x, t_{a_1}, t_{a_2}, \dots) = \frac{1}{1 - \sum_{a \in A} t_a x^a}.$$

Note that if $t_a = 1$ for all $a \in A$, then Theorem 2.10 gives Theorems 2.1 and 2.4.

Proof. The argument used in Theorem 2.1 needs to be refined. We create the compositions of n with j parts in which the part $a_k \in A$ occurs exactly r_k times by adding an $a_i \in A$ to a composition of $n - a_i$ with $j - 1$ parts, in which the part a_k occurs $\tilde{r}_k$ times where $\tilde{r}_k = r_k$ for $k \neq i$ and $\tilde{r}_i = r_i - 1$. Thus, $r_1 + r_2 + \cdots = j$ and $\tilde{r}_1 + \tilde{r}_2 + \cdots = j - 1$ and

$$D_A(j; n, r_1, r_2, \dots) = \sum_{a_i \in A} D_A(j - 1; n - a_i, \tilde{r}_1, \tilde{r}_2, \dots).$$

Multiplying this equation by $x^n \prod_{i\geq 0} t_{a_i}^{r_i}$ and summing over $n \geq 0$ and $r_1 + r_2 + \ldots = j$ yields

$$\begin{aligned} D_A(j; x, t_{a_1}, t_{a_2}, \ldots) &= \sum_{a_i \in A} x^{a_i} t_{a_i} \sum_{n\geq 0} \sum_{\tilde{r}_1+\tilde{r}_2+\ldots=j-1} D_A(j-1; n-a_i, \tilde{r}_1, \tilde{r}_2, \ldots) x^{n-a_i} \prod_{k\geq 0} t_{a_k}^{\tilde{r}_k} \\ &= \sum_{a_i \in A} x^{a_i} t_{a_i} D_A(j-1; x, t_{a_1}, t_{a_2}, \ldots). \end{aligned}$$

Iterating this equation and using the fact that $D_A(0; x, 0, 0, \ldots) = 1$ we get the desired result for $D_A(x, t_{a_1}, t_{a_2}, \ldots)$. The second result follows as in the proof of Theorem 2.4. □

Example 2.11. (see [CH1, Theorems 2-3]) *Chinn and Heubach counted the number of times the part i occurs in all compositions of n without occurrence of k. One way to count this quantity is to first count the number of compositions of n without occurrence of k which have exactly m occurrences of i. Using Theorem 2.10 with $t_i = t$, $t_k = 0$, and $t_l = 1$ for all $l \neq i, k$, we get*

$$\begin{aligned} D_{\mathbb{N}_k}(x, t_1, t_2, \ldots) &= \frac{1}{1 - \frac{x}{1-x} + x^i + x^k - tx^i} = \frac{1}{\left(\frac{1-2x}{1-x} + x^i + x^k\right) - tx^i} \\ &= \sum_{m\geq 0} \frac{(tx^i)^m}{\left(\frac{1-2x}{1-x} + x^i + x^k\right)^{m+1}}. \end{aligned}$$

Therefore, the generating function for the number of compositions of n without occurrence of k and m occurrences of i is given by

$$\frac{x^{mi}}{\left(\frac{1-2x}{1-x} + x^i + x^k\right)^{m+1}}.$$

Note that the denominator of this generating function is symmetric in i and k; thus, if we interchange the roles of i and k and compare coefficients in the respective generating functions, we get that the number of compositions of n without occurrence of k and with m occurrences of i is the same as the number of compositions of $n + m(k - i)$ without occurrence of i and with m occurrences of k, as long as $n + m(k - i) \geq 0$. Combinatorially, we can show the equivalence of these counts by adding $k - i$ to each part i in a composition of n without k. This creates m occurrences of k and takes out any occurrence of i in the new composition of $n + m(k - i)$.

Theorem 2.10 also allows us to count the number of compositions $C_A^B(n, m)$ of n with parts in A such that m parts are in $B \subseteq A$. We denote the generating function for these compositions by $C_A^B(x, y)$, where $C_A^B(x, y) = \sum_{n\geq 0} \sum_{m\geq 0} C_A^B(n, m) x^n y^m$. Setting $t_a = 1$ for all $a \in A\backslash B$ and $t_a = y$ for all $a \in B$ in Theorem 2.10 yields the following result.

Corollary 2.12. *Let A, B be two sets such that $B \subseteq A \subseteq \mathbb{N}$. Then*

$$C_A^B(x, y) = \frac{1}{1 - \sum_{a\in A\backslash B} x^a - y \sum_{a\in B} x^a} = \sum_m \frac{(y \sum_{a\in B} x^a)^m}{\left(1 - \sum_{a\in A\backslash B} x^a\right)^{m+1}}.$$

Therefore, the generating function for the number of compositions of n with parts in A such that m parts are in B is given by

$$\frac{\left(\sum_{a\in B} x^a\right)^m}{\left(1 - \sum_{a\in A\backslash B} x^a\right)^{m+1}}.$$

Example 2.13. *Let $A = \mathbb{N}$ and $B = \{2n|n \in \mathbb{N}\}$, then the second part of Corollary 2.12 gives the generating function for the number of compositions of n with m even parts is given by*

$$\frac{x^{2m}(1-x^2)}{(1-x-x^2)^{m+1}} = \left(\frac{x^2}{1-x-x^2}\right)^m \cdot \frac{1-x^2}{1-x-x^2} = \left(\sum_{n\geq 1} F_{n-1}x^n\right)^m \cdot \sum_{n\geq 0} \hat{F}_n x^n,$$

i.e., a convolution of m shifted Fibonacci sequences with the sequence $\hat{F}$ defined in Section 1. If we let $m = 0$, then all the parts are odd, and we get once more that the number of such compositions is given by $\hat{F}_n = F_n$ for $n \geq 1$ (see Example 2.9). For larger values of m, an explicit formula in terms of Fibonacci sequences can be obtained by using [M] *and the references therein. For example, the number of compositions of n with exactly one even part is given by $\frac{1}{5}((n+3)F_{n-1} + 2(n-1)F_{n-2})$ for all $n \geq 0$.*

Example 2.14. *Let $A = \{1, 2, k\}$ and $B = \{k\}$ for $k \geq 3$, then applying the above corollary we get that the number of compositions of n with parts in A and exactly m occurrences of k is*

$$\frac{x^{km}}{(1-x-x^2)^{m+1}} = \left(\frac{x^k}{1-x-x^2}\right)^m \cdot \frac{1}{(1-x-x^2)} = \left(\sum_{n\geq k-1} F_{n-k+1}x^n\right)^m \cdot \sum_{n\geq 0} F_{n+1}x^n,$$

i.e., once more we get a convolution of shifted Fibonacci sequences.

3. The number of palindromic compositions with parts in a given set

In the following theorem we will present the generating function for the number of palindromic compositions of n with j parts in A.

Theorem 3.1. *Let $A \subseteq \mathbb{N}$ and $j \geq 0$. Then the generating function for the number of palindromic compositions of n with j parts in A is given by*

$$P_A(j;x) = \begin{cases} \left(\sum_{a\in A} x^{2a}\right)^{j/2} & \text{if } j \text{ even} \\ \left(\sum_{a\in A} x^{a}\right)\left(\sum_{a\in A} x^{2a}\right)^{(j-1)/2} & \text{otherwise.} \end{cases}$$

Proof. The theorem holds for $j = 0$ since $P_A(0;0) = 1$ and $P(0;n) = 0$ for $n \geq 1$. The palindromic compositions of n with one part are exactly those for which $n = a \in A$, so $P_A(1;x) = \sum_{a\in A} x^a$, thus, the theorem holds for $j = 1$. We now generate the palindromic compositions of n with j parts in A recursively by the following process: to any palindromic composition of $n - 2a$ with $j - 2$ parts in A, add $a \in A$ on both ends. Thus, for $n \geq 1$,

$$P_A(j;n) = \sum_{a\in A} P_A(j-2; n-2a).$$

Multiplying by x^n and summing over $n \geq 1$, then changing the order of summation and simplifying results in

$$P_A(j;x) = P_A(j-2;x) \sum_{a\in A} x^{2a}.$$

Iterating this equation and using either $P_A(0;x) = 1$ (j even) or $P_A(1;x) = \sum_{a\in A} x^a$ (j odd) gives the desired result. □

As a consequence of Theorem 3.1 we get the following result.

Theorem 3.2. *Let $A \subseteq \mathbb{N}$. Then the generating function for the number of palindromic compositions of n with parts in A is given by*

$$P_A(x) = \frac{1 + \sum_{a\in A} x^a}{1 - \sum_{a\in A} x^{2a}}.$$

Proof. Since $P_A(n) = \sum_{j\geq 0} P_A(j;n)$ we get that $P_A(x) = \sum_{j\geq 0} P_A(j;x)$. Using Theorem 3.1 we get the desired result after simplification. □

Example 3.3. *Chinn and Heubach* (see [CH1]) *derived explicit formulas in terms of compositions and recursive formulas in terms of palindromes for the number of palindromes of n without k, but did not give a generating function for these quantities. Theorem 3.2 with $A = \mathbb{N}_k = \mathbb{N}\backslash\{k\}$ yields after simplification*

$$P_{\mathbb{N}_k}(x) = \frac{1 + x - x^k - x^{k+2}}{1 - 2x^2 + x^{2k} - x^{2(k+1)}}.$$

Example 3.4. (see [CH2, Lemma 2, Part 1]) *Let $A = \{1, k\}$. Then Theorem 3.2 yields*

$$P_A(x) = \frac{1 + x + x^k}{1 - x^2 - x^{2k}}.$$

Example 3.5. *If we consider palindromes of n with only odd summands, then Theorem 3.2 gives*

$$P_{\{1,3,5,\dots\}}(x) = \frac{1 + \sum_{i\geq 0} x^{2i+1}}{1 - \sum_{i\geq 0} x^{2(2i+1)}} = \frac{(1 + x - x^2)(1 + x^2)}{1 - x^2 - x^4}.$$

Thus, the number of palindromes of n with odd parts for $n \geq 1$ is given by $F_{n/2}$ if n is even, and $F_{(n+3)/2}$ if n is odd, as was shown by a direct argument not involving the generating function in [G].

The above theorems can be generalized. Let $A = \{a_1, a_2, \dots\}$, and let $Q_A(n, r_1, r_2, \dots)$ (respectively $Q_A(j; n, r_1, r_2, \dots)$) be the number of palindromic compositions of n (respectively with j parts) such that the part $a_i \in A$ occurs exactly r_i times in the composition (with $\sum_i r_i = j$). We define

$$Q_A(j; x, t_{a_1}, t_{a_2}, \dots) = \sum_{n\geq 0} \sum_{r_1+r_2+\ldots=j} Q_A(j; n, r_1, r_2, \dots) x^n \prod_{i\geq 0} t_{a_i}^{r_i},$$

and

$$Q_A(x, t_{a_1}, t_{a_2}, \dots) = \sum_{n\geq 0} \sum_{r_1+r_2+\ldots\geq 0} Q_A(n, r_1, r_2, \dots) x^n \prod_{i\geq 0} t_{a_i}^{r_i},$$

where x and t_{a_i} are indeterminate variables.

Using arguments similar to those in the proofs of Theorems 3.1 and 3.2, with the type of refinement (now symmetric for both ends of the composition) used in the proof of Theorem 2.10 we get the following generalization.

Theorem 3.6. *Let $A \subseteq \mathbb{N}$ and $j \geq 0$. Then*

$$Q_A(j; x, t_{a_1}, t_{a_2}, \dots) = \begin{cases} \left(\sum_{a\in A} t_a^2 x^{2a}\right)^{j/2} & \text{if } j \text{ even} \\ \left(\sum_{a\in A} t_a x^a\right)\left(\sum_{a\in A} t_a^2 x^{2a}\right)^{(j-1)/2} & \text{otherwise.} \end{cases}$$

Moreover,

$$Q_A(x, t_{a_1}, t_{a_2}, \dots) = \frac{1 + \sum_{a\in A} t_a x^a}{1 - \sum_{a\in A} t_a^2 x^{2a}}.$$

If $t_a = 1$ for all $a \in A$, then Theorem 3.6 gives Theorems 3.1 and 3.2.

Theorem 3.6 also allows us to count the number of palindromic compositions $Q_A^B(n,m)$ of n with parts in A such that m parts are in $B \subseteq A$. We denote the generating function for these compositions by $Q_A^B(x,y)$, where $Q_A^B(x,y) = \sum_{n\geq 0}\sum_{m\geq 0} Q_A^B(n,m)x^n y^m$. Setting $t_a = 1$ for all $a \in A\backslash B$ and $t_a = y$ for all $a \in B$ in Theorem 3.6 yields the following result.

Corollary 3.7. *Let A, B be two sets such that $B \subseteq A$. Then*

$$Q_A^B(x,y) = \frac{1+\sum_{a\in A\backslash B} x^a + y\sum_{a\in B} x^a}{1-\sum_{a\in A\backslash B} x^{2a} - y^2\sum_{a\in B} x^{2a}}.$$

Moreover,

$$Q_A^B(x,y) = \sum_{m\geq 0}\left[\frac{\left(\sum_{a\in B} x^{2a}\right)^m\left(1+\sum_{a\in A\backslash B} x^a\right)}{\left(1-\sum_{a\in A\backslash B} x^{2a}\right)^{m+1}} y^{2m} + \frac{\left(\sum_{a\in B} x^{2a}\right)^m\left(\sum_{a\in B} x^a\right)}{\left(1-\sum_{a\in A\backslash B} x^{2a}\right)^{m+1}} y^{2m+1}\right].$$

Proof. The second formula for $Q_A^B(x,y)$ follows from the first by splitting the numerator into summands with and without a factor of y, and then applying algebraic manipulations to utilize the series expansion of $1/(1-x)$. ☐

Example 3.8. *Let $A = \mathbb{N}$ and $B = \{2n|n \in \mathbb{N}\}$. Since*

$$\sum_{a\ even} x^a = \frac{x^2}{1-x^2},\quad \sum_{a\ odd} x^a = \frac{x}{1-x^2},\quad \sum_{a\ even} x^{2a} = \left(\frac{x^4}{1-x^4}\right)^m \quad \text{and} \quad \sum_{a\ odd} x^{2a} = \frac{x^2}{1-x^4},$$

the generating function for the number palindromic compositions of n with $2m$ and $2m+1$ even parts, respectively, is given by

$$\frac{x^{4m}(1+x-x^2)(1+x^2)}{(1-x^2-x^4)^{m+1}} \quad \text{and} \quad \frac{x^{4m+2}(1+x^2)}{(1-x^2-x^4)^{m+1}}.$$

In particular, if we set $m = 0$ in the first expression, then we recover Example 3.5.

Example 3.9. *Grimaldi* [G] *has counted the number of palindromic compositions of n with exactly m parts, but did not give a generating function for this quantity. We can easily read off these values as the coefficient of $x^n y^m$ by using $B = A$ in the first expression of Corollary 3.7. In this case, the generating function reduces to*

$$Q_A^A(x,y) = \frac{1+y\sum_{a\in B} x^a}{1-y^2\sum_{a\in B} x^{2a}} = \frac{(1+xy-x^2)(1+x^2)}{1+x^2y^2-x^4}.$$

4. Carlitz Compositions

A *Carlitz composition of n*, introduced in [C], is a composition of n in which no adjacent parts are the same. We denote the number of Carlitz compositions of n with parts in A (respectively with j parts in A) by $E_A(n)$ (respectively $E_A(j;n)$). The corresponding generating functions are given by

$$E_A(x) = \sum_{n\geq 0} E_A(n)x^n \quad \text{and} \quad E_A(j;x) = \sum_{n\geq 0} E_A(j;n)x^n,$$

where we define $E_A(0) = E_A(0;0) = 1$ and $E_A(j;0) = 0$ for $j \geq 1$. In particular, $E_A(0;x) = 1$.

Theorem 4.1. *Let $A \subseteq \mathbb{N}$. Then the generating function for the number of Carlitz compositions of n with parts in A is given by*

$$E_A(x) = \frac{1}{1 - \sum_{a\in A} \frac{x^a}{1+x^a}}.$$

Proof. We create the Carlitz compositions of n recursively by adding $a \in A$ to any Carlitz composition of $n-a$, except for those compositions of $n-a$ that end in a. Let $E_A(j,a;n)$ denote the number of Carlitz compositions of n that end in a, and let $E_A(j,a;x)$ denote the corresponding generating function, where $E_A(1,a;n) = 1$ if $n = a$ and 0 otherwise. This implies that $E_A(1,a;x) = x^a$. The recursive creation leads to

$$E_A(j,a;x) = x^a E_A(j-1;x) - x^a E_A(j-1,a;x) \quad \text{for } j \geq 1. \tag{4.1}$$

We now show by induction on j that for $j \geq 1$

$$E_A(j,a;x) = \sum_{k=1}^{j} (-1)^{k-1} x^{ka} E_A(j-k;x). \tag{4.2}$$

For $j = 1$, $E_A(1,a;x) = x^a = (-1)^0 x^a E_A(0;x)$. Now assume that the induction hypothesis is true for $j-1$. Substituting Eq. (4.2) into Eq. (4.1) gives

$$\begin{aligned} E_A(j,a;x) &= x^a E_A(j-1;x) - x^a \sum_{i=1}^{j-1} (-1)^{i-1} x^{ia} E_A(j-1-i;x) \\ &= x^a E_A(j-1;x) + \sum_{i=1}^{j-1} (-1)^{(i+1)-1} x^{(i+1)a} E_A(j-(i+1);x) \\ &= \sum_{k=1}^{j} (-1)^{k-1} x^{ka} E_A(j-k;x). \end{aligned}$$

By definition, $E_A(x) = \sum_{a\in A}\sum_{j\geq 0} E_A(j,a;x)$. Using that $\sum_{a\in A} E_A(0,a;x) = E_A(0;x) = 1$ and Eq. (4.2) for the terms with $j \geq 1$ we get

$$E_A(x) = 1 + \sum_{a\in A}\sum_{j\geq 1}\sum_{k=1}^{j} (-1)^{k-1} x^{ka} E_A(j-k;x).$$

Changing the order of summation for the two innermost sums and using that by definition $E_A(j;x) = 0$ for $j < 0$, we obtain

$$\begin{aligned} E_A(x) &= 1 + \sum_{a\in A}\sum_{k\geq 1} (-1)^{k-1} x^{ka} \sum_{j\geq 1} E_A(j-k;x) \\ &= 1 + E_A(x) \sum_{a\in A}\sum_{k\geq 1} (-1)^{k-1} x^{ka} = 1 + E_A(x) \sum_{a\in A} \frac{x^a}{1+x^a}. \end{aligned}$$

Solving for $E_A(x)$ gives the result. □

Example 4.2. *If $A = \mathbb{N}$, then the above theorem gives the generating function for the number of Carlitz compositions of n, namely*

$$E_{\mathbb{N}}(x) = \frac{1}{1 - \sum_{j\geq 1} \frac{x^j}{1+x^j}}.$$

This can be shown to agree with the function given in [KP], *namely* $C(x) = \frac{1}{1-\sigma(x)}$, *where* $\sigma(x) = \sum_{j\geq 1} \frac{x^j(-1)^{j-1}}{1-x^j}$ *as follows:*

$$\sum_{j\geq 1} \frac{x^j}{1+x^j} = \sum_{j\geq 1} x^j \sum_{k\geq 0} ((-x)^j)^k = \sum_{j\geq 1}\sum_{k\geq 0}(-1)^k (x^j)^{k+1} = \sum_{j\geq 1}\sum_{k\geq 1}(-1)^{k-1}(x^j)^k$$

and

$$\sum_{j\geq 1} \frac{x^j(-1)^{j-1}}{1-x^j} = \sum_{j\geq 1} x^j(-1)^{j-1} \sum_{k\geq 0} (x^j)^k = \sum_{j\geq 1}\sum_{k\geq 0}(-1)^{j-1} (x^j)^{k+1} = \sum_{j\geq 1}\sum_{k\geq 1}(-1)^{j-1}(x^j)^k.$$

Theorem 4.1 can be generalized as follows. Let $A = \{a_1, a_2, \dots\}$, and let $E_A(n, r_1, r_2, \dots)$ (respectively $E_A(j; n, r_1, r_2, \dots)$) be the number of Carlitz compositions of n (respectively with j parts) such that the part $a_i \in A$ occurs exactly r_i times in the composition (with $\sum_i r_i = j$) and $E_A(0; 0, 0, \dots) = 1$ and $E_A(0; n, 0, \dots) = 0$ for $n \neq 0$. We define the corresponding generating functions

$$E_A(j; x, t_{a_1}, t_{a_2}, \dots) = \sum_{n\geq 0} \sum_{r_1+r_2+\dots=j} E_A(j; n, r_1, r_2, \dots) x^n \prod_{i\geq 0} t_{a_i}^{r_i}$$

and

$$E_A(x, t_{a_1}, t_{a_2}, \dots) = \sum_{n\geq 0} \sum_{r_1+r_2+\dots\geq 0} E_A(n, r_1, r_2, \dots) x^n \prod_{i\geq 0} t_{a_i}^{r_i},$$

where x and t_{a_i} are indeterminate variables.

Theorem 4.3. *Let* $A = \{a_1, a_2, \dots\} \subseteq \mathbb{N}$. *Then*

$$E_A(x; t_{a_1}, t_{a_2}, \dots) = \frac{1}{1 - \sum_{a\in A} \frac{t_a x^a}{1+t_a x^a}}.$$

Proof. As before, we need to look at the last summand, so we let $E_A(j, a; n, r_1, r_2, \dots)$ denote the number of Carlitz compositions of n with j parts that end in a such that the part $a_i \in A$ occurs exactly r_i times in the composition, with $\sum_i r_i = j$, with corresponding generating function $E_A(j, a; x, t_{a_1}, t_{a_2}, \dots)$. Initial conditions give $E_A(0, a; x, t_{a_1}, t_{a_2}, \dots) = 1$ and $E_A(1, a; x, t_{a_1}, t_{a_2}, \dots) = x^a t_a$. With an argument similar to that in Theorem 2.10 with respect to the number of times a part occurs, we get a corresponding recursive equation for $j \geq 1$:

$$E_A(j, a; x, t_{a_1}, t_{a_2}, \dots) = x^a t_a E_A(j-1; x, t_1, t_2, \dots) - x^a t_a E_A(j-1, a; x, t_1, t_2, \dots).$$

Using induction, this recursion yields

$$E_A(j, a; x, t_{a_1}, t_{a_2}, \dots) = \sum_{k=1}^{j} (-1)^{k-1} x^{ka} t_a^k E_A(j-k; x, t_{a_1}, t_{a_2}, \dots) \quad \text{for } j \geq 1.$$

Replacing $E_A(x)$ by $E_A(x, t_{a_1}, t_{a_2}, \dots)$ in the remainder of the proof of Theorem 4.1 yields

$$E_A(x, t_{a_1}, t_{a_2}, \dots) = 1 + E_A(x, t_{a_1}, t_{a_2}, \dots) \cdot \sum_{a\in A} \frac{t_a x^a}{1+t_a x^a},$$

from which the result follows. $\square$

Theorem 4.3 also allows us to count the number of Carlitz compositions $E_A^B(n,m)$ of n with parts in A such that m parts are in $B \subseteq A$. We denote the generating function for these compositions by $E_A^B(x,y)$, where $E_A^B(x,y) = \sum_{n\geq 0}\sum_{m\geq 0} E_A^B(n,m)x^n y^m$. Setting $t_a = 1$ for all $a \in A\backslash B$ and $t_a = y$ for all $a \in B$ in Theorem 4.3 yields the following result.

Corollary 4.4. *Let $B \subseteq A \subseteq \mathbb{N}$. Then*

$$E_A^B(x,y) = \frac{1}{1 - \sum_{a\in A\backslash B}\frac{x^a}{1+x^a} - \sum_{a\in B}\frac{yx^a}{1+yx^a}}.$$

Example 4.5. *Let $A = \mathbb{N}$, $B = \{k\}$ and define $W = \sum_{j\geq 1, j\neq k}\frac{x^j}{1+x^j}$. Then*

$$\begin{aligned}
E_A^B(x,y) &= \frac{1}{1 - W - \frac{yx^k}{(1+yx^k)}} = \frac{1+yx^k}{1-W-yWx^k} = \frac{(1+yx^k)}{(1-W)}\sum_{m\geq 0}\frac{W^m x^{km}y^m}{(1-W)^m} \\
&= (1+yx^k)\sum_{m\geq 0}\frac{W^m x^{mk}}{(1-W)^{m+1}}y^m = \sum_{m\geq 0}\frac{W^m x^{mk}}{(1-W)^{m+1}}y^m + \sum_{m\geq 0}\frac{W^m x^{mk+k}}{(1-W)^{m+1}}y^{m+1} \\
&= \left(\frac{1}{1-W} + \sum_{m\geq 1}\frac{W^m x^{mk}}{(1-W)^{m+1}}y^m\right) + \sum_{m\geq 0}\frac{W^m x^{mk+k}}{(1-W)^{m+1}}y^{m+1} \\
&= \frac{1}{1-W} + \sum_{m\geq 0}\frac{W^{m+1}x^{mk+k}}{(1-W)^{m+2}}y^{m+1} + \sum_{m\geq 0}\frac{(1-W)W^m x^{mk+k}}{(1-W)^{m+2}}y^{m+1} \\
&= \frac{1}{1-W} + \sum_{m\geq 1}\frac{W^m x^{mk+k}}{(1-W)^{m+2}}y^{m+1}.
\end{aligned}$$

Thus, the generating function for the number of Carlitz compositions of n with exactly m occurrences of k is given by

$$\frac{x^{km}\left(\sum_{j\neq k, j\geq 1}\frac{x^j}{1+x^j}\right)^{m-1}}{\left(1 - \sum_{j\neq k, j\geq 1}\frac{x^j}{1+x^j}\right)^{m+1}}$$

for all $m \geq 1$.

5. Carlitz palindromic compositions

A *Carlitz palindromic composition of n* is a palindromic composition of n in which no adjacent parts are the same. Note that a palindromic composition of even n has an even number of parts, thus the two middle parts are the same, and no Carlitz palindromic compositions of even n exist.

We denote the number of Carlitz palindromic compositions of n with parts in A (respectively with j parts in A) by $F_A(n)$ (respectively $F_A(j;n)$). The corresponding generating functions are given by

$$F_A(x) = \sum_{n\geq 0} F_A(n)x^n \quad \text{and} \quad F_A(j;x) = \sum_{n\geq 0} F_A(j;n)x^n,$$

where we define $F_A(0) = F_A(0;0) = 1$ and $F_A(j;0) = 0$ for $j \neq 0$. Furthermore, $F_A(1;n) = 1$ if $n = a \in A$ and $F_A(1;n) = 0$ otherwise. Thus, $F_A(0;x) = 1$ and $F_A(1;x) = \sum_{a\in A} x^a$, and from the remark above it follows that $F_A(2j;x) = 0$ for $j \geq 1$.

The following theorem gives the generating function for the number of Carlitz palindromic compositions of n.

Theorem 5.1. *Let $A \subseteq \mathbb{N}$. Then the generating function for the number of Carlitz palindromic compositions of n with parts in A is given by*

$$F_A(x) = \frac{\sum_{a\in A} \frac{x^a}{1+x^{2a}}}{1 - \sum_{a\in A} \frac{x^{2a}}{1+x^{2a}}}.$$

Proof. Again, we use a recursive method to create the Carlitz palindromic compositions of n, by adding $a \in A$ to both ends of a Carlitz palindromic composition of $n-2a$, except for those that begin and end in a. Let $F_A(j,a;x)$ be the generating function for the number of Carlitz palindromic compositions of n with j parts that start and end in a. Then, for $j \geq 1$, we get

$$F_A(2j+1,a;x) = x^{2a}F_A(2j-1;x) - x^{2a}F_A(2j-1,a;x).$$

Using induction on j as in the proof of Theorem 4.1 and utilizing the initial condition $F_A(1,a;x) = x^a$ we can prove that

$$F_A(2j+1,a;x) = \left(\sum_{k=1}^{j}(-1)^{k-1}(x^{2a})^k F_A((2j+1)-2k;x)\right) + (-1)^j x^{(2j+1)a}.$$

Note that this formula also holds for $j = 0$. Since $F_A(x) = \sum_{a\in A}\sum_{j\geq 0} F_A(2j+1,a;x)$, we get

$$F_A(x) = \sum_{a\in A}\sum_{j\geq 0}\sum_{k=1}^{j}(-1)^{k-1}(x^{2a})^k F_A((2j+1)-2k;x) + \sum_{a\in A}\sum_{j\geq 0}(-1)^j x^{(2j+1)a}.$$

For the first sum we proceed as in the proof of Theorem 4.1, which yields

$$F_A(x) = F_A(x)\cdot\left(\sum_{a\in A}\frac{x^{2a}}{1+x^{2a}}\right) + \frac{x^a}{1+x^{2a}}.$$

Solving for $F_A(x)$ gives the result. □

Example 5.2. *If $A = \mathbb{N}$, then the above theorem gives the generating function for the number of Carlitz palindromic compositions of n, namely $F_{\mathbb{N}}(x) = \frac{\sum_{n\geq 1}\frac{x^n}{1+x^{2n}}}{1-\sum_{n\geq 1}\frac{x^n}{1+x^n}}$.*

As for Carlitz compositions, we can generalize Theorem 5.1. Let $A = \{a_1, a_2, \dots\}$, and denote by $F_A(n, r_1, r_2, \dots)$ (respectively $F_A(j; n, r_1, r_2, \dots)$) the number of Carlitz palindromic compositions of n (respectively with j parts) such that the part $a_i \in A$ occurs exactly r_i times in the composition (with $\sum_i r_i = j$) and define $F_A(0; 0, 0, \dots) = 1$ and $F_A(0; n, 0, \dots) = 0$ for $n \neq 0$. The corresponding generating functions are given by

$$F_A(j; x, t_{a_1}, t_{a_2}, \dots) = \sum_{n\geq 0}\sum_{r_1+r_2+\ldots=j} F_A(j; n, r_1, r_2, \dots)x^n \prod_{i\geq 0} t_{a_i}^{r_i}$$

and

$$F_A(x, t_{a_1}, t_{a_2}, \dots) = \sum_{n\geq 0}\sum_{r_1+r_2+\ldots\geq 0} F_A(n, r_1, r_2, \dots)x^n \prod_{i\geq 0} t_{a_i}^{r_i},$$

where x and t_{a_i} are indeterminate variables.

Theorem 5.3. *Let $A = \{a_1, a_2, \dots\} \subseteq \mathbb{N}$. Then*

$$F_A(x, t_{a_1}, t_{a_2}, \dots) = \frac{\sum_{a\in A} \frac{t_a x^a}{1+t_a^2 x^{2a}}}{1 - \sum_{a\in A} \frac{t_a^2 x^{2a}}{1+t_a^2 x^{2a}}}.$$

Proof. The result follows as in the proof of Theorem 4.3, with the appropriate modifications due to the symmetry of the palindromic compositions. □

Theorem 5.3 also allows us to count the number of Carlitz palindromic compositions $F_A^B(n, m)$ of n with parts in A such that m parts are in $B \subseteq A$. We denote the generating function for these compositions by $F_A^B(x, y)$, where $F_A^B(x, y) = \sum_{n\geq 0} \sum_{m\geq 0} F_A^B(n, m) x^n y^m$. Setting $t_a = 1$ for all $a \in A\backslash B$ and $t_a = y$ for all $a \in B$ in Theorem 5.3 yields the following result.

Corollary 5.4. *Let $B \subseteq A \subseteq \mathbb{N}$. Then*

$$F_A^B(x, y) = \frac{\sum_{a\in A\backslash B} \frac{x^a}{1+x^{2a}} + \sum_{a\in B} \frac{y x^a}{1+y^2 x^{2a}}}{1 - \sum_{a\in A\backslash B} \frac{x^{2a}}{1+x^{2a}} - \sum_{a\in B} \frac{y^2 x^{2a}}{1+y x^{2a}}}.$$

6. Concluding Remarks

We have derived generating functions for (palindromic) compositions and for Carlitz (palindromic) compositions of n with parts in $A \subseteq \mathbb{N}$, as well as generating functions for these compositions where each part occurs a given number of times, or where a given number of parts are in a subset of A. This very general framework can be applied to many special cases, of which we have investigated a small selection.

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