Generating functions for generating trees

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Abstract

Certain families of combinatorial objects admit recursive descriptions in terms of generating trees: each node of the tree corresponds to an object, and the branch leading to the node encodes the choices made in the construction of the object. Generating trees lead to a fast computation of enumeration sequences (sometimes, to explicit formulae as well) and provide efficient random generation algorithms. We investigate the links between the structural properties of the rewriting rules defining such trees and the rationality, algebraicity, or transcendence of the corresponding generating function.

1 Introduction

Only the simplest combinatorial structures — like binary strings, permutations, or pure involutions (*i.e.*, involutions with no fixed point) — admit product decompositions. In that case, the set Ω_n of objects of size n is isomorphic to a product set: $\Omega_n \cong [1, e_1] \times [1, e_2] \times \cdots \times [1, e_n]$. Two properties result from such a strong decomposability property: (i) enumeration is easy, since the cardinality of Ω_n is $e_1 e_2 \cdots e_n$; (ii) random generation is efficient since it reduces to a sequence of random independent draws from intervals. A simple infinite tree, called a *uniform generating tree* is determined by the e_i: the root has degree e_1, each of its e_1 descendents has degree e_2, and so on. This tree describes the sequence of all possible choices and the objects of size n are then in natural correspondence with the branches of length n, or equivalently with the nodes of generation n in the tree. The generating tree is thus fully described by its root degree (e_1) and by rewriting rules, here of the special form,

$$(e_i) \rightsquigarrow (e_{i+1})\,(e_{i+1})\,\cdots(e_{i+1}) \equiv (e_{i+1})^{e_i},$$

where the power notation is used to express repetitions. For instance binary strings, permutations, and pure involutions are determined by

$$\begin{array}{ll} \mathcal{S}: & [(2),\ (2) \rightsquigarrow (2)\,(2)] \\ \mathcal{P}: & [(1),\ \{(k) \rightsquigarrow (k+1)^k\}_{k\geq 1}] \\ \mathcal{I}: & [(1),\ \{(2k-1) \rightsquigarrow (2k+1)^{2k-1}\}_{k\geq 1}]. \end{array}$$

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A powerful generalization of this idea consists in considering unconstrained *generating trees* where any set of rules

$$\Sigma = [(s_0),\ \{(k) \rightsquigarrow (e_{1,k})\,(e_{2,k})\ \cdots\ (e_{k,k})\}] \tag{1}$$

is allowed. Here, the *axiom* (s_0) specifies the degree of the root, while the *productions* $e_{i,k}$ list the degrees of the k descendents of a node labeled k. Following Barcucci, Del Lungo, Pergola and Pinzani, we call Σ an *ECO-system* (ECO stands for "Enumerating Combinatorial Objects"). Obviously, much more leeway is available and there is hope to describe a much wider class of structures than those corresponding to product forms and uniform generating trees.

The idea of generating trees has surfaced occasionally in the literature. West introduced it in the context of enumeration of permutations with forbidden subsequences [27, 28]; this idea has been further exploited in closely related problems [6, 5, 12, 13]. A major contribution in this area is due to Barcucci, Del Lungo, Pergola, and Pinzani [4, 3] who showed that a fairly large number of classical combinatorial structures can be described by generating trees.

A form equivalent to generating trees is well worth noting at this stage. Consider the *walks* on the integer half-line that start at point (s_0) and such that the only allowable transitions are those specified by Σ (the steps corresponding to transitions with multiplicities being labeled). Then, the walks of length n are in bijective correspondence with the nodes of generation n in the tree. These walks are constrained by the consistency requirement of trees, namely, that the number of outgoing edges from point k on the half-line has to be exactly k.

EXAMPLE 1. 123-*avoiding permutations*
The method of "local expansion" sometimes gives good results in the enumeration of permutations avoiding specified patterns. Consider for example the set $\mathfrak{S}_n(123)$ of permutations of length n that *avoid the pattern* 123: there exist no integers $i < j < k$ such that $\sigma(i) < \sigma(j) < \sigma(k)$. For instance, $\sigma = 4213$ belongs to $\mathfrak{S}_4(123)$ but $\sigma = 1324$ does not, as $\sigma(1) < \sigma(3) < \sigma(4)$.

Observe that if $\tau \in \mathfrak{S}_{n+1}(123)$, then the permutation σ obtained by erasing the entry $n+1$ from τ belongs to $\mathfrak{S}_n(123)$. Conversely, for every $\sigma \in \mathfrak{S}_n(123)$, insert the value $n+1$ in each place that gives an element of $\mathfrak{S}_{n+1}(123)$ (this is the local expansion). For example, the permutation $\sigma = 213$ gives 4213, 2413 and 2143, by insertion of 4 in first, second and third place respectively. The permutation 2134, resulting from the insertion of 4 in the last place, does not belong to $\mathfrak{S}_4(123)$. This process can be described by a tree whose nodes are the permutations avoiding 123: the root is 1, and the children of any node σ are the permutations derived as above. Figure 1(a) presents the first four levels of this tree.

Let us now label the nodes by their number of children: we obtain the tree of Figure 1(b). It can be proved that the k children of any node labeled k are labeled respectively $k+1, 2, 3, \ldots, k$ (see [27]). Thus the tree we have constructed is the generating tree obtained from the following rewriting rules:

$$[(2),\ \{(k) \rightsquigarrow (2)(3)\ldots(k-1)(k)(k+1)\}_{k\geq 2}].$$

The interpretation of this system in terms of paths implies that 123-avoiding permutations are equinumerous with "walks with returns" on the half-line, themselves isomorphic to Łukasiewicz codes of plane trees (see, *e.g.*, [26, p. 31–35]). We thus recover a classic result [18]: 123-avoiding permutations are counted by Catalan numbers; more precisely, $|\mathfrak{S}_n(123)| = \binom{2n}{n}/(n+1)$. □

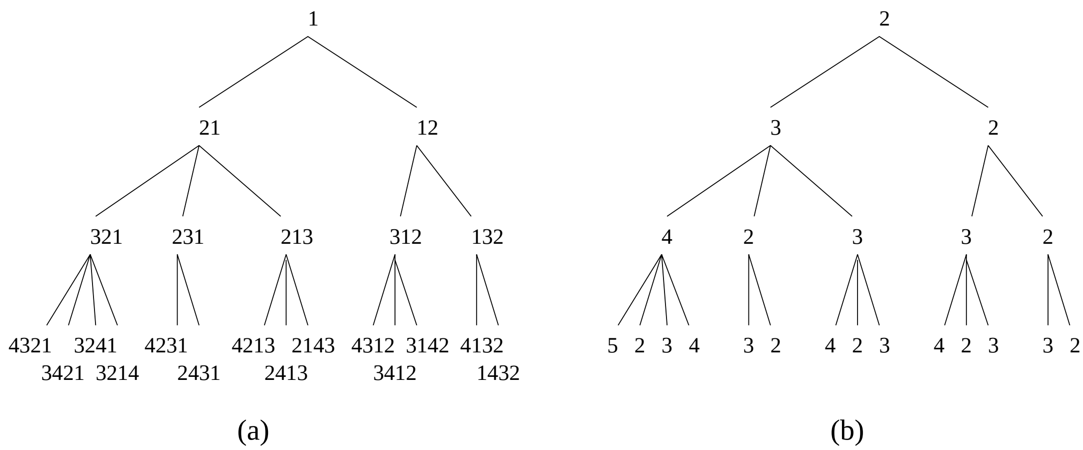

Figure 1: The generating tree of 123-avoiding permutations. (a) Nodes labeled by the permutations. (b) Nodes labeled by the numbers of children.

We shall see below that (certain) generating trees correspond to enumeration sequences of relatively low computational complexity and provide fast random generation algorithms. Hence, there is an obvious interest in delineating as precisely as possible which combinatorial classes admit a generating tree specification. Generating functions condense structural information in a simple analytic entity. We can thus wonder what kind of generating function can be obtained through generating trees. More precisely, we study in this paper the connections between the *structural* properties of the rewriting rules and the *algebraic* properties of the corresponding generating function.

We shall prove several conjectures that were presented to us by Pinzani and his coauthors in March 1998. Our main results can be roughly described as follows.

— *Rational systems.* Systems satisfying strong regularity conditions lead to rational generating functions (Section 2). This covers systems that have a finite number of allowed degrees, as well as systems like $(2.a)$, $(2.b)$, $(2.c)$ and $(2.d)$ below where the labels are constant except for a fixed number of labels that depend linearly and uniformly on k.

— *Algebraic systems.* Systems of a *factorial* form, *i.e.*, where a finite modification of the set $\{1, \ldots, k\}$ is reachable from k, lead to algebraic generating functions (Section 3). This includes in particular cases $(2.f)$ and $(2.g)$.

— *Transcendental systems.* One possible reason for a system to give a transcendental series is the fact that its coefficients grow too fast, so that its radius of convergence is zero. This is the case for System $(2.h)$ below. Transcendental generating functions are also associated with systems that are too “irregular”. An example is System $(2.e)$. We shall also discuss the holonomy of transcendental systems (Section 4).

EXAMPLE 2. *A zoo of rewriting systems*
Here is a list of examples recurring throughout this paper.

$$[(3), \{(k) \rightsquigarrow (3)^{k-3}(k+1)(k+2)(k+9)\}] \qquad (2.a)$$
$$[(3), \{(k) \rightsquigarrow (3)^{k-1}(3k+6)\}] \qquad (2.b)$$
$$[(2), \{(k) \rightsquigarrow (2)^{k-2}(2+(k \bmod 2))(k+1)\}] \qquad (2.c)$$
$$[(2), \{(k) \rightsquigarrow (2)^{k-2}(3-(k \bmod 2))(k+1)\}] \qquad (2.d)$$
$$[(2), \{(k) \rightsquigarrow (2)^{k-2}(3-[\exists p : k = 2^p])(k+1)\}] \qquad (2.e)$$
$$[(2), \{(k) \rightsquigarrow (2)(3)\dots(k-1)(k)(k+1)\}] \qquad (2.f)$$
$$[(1), \{(k) \rightsquigarrow (1)(2)\dots(k-1)(k+1)\}] \qquad (2.g)$$
$$[(2), \{(k) \rightsquigarrow (2)(3)(k+2)^{k-2}\}] \qquad (2.h)$$

(In (2.e), we make use of Iverson's brackets: $[P]$ equals 1 if P is true, 0 otherwise.) □

Notations. From now on, we adopt functional notations for rewriting rules: systems will be of the form

$$[(s_0), \quad \{(k) \rightsquigarrow (e_1(k))\ (e_2(k)) \dots (e_k(k))\}]$$

where s_0 is a constant and each e_i is a function of k. Moreover, we assume that all the values appearing in the generating tree are positive: each node has at least one descendent.

In the generating tree, let f_n be the number of nodes at level n and s_n the sum of the labels of these nodes. By convention, the root is at level 0, so that $f_0 = 1$. In terms of walks, f_n is the number of walks of length n. The generating function associated with the system is

$$F(z) = \sum_{n \geq 0} f_n z^n.$$

Remark that $s_n = f_{n+1}$, and that the sequence $(f_n)_n$ is nondecreasing.

Now let $f_{n,k}$ be the number of nodes at level n having label k (or the number of walks of length n ending at position k). The following generating functions will be also of interest:

$$F(z,u) = \sum_{n,k \geq 0} f_{n,k} z^n u^k \qquad \text{and} \quad F_k(z) = \sum_{n \geq 0} f_{n,k} z^n.$$

We have $F(z) = F(z,1) = \sum_{k \geq 1} F_k(z)$. Furthermore, the F_k's satisfy the relation

$$F_k(z) = [k = s_0] + z \sum_{j \geq 1} \pi_{j,k} F_j(z), \qquad (2)$$

where $\pi_{j,k} = |\{i \leq j : e_i(j) = k\}|$ denotes the number of one-step transitions from j to k. This is equivalent to the following recurrence for the numbers $f_{n,k}$,

$$f_{0,k} = [k = s_0] \qquad \text{and} \quad f_{n+1,k} = \sum_{j \geq 1} \pi_{j,k} f_{n,j}, \qquad (3)$$

that results from tracing all the paths that lead to k in $n+1$ steps.

Counting and random generation. The recurrence (3) gives rise to an algorithm that computes the successive rows of the matrix $(f_{n,k})$ by "forward propagation": to compute the $(n+1)$th row, propagate the contribution $f_{n,j}$ to $f_{n+1,e_i(j)}$ for all pairs (i,j) such that $i \le j$. Assume the system is *linearly bounded*: this means that the labels of the nodes that can be reached in m steps are bounded by a linear function of m. (All the systems given in Example 2, except for (2.b), are linearly bounded; more generally, systems where forward jumps are bounded by a constant are linearly bounded.) Clearly, the forward propagation algorithm provides a counting algorithm of arithmetic complexity that is at most cubic.

For a linearly bounded system, uniform random generation can also be achieved in polynomial time, as shown in [2]. We present here the general principle.

Let $g_{n,k}$ be the number of walks of length n that start from label k. These numbers are determined by the recurrence $g_{n,k} = \sum_i g_{n-1,e_i(k)}$, that traces all the possible continuations of a path given its initial step. Obviously, $f_n = g_{n,s_0}$, with s_0 the axiom of the system. As above, the $g_{n,k}$ can be determined in time $O(n^3)$ and $O(n^2)$ storage. Random generation is then achieved as follows: In order to generate a walk of length n starting from state k, pick up a transition i with probability $g_{n-1,e_i(k)}/g_{n,k}$, and generate recursively a walk of length $n-1$ starting from state $e_i(k)$. The cost of a single random generation is then $O(n^2)$ if a sequential search is used over the $O(n)$ possibilities of each of the n random drawings; the time complexity goes down to $O(n \log n)$ if binary search is used, but at the expense of an increase in storage complexity of $O(n^3)$ (arising from $O(n^2)$ arrays of size $O(n)$ that binary search requires).

2 Rational systems

We give in this section three main criteria (and a variation on one of them) implying that the generating function of a given ECO-system is rational.

Our first and simplest criterion applies to systems in which the functions e_i are uniformly bounded.

Proposition 1 *If finitely many labels appear in the tree, then $F(z)$ is rational.*

Proof. Only a finite number of F_k's are nonzero, and they are related by linear equations like Equation (2) above. ■

EXAMPLE 3. *The Fibonacci numbers*
The system $[(1), \{(k) \rightsquigarrow (k)^{k-1}((k \bmod 2)+1)\}]$ can be also written as $[(1), \{(1) \rightsquigarrow (2), (2) \rightsquigarrow (1)(2)\}]$. Hence the only labels that occur in the tree are 1 and 2. Eq. (2) gives $F_1(z) = 1 + zF_2(z)$ and $F_2(z) = z(F_1(z) + F_2(z))$. Finally,

$$F(z) = \frac{1}{1-z-z^2} = \sum_{n\ge 0} f_n z^n = 1 + z + 2z^2 + 3z^3 + 5z^4 + \cdots,$$

the well-known Fibonacci generating function. □

None of the systems of Example 2 satisfy the assumptions of Proposition 1. However, the following criterion can be applied to systems (2.a) and (2.b).

Proposition 2 *Let $\sigma(k) = e_1(k) + e_2(k) + \cdots + e_k(k)$. If σ is an affine function of k, say $\sigma(k) = \alpha k + \beta$, then the series $F(z)$ is rational. More precisely:*

$$F(z) \;=\; \frac{1 + (s_0 - \alpha)z}{1 - \alpha z - \beta z^2}.$$

Proof. Let $n \geq 0$ and let $k_1, k_2, \ldots k_{f_n}$ denote the labels of the f_n nodes at level n. Then

$$\begin{aligned} f_{n+2} \;=\; s_{n+1} &\;=\; (\alpha k_1 + \beta) + (\alpha k_2 + \beta) + \cdots + (\alpha k_{f_n} + \beta) \\ &\;=\; \alpha s_n + \beta f_n \;=\; \alpha f_{n+1} + \beta f_n. \end{aligned}$$

We know that $f_0 = 1$ and $f_1 = s_0$. The result follows. ∎

EXAMPLE 4. *Bisection of Fibonacci sequence*
The system $[(2), \{(k) \rightsquigarrow (2)^{k-1}(k+1)\}]$ gives $F(z) = \frac{1-z}{1-3z+z^2} = 1 + 2z + 5z^2 + \cdots$, the generating function for Fibonacci numbers of even index. (Changing the axiom to $(s_0) = (3)$ leads to the other half of the Fibonacci sequence.) Some other systems, like

$$\begin{aligned} &[(2),\ \{(k) \rightsquigarrow (1)^{k-1}(2k)\}], \\ &[(2),\ \{(k) \rightsquigarrow (2)^{k-2}(3 - (k \bmod 2))(k + (k \bmod 2))\}], \\ &[(2),\ \{(k) \rightsquigarrow (2)^{k-2}(3 - [k \text{ is prime}])(k + [k \text{ is prime}])\}], \end{aligned}$$

lead to the same function $F(z)$ since $\sigma(k) = 3k - 1$ and $s_0 = 2$. However, the generating trees are different, as are the bivariate functions $F(z, u)$. □

EXAMPLE 5. *Prime numbers and rational generating functions*
Amazingly, it is possible to construct a generating tree whose set of labels is the set of prime numbers but that has a rational generating function $F(z)$. This is a bit unexpected, as prime numbers are usually thought "too irregular" to be associated with rational generating functions. For $n \geq 1$, let p_n denote the nth prime; hence $(p_1, p_2, p_3, \ldots) = (2, 3, 5, \ldots)$. Assume for the moment that the Goldbach conjecture is true: every even number larger than 3 is the sum of two primes. Remember that, according to Bertrand's postulate, $p_{n+1} < 2p_n$ for all n (see, *e.g.*, [23, p. 140]).

For $n \geq 1$, the number $2p_n - p_{n+1} + 3$ is an even number larger than 3. Let q_n and r_n be two primes such that $2p_n - p_{n+1} + 3 = q_n + r_n$. In particular, $q_1 = r_1 = 2$. Consider the system

$$[(2), \{(p_n) \rightsquigarrow (p_{n+1})(q_n)(r_n)(2)^{p_n - 3}\}].$$

It satisfies the criterion of Proposition 2, with $\sigma(k) = 4k - 3$. Hence, the generating function of the associated generating tree is

$$F(z) = \frac{1 - 2z}{1 - 4z + 3z^2} = \frac{1}{2}\left[\frac{1}{1-z} + \frac{1}{1-3z}\right].$$

Consequently, the number of nodes at level n is simply $f_n = (1 + 3^n)/2$. This can be checked on the first few levels of the tree drawn in Figure 2.

Now, one can object that the Goldbach conjecture is not proved; however, it is known that every even number is the sum of at most six primes [22], and a similar example can be constructed using this result.

□

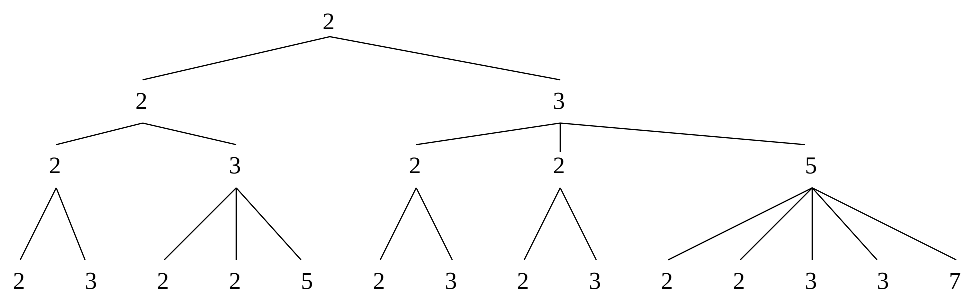

Figure 2: A generating tree with prime labels and rational generating function.

Proposition 2 can be adapted to apply to systems that "almost" satisfy the criterion of Proposition 2, like System (2.c) or (2.d). Let us consider a system of the form

$$(s_0), \quad (k) \rightsquigarrow e_1^{[0]}(k), \dots, e_k^{[0]}(k) \quad \text{if } k \text{ is even,}$$
$$(k) \rightsquigarrow e_1^{[1]}(k), \dots, e_k^{[1]}(k) \quad \text{if } k \text{ is odd.}$$

Assume, moreover, that:

(i) the corresponding functions σ_0 and σ_1 are affine and have the same leading coefficient α, say $\sigma_0(k) = \alpha k + \beta_0$ and $\sigma_1(k) = \alpha k + \beta_1$;

(ii) exactly m odd labels occur in the right-hand side of each rule, for some $m \geq 0$.

Proposition 3 *If a system satisfies properties* (i) *and* (ii) *above, then*

$$F(z) = \frac{1 + (s_0 - \alpha)z + (s_1 - \alpha s_0 - \beta_0)z^2}{1 - \alpha z - \beta_0 z^2 - m(\beta_1 - \beta_0)z^3}.$$

Of course, if $\beta_0 = \beta_1$, we recover the generating function of Proposition 2.

Proof. The proof is similar to that of Proposition 2. The only new ingredient is the fact that, for $n \geq 1$, the number of nodes of odd label at level n is mf_{n-1}. ■

System (2.c) satisfies properties (i) and (ii) above with $\alpha = 3$, $\beta_0 = -1$, $\beta_1 = 0$, $m = 1$, $s_0 = 2$ and $s_1 = 5$. Consequently, its generating function is $F(z) = \frac{1-z}{1-3z+z^2-z^3}$. System (2.$d$), although very close to (2.c), does not satisfy property (ii) above, so that Proposition 3 does not apply. However, another minor variation on the argument of Proposition 2, based on the fact that the number o_n of odd labels at level n satisfies $o_n = 2(f_{n-1} - o_{n-1})$, proves the rationality of $F(z)$.

Alternatively, rationality follows from the last criterion of this section, which is of a different nature. We consider systems $[(s_0), \{(k) \rightsquigarrow (e_1(k))(e_2(k)) \dots (e_k(k))\}]$ that can be written as

$$[(s_0), \{(k) \rightsquigarrow (c_1(k))(c_2(k)) \dots (c_{k-m}(k))(k + a_1)(k + a_2) \dots (k + a_m)\}] \tag{4}$$

where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_m$ and the functions c_i are uniformly bounded. Let $C = \max_{i,k}\{s_0, c_i(k)\}$.

Proposition 4 *Consider the system* (4), *and let* $\pi_{j,k} = |\{i \leq j : e_i(j) = k\}|$. *If all the series*

$$\sum_{j \geq 1} \pi_{j,k}\ t^j$$

for $k \leq C$ are rational, then so is the series $F(z)$.

Proof. We form an infinite system of equations defining the series $F_k(z)$ by writing Eq. (2) for all $k \geq 1$. In particular, for $k > C$, we obtain

$$F_k(z) = z \sum_{\ell=1}^{m} F_{k-a_\ell}(z),$$

with $F_j(z) = 0$ if $j \leq 0$. This part of the system is easy to solve in terms of $F_1, \ldots, F_C$. Indeed, for $k \in \mathbb{Z}$:

$$F_k(z) = \sum_{i=1}^{C} P_{i,k}(z) F_i(z) \tag{5}$$

where the $P_{i,k}$ are polynomials in z defined by the following recurrence: for all $i \leq C$,

$$P_{i,k}(z) = \begin{cases} 0 & \text{if } k \leq 0, \\ [k = i] & \text{if } 0 < k \leq C, \\ z \sum_{\ell=1}^{m} P_{i,k-a_\ell}(z) & \text{if } k > C. \end{cases} \tag{6}$$

Using (5), we find

$$F(z) = \sum_{k \geq 1} F_k(z) = \sum_{i=1}^{C} \left[F_i(z) \sum_{k \geq 1} P_{i,k}(z) \right].$$

According to (6), for all $i \leq C$, the series $\sum_{k \geq 1} P_{i,k}(z) t^k$ is a rational function of z and t, of denominator $1 - z \sum_\ell t^{a_\ell}$. At $t = 1$, it is rational in z. Hence, to prove the rationality of $F(z)$, it suffices to prove the rationality of the $F_i(z)$, for $i \leq C$.

Let us go back to the C first equations of our system; using (5), we find, for $k \leq C$:

$$F_k(z) = [k = s_0] + z \sum_{i=1}^{C} \left[F_i(z) \sum_{j \geq 1} P_{i,j}(z) \pi_{j,k} \right].$$

Again, $\sum_{j \geq 1} P_{i,j}(z) \pi_{j,k} t^j$ is a rational function of z and t (the Hadamard product of two rational series is rational). Thus the series $F_k(z)$, for $k \leq C$, satisfy a linear system with rational coefficients: they are rational themselves, as well as $F(z)$. ∎

Examples (2.a), (2.c), (2.d) and (2.e) have the form (4). The above proposition implies that the first three have a rational generating function. System (2.e) will be discussed in Section 4, and proved to have a transcendental generating function.

3 Factorial walks and algebraic systems

In this section, we consider systems that are of a *factorial form*. By this, we mean informally that the set of successors of (k) is a finite modification of the integer interval $\{1, 2, \ldots, k\}$. As was detailed in the introduction, ECO-systems can be rephrased in terms of walks over the integer half-line. We thus consider the problem of enumerating walks over the integer half-line such that the set of allowed moves from point k is a finite modification of the integer interval $[0, k]$. We shall mostly study modifications around the point k (although some examples where the interval is modified around 0 as well are given at the end of the section). Precisely, a *factorial walk* is defined by a finite (multi)set $A \subset \mathbb{Z}$ and a finite set $B \subset \mathbb{N}^+$, where $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$, specifying respectively the *allowed supplementary jumps* (possibly labeled) and the *forbidden backward jumps*. In other words, the possible moves from k are given by the rule:

$$(k) \leadsto [0, k-1] \setminus (k - B) \ \cup \ (k + A). \tag{7}$$

Observe that these walk models are not necessarily ECO-systems, first because we allow labels to be zero – but a simple translation can take us back to a model with positive labels – and second because we do not require (k) to have exactly k successors.

We say that an ECO-system is factorial if a shift of indices transforms it into a factorial walk. Hence the rules of a factorial ECO-system are of the form

$$(k + r) \leadsto [r, k + r - 1] \setminus (k + r - B) \ \cup \ (k + r + A),$$

that is,

$$(k) \leadsto [r, k-1] \setminus (k - B) \ \cup \ (k + A) \quad \text{ for } k \geq r \geq 1. \tag{8}$$

The generating function $F(z)$ for such an ECO-system, taken with axiom (s_0), equals the generating function for the walk model (7), taken with axiom $(s_0 - r)$. However, remember that the rewriting rules defining a generating tree have to obey the additional condition that a node labeled k has exactly k successors. Taking $k = r$ in (8), this implies that $r = |A|$. Taking $k > r + \max B$, this implies that $r + |B| = |A|$, so that finally $B = \emptyset$. Hence, strictly speaking, either one has a "fake" factorial ECO-system (that is some of its initial rules are not of the factorial type), either one has a "real" factorial ECO-system and then it is given by rules of the form

$$(k) \leadsto [r, k-1] \ \cup \ (k + A) \quad \text{ for } k \geq r \geq 1,$$

where A is a multiset of integers of cardinality r. For instance, Systems $(2.f)$ and $(2.g)$ are factorial. We shall prove that all factorial walks have an algebraic generating function. The result naturally applies to factorial ECO-systems.

We consider again the generating function $F(z, u) = \sum_{n,k \geq 0} f_{n,k} z^n u^k$, where $f_{n,k}$ is the number of walks of length n ending at point k. We also denote by $F_k(z)$ the coefficient of u^k in this series, and by $f_n(u)$ the coefficient of z^n. The first ingredient of the proof is a linear operator M, acting on formal power series in u, that encodes the possible moves. More precisely, for all $n \geq 0$, we will have:

$$M[f_n](u) = f_{n+1}(u).$$

The operator M is constructed step by step as follows.

— The set of moves from k to all the positions $0, 1, \ldots, k-1$ is described by the operator L_0 that maps u^k to $u^0 + u^1 + \cdots + u^{k-1} = (1 - u^k)/(1 - u)$. As L_0 is a linear operator, we have, for any series $g(u)$:

$$L_0[g](u) = \frac{g(1) - g(u)}{1 - u}.$$

— The fact that transitions near k are modified, with those of type $k+\alpha$ (with $\alpha \in A$) allowed and those of type $k-\beta$ (with $\beta \in B$) forbidden, is expressed by a Laurent polynomial

$$P(u) = \sum_{k=-b}^{a} p_k u^k = A(u) - B(u) \quad \text{with} \quad A(u) = \sum_{\alpha \in A} u^\alpha \quad \text{and} \quad B(u) = \sum_{\beta \in B} u^{-\beta}.$$

The degree of P is $a := \max A$, the largest forward jump and $b := \max(0, -B, -A)$ is largest forbidden backward jump or the largest supplementary backward jumps (we take $b = 0$ if the set B is empty).

The operator

$$L[g](u) := L_0[g](u) + P(u)g(u)$$

describes the extension of a walk by one step.

— Finally, the operator M is given by

$$M[g](u) = L[g](u) - \{u^{<0}\}L[g](u),$$

where $\{u^{<0}\}h(u)$ is the sum of all the monomials in $h(u)$ having a negative exponent. Hence M is nothing but L stripped of the negative exponent monomials, which correspond to walks ending on the nonpositive half-line. Observe that, for any series $g(u)$, the only part of $L[g](u)$ that is likely to contain monomials with negative exponents is $P(u)g(u)$. Consequently,

$$M[g](u) = L[g](u) - \{u^{<0}\}[P(u)g(u)]$$

and if $g(u) = \sum_k g_k u^k$, then

$$\{u^{<0}\}[P(u)g(u)] = \sum_{i=1}^{b}\sum_{k=0}^{i-1} g_k p_{-i}\, u^{k-i} = \sum_{k=0}^{b-1} g_k r_k(u). \tag{9}$$

Assume for simplicity that the initial point of the walk is 0; other cases follow the same argument. The linear relation $f_{n+1}(u) = M[f_n](u)$, together with $f_0(u) = 1$, yields

$$\begin{aligned} F(z,u) &= 1 + zM[F](z,u) \\ &= 1 + z\left(\frac{F(z,1) - F(z,u)}{1-u} + P(u)F(z,u) + \{u^{<0}\}[P(u)F(z,u)]\right). \end{aligned}$$

Thanks to (9), we can write

$$\{u^{<0}\}[P(u)F(z,u)] = \sum_{k=0}^{b-1} r_k(u)F_k(z),$$

where $r_k(u)$ is a Laurent polynomials (defined by Equation 9) whose exponents belong to $[k-b, -1]$. Thus, $F(z,u)$ satisfies the following functional equation:

$$F(z,u)\left(1 + \frac{z}{1-u} - zP(u)\right) = 1 + \frac{zF(z,1)}{1-u} + z\sum_{k=0}^{b-1} r_k(u)F_k(z). \tag{10}$$

Let us take an example. The moves

$$(k) \rightsquigarrow (0)(1)\cdots(k-5)(k-3)(k-1)(k)(k+7)(k+9),$$

lead to $A(u) = u^0 + u^7 + u^9$ and $B(u) = u^{-4} + u^{-2}$. Moreover,

$$\{u^{<0}\}[B(u)F(z,u)] = (u^{-2} + u^{-4})F_0(z) + (u^{-1} + u^{-3})F_1(z) + u^{-2}F_2(z) + u^{-1}F_3(z),$$

so that the functional equation defining $F(z,u)$ is

$$F(z,u)\left(1 + \frac{z}{1-u} - z(1 + u^7 + u^9 - u^{-4} - u^{-2})\right) =$$

$$1 + \frac{zF(z,1)}{1-u} + z(u^{-2} + u^{-4})F_0(z) + z(u^{-1} + u^{-3})F_1(z) + zu^{-2}F_2(z) + zu^{-1}F_3(z).$$

The second ingredient of the proof, sometimes called the *kernel method*, seems to belong to the "mathematical folklore" since the 1970's. It has been used in various combinatorial problems [10, 18, 20] and in probabilities [14]. See also [8, 9, 21] for more recent and systematic applications. This method consists in cancelling the left-hand side of the fundamental functional equation (10) by coupling z and u, so that the coefficient of the (unknown) quantity $F(z,u)$ is zero. This constraint defines u as one of the branches of an algebraic function of z. Each branch that can be substituted analytically into the functional equation yields a linear relation between the unknown series $F(z,1)$ and $F_k(z)$, $0 \le k < b$. If enough branches can be substituted analytically, we obtain a system of linear equations, whose solution gives $F(z,1)$ and the $F_k(z)$ as algebraic functions. From there, an expression for $F(z,u)$ also results in the form of a bivariate algebraic function.

Let us multiply Eq. (10) by $u^b(1-u)$ to obtain an equation with polynomial coefficients (remind that we take $b = 0$ if the set B of forbidden backward steps is empty). The new equation reads $K(z,u)F(z,u) = R(z,u)$, where $K(z,u)$ is the *kernel* of the equation:

$$\begin{aligned} K(z,u) &= u^b(1-u)\left(1 + \frac{z}{1-u} - zP(u)\right), \\ &= u^b(1-u) + zu^b - z(1-u)\sum_{\alpha\in A} u^{\alpha+b} + z(1-u)\sum_{\beta\in B} u^{b-\beta}. \end{aligned} \tag{11}$$

This polynomial has degree $a+b+1$ in u, and hence, admits $a+b+1$ solutions, which are algebraic functions of z. The classical theory of algebraic functions and the Newton polygon construction enable us to expand the solutions near any point as Puiseux series (that is, series involving fractional exponents; see [11]). The $a+b+1$ solutions, expanded around 0, can be classified as follows:

— the "unit" branch, denoted by u_0, is a power series in z with constant term 1;

— b "small" branches, denoted by $u_1, \ldots, u_b$, are power series in $z^{1/b}$ whose first nonzero term is $\zeta z^{1/b}$, with $\zeta^b + 1 = 0$;

— a "large" branches, denoted by $v_1, \ldots, v_a$, are Laurent series in $z^{1/a}$ whose first nonzero term is $\zeta z^{-1/a}$, with $\zeta^a + 1 = 0$.

In particular, all the roots are distinct. (It is not difficult to check "by hand" the existence of these solutions: for instance, plugging $z = t^b$ and $u = tw(t)$ in $K(z,u) = 0$ confirms the existence of the b small branches.) Note that there are exactly $b+1$ finite branches: the unit branch u_0 and the b small branches $u_1, \ldots, u_b$. As $F(z,u)$ is a series in z with *polynomial* coefficients in u, these $b+1$ series u_i, having no negative exponents, can be substituted for u in $F(z,u)$. More specifically, let us replace u by u_i in (10): the right-hand side of the equation vanishes, giving a linear equation relating the $b+1$ unknown series $F(z,1)$ and $F_k(z)$, $0 \le k < b$. Hence the $b+1$ finite branches give a set of $b+1$ linear equations relating the $b+1$ unknown series. One could solve directly this system, but the following argument is more elegant.

The right-hand side of (10), once multiplied by $u^b(1-u)$, is

$$R(z,u) = u^b(1-u)\left(1 + \frac{z}{1-u}F(z,1) + z\sum_{k=0}^{b-1} r_k(u)F_k(z)\right).$$

By construction, it is a *polynomial* in u of degree $b+1$ and leading coefficient -1. Hence, it admits $b+1$ roots, which depend on z. Replacing u by the series $u_0, u_1, \ldots, u_b$ in Eq. (10) shows that these series are exactly the $b+1$ roots of R, so that

$$R(z,u) = -\prod_{i=0}^{b}(u-u_i).$$

Let $p_a := [u^a]P(u)$ be the multiplicity of the largest forward jump. Then the coefficient of u^{a+b+1} in $K(z,u)$ is $p_a z$, and we can write

$$K(z,u) = p_a z\prod_{i=0}^{b}(u-u_i)\prod_{i=1}^{a}(u-v_i).$$

Finally, as $K(z,u)F(z,u) = R(z,u)$, we obtain

$$F(z,u) = \frac{-\prod_{i=0}^{b}(u-u_i)}{u^b(1-u) + zu^b - zu^b(1-u)P(u)} = -\frac{1}{p_a z\prod_{i=1}^{a}(u-v_i)}. \tag{12}$$

We have thus proved the following result.

Proposition 5 *The generating function $F(z,u)$ for factorial walks defined by* (7) *and starting from* 0 *is algebraic; it is given by* (12)*, where $u_0, \ldots, u_b$ (resp. $v_1, \ldots, v_a$) are the finite (resp. infinite) solutions at $z = 0$ of the equation $K(z,u) = 0$ and the kernel K is defined by* (11)*. In particular, the generating function for all walks, irrespective of their endpoint, is*

$$F(z,1) = -\frac{1}{z}\prod_{i=0}^{b}(1-u_i),$$

and the generating function for excursions, *i.e., walks ending at* 0*, is, for $b < 0$:*

$$F(z,0) = \frac{(-1)^b}{z}\prod_{i=0}^{b}u_i,$$

(for $b = 0$, the relation becomes $F(z,0) = \frac{(-1)^b}{1+z-p_0 z}\prod_{i=0}^{b}u_i$.)

These results could be derived by a detour via multivariate linear recurrences, and the present treatment is closely related to [9, 21]; however, our results were obtained independently in March 1998 [1].

The asymptotic behaviour of the number of n-step walks can be established via singularity analysis or saddle point methods. The series u_i have "in general" a square root singularity, yielding an asymptotic behaviour of the form $A\mu^n n^{-3/2}$. We plan to develop this study in a forthcoming paper.

EXAMPLE 6. *Catalan numbers*
This is the simplest factorial walk, $(k) \rightsquigarrow (0)(1)\dots(k)(k+1)$, which corresponds to the ECO-system $(2.f)$. The operator M is given by

$$M[f](u) = \frac{f(1) - f(u)}{1-u} + (1+u)f(u).$$

The kernel is $K(z,u) = 1 - u + z - z(1-u)(1+u) = 1 - u + zu^2$, hence $u_0(z) = \frac{1-\sqrt{1-4z}}{2z}$, so that

$$F(z,1) = -\frac{1-u_0}{z} = \frac{1-2z-\sqrt{1-4z}}{2z^2} = \sum_{n\geq 1} \binom{2n}{n} \frac{z^{n-1}}{n+1},$$

the generating function of the Catalan numbers (sequence **M1459**[1]). This result could be expected, given the obvious relation between these walks and Łukasiewicz codes. □

EXAMPLE 7. *Motzkin numbers*
This example, due to Pinzani and his co-authors, is derived from the previous one by forbidding "forward" jumps of length zero. The rule is then

$$(k) \rightsquigarrow (0)\cdots(k-1)(k+1).$$

The operator M is

$$M[f](u) = \frac{f(1) - f(u)}{1-u} + uf(u).$$

The kernel is $K(z,u) = 1 - u + z - zu(1-u) = 1 + z - u(1+z) + zu^2$, leading to

$$F(z,1) \quad = \quad \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2} = 1 + z + 2z^2 + 4z^3 + 9z^4 + 21z^5 + O(z^6),$$

the generating function for Motzkin numbers (sequence **M1184**). □

EXAMPLE 8. *Schröder numbers*
This example is also due to the Florentine group. The rule is $(k) \rightsquigarrow (0)\dots(k-1)(k)(k+1)^2$. From Proposition 5, we derive

$$F(z,1) = \frac{1-3z-\sqrt{1-6z+z^2}}{4z^2} = 1 + 3z + 11z^2 + 45z^3 + 197z^4 + O(z^5).$$

The coefficients are the Schröder numbers (**M2898**: Schröder's second problem). We give in Table 1 at the end of the paper a generalization of Catalan and Schröder numbers, corresponding to the rule $(k) \rightsquigarrow (0)\dots(k-1)(k)(k+1)^m$. This generalized rule has recently been shown to describe a set of permutations avoiding certain patterns [19]. □

[1]The numbers **Mxxxx** are identifiers of the sequences in *The Encyclopedia of Integer Sequences* [24].

The above examples were all quadratic. However, it is clear from our treatment that algebraic functions of arbitrary degree can be obtained: it suffices that the set of "exceptions" around k have a span greater than 1. Let us start with a family of ECO-systems where supplementary forward jumps of length larger than one are allowed.

EXAMPLE 9. *Ternary trees, dissections of a polygon, and m-ary trees*
The ECO-system with axiom $(s_0) = (3)$ and rule

$$(k) \leadsto (3)(4)\cdots(k)(k+1)(k+2)$$

is equivalent to the walk

$$(k) \leadsto (0)(1)\cdots(k)(k+1)(k+2).$$

The kernel is $K(z,u) = 1 - u + zu^3$, and the generating function

$$F(z,1) = \sum_{n\geq 1} \binom{3n}{n} \frac{z^{n-1}}{2n+1}$$

counts ternary trees (**M2926**).

More generally, the system with axiom (m) and rewriting rules

$$(k) \leadsto (m)\cdots(k)(k+1)(k+2)\cdots(k+m-1)$$

yields the m-Catalan numbers, $\binom{mn}{n}/((m-1)n+1)$, that count m-ary trees. The kernel is $1 - u + zu^m$ and the generating function $F(z,1)$ satisfies $F(z,1) = (1 + zF(z,1))^m$. In particular, the 4-Catalan numbers $\binom{4n}{n}/(3n+1)$ appear in [24] (sequence **M3587**) and count dissections of a polygon.

$\square$

In the above examples, all backward jumps are allowed. In other words, each of these examples corresponds to an ECO-system. Let us now give an example where backward jumps of length 1 are forbidden.

EXAMPLE 10.
Consider the following modification of the Motzkin rule:

$$(k) \leadsto (0)\cdots(k-2)(k+1).$$

The kernel is now $K(z,u) = u(1-u) + zu - z(1-u)(u^2-1)$, and, according to (12), the series $F(z) = F(z,1)$ is given by $F(z) = 1/[z(v_1 - 1)]$, where v_1 satisfies $K(z,v_1) = 0$ and is infinite at $z = 0$. Denoting $G = zF(z)$, we find that the algebraic equation defining G is:

$$G = z\,\frac{1 + 2G + G^2 + G^3}{1+G}.$$

$\square$

So far, we have only dealt with walks for which the set of allowed moves was obtained by modifying the interval $[0, k]$ around k. One can also modify this interval around 0: we shall see – in examples – that the generating function remains algebraic. However, it is interesting to note that in these examples, the kernel method does not immediately provide enough equations between the "unknown functions" to solve the functional equation.

Let us first explain how we modify the interval $[0, k]$ around 0. The walks we wish to count are still specified by a multiset A of allowed supplementary jumps and a set B of forbidden backward jumps. But, in addition, we forbid backward jumps to end up in C, where C is a given finite subset of $\mathbb{N}$. In other words, the possible moves from k are given by the rule

$$(k) \rightsquigarrow [0, k-1] \setminus (C \cup (k - B)) \ \cup \ (k + A).$$

Again, we can write a functional equation defining $F(z, u)$:

$$F(z,u) = 1 + z\left(\frac{F(z,1) - F(z,u)}{1-u} + P(u)F(z,u) + \sum_{k=0}^{b-1} r_k(u)F_k(z) - \sum_{\gamma \in C} u^\gamma G_\gamma(z)\right), \quad (13)$$

where, as above,

$$P(u) = \sum_{\alpha \in A} u^\alpha - \sum_{\beta \in B} u^{-\beta} \quad \text{and} \quad r_k(u) = \sum_{\beta > k,\ \beta \in B} u^{k-\beta},$$

the new terms in the equations being

$$G_\gamma(z) = F(z,1) - \sum_{k=0}^{\gamma} F_k(z) - \sum_{\beta \in B} F_{\beta+\gamma}(z).$$

Observe that the first three terms are the same as in the case $C = \emptyset$. The equation, once multiplied by $u^b(1-u)$, reads $K(z,u)F(z,u) = R(z,u)$ where $K(z,u)$ is given by (11) and

$$R(z,u) = u^b(1-u)\left(1 + \frac{zF(z,1)}{1-u} + z\sum_{k=0}^{b-1} r_k(u)F_k(z) - z\sum_{\gamma \in C} u^\gamma G_\gamma(z)\right).$$

The kernel is not modified by the introduction of C. As above, it has degree $a+b+1$ in u, and admits $b+1$ finite roots $u_0, \ldots, u_b$ around $z = 0$. However, $R(z,u)$ now involves $b + 1 + |C|$ unknown functions, namely $F(z,1)$, the $F_k(z)$, $0 \le k < b$ and the $G_\gamma(z)$, $\gamma \in C$. The degree of R in u is no longer $b+1$ but $b+c+1$, where $c = \max C$. The $b+1$ roots of K that can be substituted for u in Eq. (13) provide $b+1$ linear equations between the $b + |C| + 1$ unknown functions. Additional equations will be obtained by extracting the coefficient of u^j from Eq. (13), for some values of j. In general, we have:

$$F_j(z) = [j=0] + z\sum_{\alpha \in A} F_{j-\alpha}(z) + z[j \notin C]\left(F(z,1) - \sum_{k=0}^{j} F_k(z) - \sum_{\beta \in B} F_{j+\beta}(z)\right). \quad (14)$$

It is possible to construct a finite subset $S \subset \mathbb{N}$ such that the combination of the $b+1$ equations obtained via the kernel method and the equations (14) written for $j \in S$ determines all unknown functions as algebraic functions of z – more precisely, as rational functions of z and the roots $u_0, \ldots, u_b$ of the kernel. However, this is a long development, and these classes of walks play a marginal role in the context of ECO-systems. For these reasons, we shall merely give two examples. The details on the general procedure for constructing the set S can be found in [7].

EXAMPLE 11.
This example is obtained by modifying the Motzkin rule of Example 7 around the point 0. Take $A = C = \{1\}$ and $B = \emptyset$. The rewriting rule is

$$(k) \leadsto (0)(2)(3)\cdots(k-1)(k+1).$$

The functional equation reads

$$(1-u+z-zu(1-u))F(z,u) = 1-u+zF(z,1)-zu(1-u)G_1(z), \tag{15}$$

with $G_1(z) = F(z,1) - F_0(z) - F_1(z)$. The kernel has a *unique* finite root at $z = 0$:

$$u_0 = \frac{1+z-\sqrt{1-2z-3z^2}}{2z},$$

whereas the right-hand side of Eq. (15) contains *two* unknown functions. Writing Eq. (14) for $j = 0$ and $j = 1$ yields

$$F_0(z) = 1 + z(F(z,1) - F_0(z)) \quad \text{and} \quad F_1(z) = zF_0(z).$$

These two equations allow us to express F_0 and F_1, and hence G_1, in terms of $F(z,1)$:

$$G_1(z) = (1-z)F(z,1) - 1.$$

This equation relates the two unknown functions of Eq. (15). We replace $G_1(z)$ by the above expression in (15), so that only one unknown function, namely $F(z,1)$, is left. The kernel method finally gives:

$$F(z,1) = \frac{3-3z^2-2z^3-(1+z)\sqrt{1-2z-3z^2}}{2(1-z-z^2+z^3+z^4)} = 1+z+2z^2+3z^3+6z^4+12z^5+O(z^6).$$

□

EXAMPLE 12.
Let us choose $A = \{1\}$, $B = \{2\}$ et $C = \{2\}$. The rewriting rule is now:

$$(k) \to (0)(1)(3)(4)(5)\ldots(k-3)(k-1)(k+1).$$

The functional equation reads

$$\begin{aligned}&\left[u^2(1-u)+zu^2-zu^3(1-u)+z(1-u)\right]F(z,u)\\ &\qquad = u^2(1-u)+zu^2F(z,1)+z(1-u)\left[F_0(z)+uF_1(z)\right]-zu^4(1-u)G_2(z),\end{aligned} \tag{16}$$

with $G_2(z) = F(z,1) - F_0(z) - F_1(z) - F_2(z) - F_4(z)$. Only three roots, u_0, u_1, u_2 can be substituted for u in the kernel, while the right-hand side of the equation contains four unknown functions, $F(z,1), F_0(z), F_1(z)$ and $G_2(z)$. Writing (14) for $j = 0, 1$ and 2 yields

$$\begin{aligned}F_0(z) &= 1 + z\left[F(z,1) - F_0(z) - F_2(z)\right],\\ F_1(z) &= zF_0(z) + z\left[F(z,1) - F_0(z) - F_1(z) - F_3(z)\right],\\ F_2(z) &= zF_1(z).\end{aligned}$$

The second equation is not of much use but, by combining the first and third one, we find

$$F_0(z) = \frac{1 + z\,[F(z,1) - zF_1(z)]}{1+z}.$$

Replacing $F_0(z)$ by this expression in (16) gives:

$$\Big[u^2(1-u) + zu^2 - zu^3(1-u) + z(1-u)\Big]\, F(z,u) = u^2(1-u) + \frac{z(1-u)}{1+z}$$

$$+zF(z,1)\left[u^2 + \frac{z(1-u)}{1+z}\right] + z(1-u)F_1(z)\left[u - \frac{z^2}{1+z}\right] - zu^4(1-u)G_2(z). \qquad (17)$$

We are left with three unknown functions, related by three linear equations obtained by cancelling the kernel. Solving these equations would give $F(z,1)$ as an enormous rational function of z, u_0, u_1 and u_2, symmetric in the u_i. This implies that $F(z,1)$ can also be written as a rational function of z and $v \equiv v_1$, the fourth and last root of the kernel. In particular, $F(z,1)$ is algebraic of degree at most 4.

In order to obtain directly an expression of $F(z,1)$ in terms of z and v, we can proceed as follows. Let $R'(z,u)$ denote the right-hand side of Eq. (17). Then $R'(z,u)$ is a polynomial in u of degree 5, and three of its roots are u_0, u_1, u_2. Consequently, as a polynomial in u, the kernel $K(z,u)$ divides $(u-v)R'(z,u)$.

Let us evaluate $(u-v)R'(z,u)$ modulo $K(z,u)$: we obtain a polynomial of degree 3 in u, whose coefficients depend on $z, v, F(z,1), F_1(z)$ and $G_2(z)$. This polynomial has to be zero: this gives a system of four (dependent) equations relating the three unknown functions $F(z,1), F_1(z)$ and $G_2(z)$. Solving the first three of these equations yields

$$\begin{aligned} F(z,1) &= \frac{1+z+z^2-(z+1)zv+(z+1)zv^2-z^2v^3}{1-z^2-z(1-z^2)v+z^3v^3} \\ &= 1+z+2z^2+3z^3+6z^4+11z^5+23z^6+47z^7+101x^8+O(z^9). \end{aligned}$$

Eliminating v between this expression and $K(z,v)=0$ gives a quartic equation satisfied by $F(z,1)$. $\square$

4 Transcendental systems

4.1 Transcendence

The radius of convergence of an algebraic series is always positive. Hence, one possible reason for a system to give a transcendental series is the fact that its coefficients grow too fast, so that its radius of convergence is zero. This is the case for System (2.h), as proved by the following proposition.

Proposition 6 *Let b be a nonnegative integer. For $k \geq 1$, let $m(k) = |\{i : e_i(k) \geq k-b\}|$. Assume that:*

1. *for all k, there exists a forward jump from k (i.e., $e_i(k) > k$ for some i),*
2. *the sequence $(m(k))_k$ is nondecreasing and tends to infinity.*

Then the (ordinary) generating function of the system has radius of convergence 0.

Proof. Let s_0 be the axiom of the system. Let us denote by h_n the product $m(s_0+b)m(s_0+2b)\cdots m(s_0+nb)$. Let us prove that the generating tree contains at least h_n nodes at level $n(b+1)$. At level nb, take a node v labeled k, with $k \geq s_0+nb$. Such a node exists thanks to the first assumption. By definition of $m(k)$, this node v has $m(k)$ sons whose label is at least $k-b$. As m is non decreasing, v has at least $m(s_0+nb)$ sons of label at least $s_0+(n-1)b$. Iterating this procedure shows that, at level $nb+i$, at least $m(s_0+(n-i+1)b)\cdots m(s_0+nb)$ descendents of v have a label larger than or equal to $s_0+(n-i)b$, for $0<i\leq n$. In particular, for $i=n$, we obtain at level $n(b+1)$ at least h_n descendents of v whose label is at least s_0.

Hence $f_{n(b+1)} \geq h_n$. But as $h_n/h_{n-1} = m(s_0+nb)$ goes to infinity with n, the series $\sum_n h_n z^{n(b+1)}$ has radius of convergence 0, and the same is true for $F(z)=\sum_n f_n z^n$. ■

In particular, this proposition implies that *the generating function of any ECO-system in which the length of backward jumps is bounded has radius of convergence* 0. Many examples of this type will be given in the next subsection, in which we shall study whether the corresponding generating function is holonomic or not. The following example, in which backward jumps are not bounded, was suggested by Nantel Bergeron.

Example 13. *A fake factorial walk*
Consider the system with axiom (1) and rewriting rules $\{(k) \rightsquigarrow (2)(4)\cdots(2k)\}$. Proposition 6 applies with $b=0$ and $m(k)=1+\lfloor k/2 \rfloor$. Note that the radius of convergence of $F(z)$ is zero although *all* the functions e_i are bounded, and indeed constant: $e_i(k)=2i$ for all $k\geq i$. The series $F(z)$ is of course transcendental. Note, however, that $F(z,u)$ satisfies a functional equation that is at first sight reminiscent of the equations studied in Section 3:

$$F(z,u)=u+zu^2\,\frac{F(z,1)-F(z,u^2)}{1-u^2}.$$

□

The following example shows that Proposition 6 is not far from optimal: an ECO-system in which all functions e_i grow linearly can have a finite radius of convergence.

Example 14.
The system with axiom (1) and rules $(k) \rightsquigarrow (\lceil k/2 \rceil)^{k-1}(k+1)$ leads to a generating function with a positive radius of convergence.
Let us start from the recursion defining the numbers $f_{n,k}$. We have $f_{0,1}=1$ and for $n\geq 1$,

$$f_{n+1,k}=f_{n,k-1}+(2k-1)f_{n,2k}+(2k-2)f_{n,2k-1}.$$

The largest label occurring at level n in the tree is $n+1$. Let us introduce the numbers $g_{n,k}=f_{n,n-k+1}$, for $k\leq n$. The above recursion can be rewritten as:

$$g_{n+1,k}=g_{n,k}+(2n-2k+3)g_{n,2k-n-3}+(2n-2k+2)g_{n,2k-n-2}. \tag{18}$$

We have $g_{n,k}=0$ for $k<0$. Hence Eq. (18) implies that for $k\geq 0$, the sequence $(g_{n,k})_n$ is nondecreasing and reaches a constant value $g(k)$ as soon as $n\geq 2k-1$ (see Table 1).

Going back to the number f_n of nodes at level n, we have

$$f_n=\sum_{k=0}^{n} g_{n,k} \leq \sum_{k=0}^{n} g(k).$$

n k	1	2	3	4	5	6
0	**1**					
1	**0**	**1**				
2	1	**0**	**1**			
3	0	**3**	**0**	**1**		
4	3	3	**3**	**0**	**1**	
5	3	9	**7**	**3**	**0**	**1**

n k	0	1	2	3	4	5
0	**1**					
1	**1**	**0**				
2	**1**	**0**	1			
3	**1**	**0**	**3**	0		
4	**1**	**0**	**3**	3	3	
5	**1**	**0**	**3**	**7**	9	3

Table 1: The numbers $f_{n,k}$ and $g_{n,k}$. Observe the convergence of the coefficients.

But

$$\sum_{n\geq 0} z^n \sum_{k=0}^{n} g(k) = \frac{1}{1-z}\sum_{k=0}^{n} g(k)z^k,$$

and hence it suffices to prove that the generating function for the numbers $g(k)$ has a finite radius of convergence, that is, that these numbers grow at most exponentially.

Writing (18) for $n+1=2k-i$, for $1\leq i\leq k$, we obtain:

$$g_{2k-i,k} = g_{2k-i-1,k} + (2k-2i+1)g_{2k-i-1,i-2} + (2k-2i)g_{2k-i-1,i-1}.$$

Iterating this formula for i between 1 and k yields

$$\begin{aligned} g(k) &= g_{2k-1,k} = \sum_{i=1}^{k}\left[(2k-2i+1)g_{2k-i-1,i-2} + (2k-2i)g_{2k-i-1,i-1}\right] \\ &\leq \sum_{i=1}^{k}\left[(2k-2i+1)g(i-2) + (2k-2i)g(i-1)\right] = \sum_{i=0}^{k-2}(4k-4i-5)g(i). \end{aligned}$$

This inequality, together with the fact that $g(0)=1$, implies that for all $k\geq 0$, $g(k)\leq \widetilde{g}(k)$, where the sequence $\widetilde{g}(k)$ is defined by $\widetilde{g}(0)=1$ and $\widetilde{g}(k)=\sum_{i=0}^{k-2}(4k-4i-5)\widetilde{g}(i)$ for $k>0$. But the series $\sum_k \widetilde{g}(k)z^k$ is rational, equal to $(1-z)^2/(1-2z-2z^2-z^3)$, and has a finite radius of convergence. Consequently, the numbers $\widetilde{g}(k)$ and $g(k)$ grow at most exponentially. $\square$

Algebraic generating functions are strongly constrained in their algebraic structure (by a polynomial equation) as well as in their analytic structure (in terms of singularities and asymptotic behaviour). In particular, they have a finite number of singularities, which are algebraic numbers, and they admit local asymptotic expansions that involve only rational exponents. *A contrario*, a generating function that has infinitely many singularities (*e.g.*, a natural boundary) or that involves a transcendental element (*e.g.*, a logarithm) in a local asymptotic expansion is by necessity transcendental; see [16] for a discussion of such transcendence criteria. In the case of generating trees, this means that the presence of a condition involving a transcendental element is expected to lead to a transcendental generating function. This is the case in the following example.

EXAMPLE 15. *A Fredholm system*
We examine System (2.e), in which the rules are irregular at powers of 2:

$$(s_0)=(2), \qquad (k) \rightsquigarrow (2)^{k-2}(3-[\exists p\colon k=2^p])(k+1), \quad k\geq 2.$$

This example will involve the Fredholm series $h(z) := \sum_{p\geq 1} z^{2^p}$, which is well-known to admit the unit circle as a natural boundary. (This can be seen by way of the functional equation $h(z) = z^2 + h(z^2)$, from which there results that $h(z)$ is infinite at all iterated square-roots of unity.) According to Eq. (2), we have, for $k > 3$, $F_k(z) = zF_{k-1}(z)$, so that

$$F_k(z) = z^{k-3}F_3(z) \quad \text{for } k \geq 3.$$

Now, writing Eq. (2) for $k = 2$ gives

$$\begin{aligned} F_2(z) &= 1 + z\sum_{k\geq 3}(k-2)F_k(z) + z\sum_{p\geq 1} F_{2^p}(z) \\ &= 1 + \frac{z}{(1-z)^2}F_3(z) + zF_2(z) + F_3(z)\left(\frac{h(z)}{z^2} - 1\right) \\ &= 1 + zF_2(z) + F_3(z)\left(\frac{z}{(1-z)^2} + \frac{h(z)}{z^2} - 1\right). \end{aligned}$$

For $k = 3$, we obtain:

$$\begin{aligned} F_3(z) &= zF_2(z) + z\sum_{k\geq 3,\ k\neq 2^p} F_k(z) \\ &= zF_2(z) + F_3(z)\left(\frac{1}{1-z} - \frac{h(z)}{z^2}\right). \end{aligned}$$

Solving for $F_2(z)$ and $F_3(z)$, then summing ($F(z) = F_2(z) + F_3(z)/(1-z)$), we obtain:

$$F(z) = \frac{(1-z)^2h(z)}{(1-2z)(1-z)^2h(z) - z^4} = 1 + 2z + 5z^2 + 14z^3 + 39z^4 + 108z^5 + O(x^6).$$

The functions $h(z)$ and $F(z)$ are rationally related, so that $F(z)$ is itself transcendental. The series h has radius 1, but the denominator of F vanishes before z reaches 1 – actually, before z reaches $1/2$. Hence the radius of F is the smallest root of its denominator. Its value is easily determined numerically and found to be about 0.360102. □

4.2 Holonomy

In the transcendental case, one can also discuss the *holonomic* character of the generating function $F(z)$.

A series is said to be *holonomic*, or *D-finite* [25], if it satisfies a linear differential equation with polynomial coefficients in z. Equivalently, its coefficients f_n satisfy a linear recurrence relation with polynomial coefficients in n. Consequently, given a sequence f_n, the ordinary generating function $\sum_n f_n z^n$ is holonomic if and only if the exponential generating function $\sum_n f_n z^n/n!$ is holonomic. The set of holonomic series has nice closure properties: the sum or product of two of them is still holonomic, and the substitution of an algebraic series into an holonomic one gives an holonomic series. Holonomic series include algebraic series, and have a finite number of singularities. This implies that Example 15, for which $F(z)$ has a natural boundary, is not holonomic.

We study below five ECO-systems that, at first sight, do not look to be very different. In particular, for each of them, forward and backward jumps are bounded. Consequently, Proposition 6 implies that the corresponding ordinary generating function has radius of convergence zero. However, we shall see that the first three systems have an holonomic generating function, while the last two have not. We have no general criterion that would allow us to distinguish between systems leading to holonomic generating functions and those leading to nonholonomic generating functions.

Among the systems with bounded jumps, those for which $e_i(k) - k$ belongs to $\{-1, 0, 1\}$ for all $i \leq k$ have a nice property: the generating function for the corresponding *excursions* (walks starting and ending at level 0) can be written as the following continued fraction [15]:

$$\cfrac{1}{1 - b_0 z - \cfrac{a_1 c_0 z^2}{1 - b_1 z - \cfrac{a_2 c_1 z^2}{1 - b_2 z - \cfrac{a_3 c_2 z^2}{\cdots}}}},$$

where the coefficients a_k, b_k and c_k are the multiplicities appearing in the rules, which read $(k) \rightsquigarrow (k-1)^{a_k} (k)^{b_k} (k+1)^{c_k}$.

Example 16. *Arrangements*
The system $(k) \rightsquigarrow (k)(k+1)^{k-1}$ with axiom $(s_0) = (2)$ generates a sequence that starts with $1, 2, 5, 16, 65, 326$ (**M1497**). It is not hard to see that the triangular array $f_{n,k+2}$ is given by the arrangement numbers $k!\binom{n}{k}$, so that the *exponential* generating function (EGF) of the sequence is

$$\widetilde{F}(z, u) = \sum_{n \geq 0, k \geq 2} f_{n,k} u^k \frac{z^n}{n!} = \frac{u^2 e^z}{1 - uz}.$$

This system satisfies the conditions of Proposition 6 with $b = 0$ and $m(k) = k$. Accordingly, one has $f_n \sim e\, n!$, so that the *ordinary* generating function $F(z)$ has radius of convergence 0 and cannot be algebraic. However, $\widetilde{F}(z, 1) = e^z/(1 - z)$ is holonomic, and so is $F(z)$. □

Example 17. *Involutions and Hermite polynomials*
The system $(k) \rightsquigarrow (k-1)^{k-1}(k+1)$ with axiom $(s_0) = (1)$ generates a sequence that starts with $1, 1, 2, 4, 10, 26, 76$ (**M1221**). These numbers count involutions: more precisely, one easily derives from the recursion satisfied by the coefficients $f_{n,k}$ that $f_{n,k}$ is the number of involutions on n points, $k - 1$ of which are fixed. Proposition 6 applies with $b = 1$ and $m(k) = k$.

The corresponding EGF is

$$\widetilde{F}(z, u) = \sum_{n \geq 0, k \geq 1} f_{n,k} u^k \frac{z^n}{n!} = u \exp\left(zu + \frac{z^2}{2}\right), \tag{19}$$

and its value at $u = 1$ is holonomic.

The polynomials $f_n(u) = \sum_k f_{n,k} u^k$ counting involutions on n points are in fact closely related to the Hermite polynomials, defined by:

$$\sum_{n \geq 0} H_n(x) \frac{t^n}{n!} = \exp\left(xt - \frac{t^2}{2}\right).$$

Indeed, comparing the above identity with (19) shows that $f_n(u) = u\ i^n H_n(-iu)$. □

EXAMPLE 18. *Partial permutations and Laguerre polynomials*
The rewriting rule $(k) \rightsquigarrow (k+1)^{k-1}(k+2)$, taken with axiom (2), generates a sequence that starts with $1, 2, 7, 34, 209, \ldots$ (**M1795**). From the recursion satisfied by the coefficients $f_{n,k}$, we derive that $f_{n,n+k}$ is the number of partial injections of $\{1, 2, \ldots, n\}$ into itself in which $k-2$ points are unmatched. From this, we obtain:

$$\widetilde{F}(z,u) = \frac{u^2}{1-uz} \exp\left(\frac{u^2 z}{1-uz}\right) = u^2 \sum_{n \geq 0} L_n(-u) \frac{(uz)^n}{n!}$$

where $L_n(u)$ is the nth Laguerre polynomial. Again, $\widetilde{F}(z,1)$ is holonomic. □

The next two systems, as announced, lead to nonholonomic generating functions.

EXAMPLE 19. *Set partitions and Stirling polynomials*
Let us consider the system $[(1), (k) \rightsquigarrow (k)^{k-1}(k+1)]$. From the recursion satisfied by the coefficients $f_{n,k}$, we derive that $f_{n,k+1}$ is equal to the Stirling number of the second kind $\left\{ n \atop k \right\}$, which counts partitions of n objects into k nonempty subsets. The corresponding EGF is

$$\widetilde{F}(z,u) = u \exp\left(u(\exp z - 1)\right).$$

At $u = 1$, this generating function specializes to

$$\widetilde{F}(z,1) = \exp(\exp(z) - 1)) = \sum_{n \geq 0} B_n \frac{z^n}{n!} = 1 + z + 2\frac{z^2}{2!} + 5\frac{z^3}{3!} + 15\frac{z^4}{4!} + 52\frac{z^5}{5!} + 203\frac{z^6}{6!} + \ldots$$

This is the exponential generating function of the Bell numbers (**M1484**). It is known that $\log B_n = n \log n - n \log\log n + O(n)$ (see [20]), and this cannot be the asymptotic behaviour of the logarithm of the coefficients of an holonomic series (see [29] for admissible types). Hence, $\widetilde{F}(z,1)$, as well as $F(z,1)$, is nonholonomic. □

EXAMPLE 20. *Bessel numbers*
We study the system with axiom (2) and rewriting rules

$$(2) \rightsquigarrow (2)(3), \quad (k) \rightsquigarrow (k-1)(k)^{k-2}(k+1), \quad k \geq 3. \tag{20}$$

We shift the labels by 2 to obtain a walk model with axiom (0) and rules

$$(0) \rightsquigarrow (0)(1), \quad (k) \rightsquigarrow (k-1)(k)^k(k+1), \quad k \geq 1.$$

The corresponding bivariate generating function $F(z,u)$ satisfies the functional differential equation

$$F(z,u)\left(1 - z(u + u^{-1})\right) = 1 + z(1 - u^{-1})F(z,0) + zu\frac{\partial F}{\partial u}(z,u),$$

which is certainly not obvious to solve. However, as observed in [15], it is easy to obtain a continued fraction expansion of the excursion generating function:

$$F(z,0) = 1+z+2z^2+4z^3+9z^4+\cdots = \cfrac{1}{1 - z - \cfrac{z^2}{1 - z - \cfrac{z^2}{1 - 2z - \cfrac{z^2}{1 - 3z - \ddots}}}} = \frac{1}{1 - z - z^2 B(z)},$$

where $B(z) = \sum_n B_n^* z^n = 1 + z + 2z^2 + 5z^3 + 14z^4 + 43z^5 + 143z^6 + \cdots$ is the generating function of Bessel numbers (**M1462**) and counts non-overlapping partitions [17]. As $F(z,0)$ itself, the series $B(z)$ has radius of convergence zero. The fast increase of B_n^* entails

$$[z^n]F(z,0) \sim B_{n-2}^*.$$

From [17], we know that $\log B_n^* = n \log n - n \log\log n + O(n)$. Again, this prevents $F(z,0)$ from being holonomic.

In order to prove that $F(z,1)$ itself is nonholonomic, we are going to prove that its coefficients f_n have the same asymptotic behaviour as the coefficients of $F(z,0)$. Clearly,

$$[z^n]F(z,0) = f_{n,0} \leq \sum_k f_{n,k} = f_n.$$

To find an upper bound for f_n, we compare the system (20) (denoted Σ_1 below) to the system Σ_2 with axiom (2) and rule $(k) \leadsto (k)^{k-1}(k+1)$. This system generates a tree with counting sequence g_n. The form of the rules implies that the (unlabeled) tree associated with Σ_1 is a subtree of the tree associated with Σ_2. Hence $f_n \leq g_n$. Comparing Σ_2 to the system studied in the previous example shows that g_n is the Bell number B_{n+1}, the logarithm of which is also known to be $n \log n - n \log\log n + O(n)$ (see [20]). Hence $\log f_n = n \log n - n \log\log n + O(n)$, and this prevents the series $F(z,1)$ from being holonomic. $\square$

Axiom	System	Name	Id.	Generating Function
	Rational OGF			**OGF**
(1)	$(k) \rightsquigarrow (k)^{k-1}((k \bmod 2)+1)$	Ex. 3: Fibonacci	M0692	$\frac{1}{1-z-z^2}$
(2)	$(k) \rightsquigarrow (2)^{k-1}(k+1)$	Ex. 4: even Fibonacci	M1439	$\frac{1-z}{1-3z+z^2}$
(3)	$(k) \rightsquigarrow (2)^{k-1}(k+1)$	Ex. 4: odd Fibonacci	M2741	$\frac{1}{1-3z+z^2}$
	Algebraic OGF			**OGF**
(1)	$(k) \rightsquigarrow (1)\cdots(k-1)(k+1)$	Ex. 7: Motzkin numbers	M1184	$\frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$
(2)	$(k) \rightsquigarrow (2)\cdots(k)(k+1)$	Ex. 6: Catalan numbers	M1459	$\frac{1-2z-\sqrt{1-4z}}{2z^2}$
(3)	$(k) \rightsquigarrow (3)\cdots(k)(k+1)^2$	Ex. 8: Schröder numbers	M2898	$\frac{1-3z-\sqrt{1-6z+z^2}}{4z^2}$
(4)	$(k) \rightsquigarrow (4)\cdots(k)(k+1)^3$	—	M3556	$\frac{1-4z-\sqrt{1-8z+4z^2}}{6z^2}$
(m)	$(k) \rightsquigarrow (m)\cdots(k)(k+1)^{m-1}$	—	—	$\frac{1-mz-\sqrt{1-2mz+(m-2)^2z^2}}{2(m-1)z^2}$
(3)	$(k) \rightsquigarrow (3)\cdots(k+2)$	Ex. 9: Ternary trees	M2926	$F=(1+zF)^3$
(4)	$(k) \rightsquigarrow (4)\cdots(k+3)$	Ex. 9: Dissections of a polygon	M3587	$F=(1+zF)^4$
(m)	$(k) \rightsquigarrow (m)\cdots(k+m-1)$	Ex. 9: m-ary trees		$F=(1+zF)^m$
	Holonomic transcendental OGF			**EGF**
(1)	$(k) \rightsquigarrow (k+1)^k$	Permutations	M1675	$1/(1-z)$
(2)	$(k) \rightsquigarrow (k)(k+1)^{k-1}$	Ex. 16: Arrangements	M1497	$e^z/(1-z)$
(1)	$(k) \rightsquigarrow (k-1)^{k-1}(k+1)$	Ex. 17: Involutions	M1221	$e^{z+\frac{1}{2}z^2}$
(2)	$(k) \rightsquigarrow (k+1)^{k-1}(k+2)$	Ex. 18: Partial permutations	M1795	$e^{z/(1-z)}/(1-z)$
(2)	$(k) \rightsquigarrow (k-1)^{k-2}(k)(k+1)$	Switchboard problem	M1461	$e^{2z+\frac{1}{2}z^2}$
(2)	$(k) \rightsquigarrow (k-1)^{k-2}(k+1)^2$	Bicolored involutions	M1648	e^{2z+z^2}
	Nonholonomic OGF			**EGF**
(1)	$(k) \rightsquigarrow (k)^{k-1}(k+1)$	Ex. 19: Bell numbers	M1484	e^{e^z-1}
(2)	$(k) \rightsquigarrow (k)^{k-2}(k+1)^2$	Bicolored partitions	M1662	$e^{2(e^z-1)}$
(2)	$(k) \rightsquigarrow (k-1)(k)^{k-2}(k+1)$	Ex. 20: Bessel numbers	M1462	—

Table 2: Some ECO-systems of combinatorial interest.

A small catalog of ECO-systems

To conclude, we present in Table 2 a small catalog of ECO-systems that lead to sequences of combinatorial interest. Several examples are detailed in the paper; others are due to West [27, 28] or Barcucci, Del Lungo, Pergola, Pinzani [4, 6, 5, 3], or are folklore. Each of them is an instance of application of our criteria.

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