Dyck Paths with Peaks Avoiding or Restricted to a Given Set

By Sen-Peng Eu, Shu-Chung Liu, and Yeong-Nan Yeh

In this paper we focus on Dyck paths with peaks avoiding or restricted to an arbitrary set of heights. The generating functions of such types of Dyck paths can be represented by continued fractions. We also discuss a special case that requires all peak heights to either lie on or avoid a congruence class (or classes) modulo k. The case when $k = 2$ is especially interesting. The two sequences for this case are proved, combinatorially as well as algebraically, to be the Motzkin numbers and the Riordan numbers. We introduce the concept of shift equivalence on sequences, which in turn induces an equivalence relation on avoiding and restricted sets. Several interesting equivalence classes whose representatives are well-known sequences are given as examples.

1. Introduction

An *n-Dyck path* is a lattice path in the first quadrant with end points (0, 0) and $(2n, 0)$, and consists of two kinds of steps—*rise step*: $U = (1, 1)$ and *fall step*: $D = (1, -1)$. Let $\mathcal{D}_n$ denote the set of all n-Dyck paths and $\mathcal{D} = \cup_{n\geq 0}\mathcal{D}_n$, the set of all Dyck paths. It is well known that $|\mathcal{D}_n|$, the cardinality of $\mathcal{D}_n$, equals the nth Catalan number, $c_n = \frac{1}{n+1}\binom{2n}{n}$, and that the generating function

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Published by Blackwell Publishing, 350 Main Street, Malden, MA 02148, USA, and 9600 Garsington Road, Oxford, OX4 2DQ, UK.

$C(z) := \sum_{n\geq 0} c_n z^n$ satisfies the functional equation $C(z) = 1 + zC(z)^2$ and then, of course, $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ explicitly.

A similar structure is an *n-Motzkin path* [5, 6, 12], which is a lattice path in the first quadrant starting at (0, 0) and ending at (n, 0), and consists of one more type of step—*level step*: $L = (1, 0)$. Let $\mathcal{M}_n$ denote the set of all n-Motzkin paths and $\mathcal{M} = \cup_{n\geq 0}\mathcal{M}_n$, the set of all Motzkin paths. The cardinality of $\mathcal{M}_n$, denoted by m_n, is the well-known *nth Motzkin number*. It is also well known that the generating function $M(z) := \sum_{n\geq 0} m_n z^n$ satisfies $M(z) = 1 + zM(z) + zM(z)^2$ and explicitly $M(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$.

A *peak* of a Dyck path is the joint node formed by a rise step followed by a fall step. The *height* of a peak is the y-coordinate of this node. Given a Dyck path P, let H_P denote the set of heights of all peaks of P. In this article, we study the class of Dyck paths P such that every peak of P keeps away from an *avoiding set* $A \subseteq \mathbb{P}$, i.e., $H_P \cap A = \emptyset$ where $\mathbb{P}$ is the set of positive integers. Let $\mathcal{D}_{n,\overline{A}}$ denote the set of such n-Dyck paths and $\mathcal{D}_{\overline{A}} = \cup_{n\geq 0}\mathcal{D}_{n,\overline{A}}$, the set of all Dyck paths with heights avoiding A. Also let $D_{\overline{A}}(z)$ be the generating function of $|\mathcal{D}_{n,\overline{A}}|$. The concept of having heights avoiding A is dual to the concept of having heights confined to the *restricted set* $R := \mathbb{P} - A$, i.e., $H_p \subseteq R$. We shall also discuss the case of restriction, because sometimes R is very neat to use. We define, in a similar manner, $\mathcal{D}_{n,R}$, $\mathcal{D}_R$, and $D_R(z)$.

Some special cases have been studied before. For instance, Deutsch showed that $|\mathcal{D}_{n,\overline{\{1\}}}|$ is exactly the nth Fine number [3]. The general case $\mathcal{D}_{n,\overline{\{h\}}}$ for any fixed $h \in \mathbb{P}$ was considered by Peart and Woan [11]. They proved that $|\mathcal{D}_{n,\overline{\{2\}}}|$ equals the $(n-1)$st Catalan number. Mansour also studied the Dyck paths with fixed number of peaks with height h [10]. The case that $R = \{1, 2, \ldots, h\}$ was well studied and the generating function $D_R(z)$ has recently been related to pattern-avoiding permutations and Chebyshev polynomials [2, 8, 9].

The organization of the paper is as follows. In Section 2, we discuss the case that A is the set of odd/even positive integers, and show that the Dyck paths without peaks of even heights are counted by shifted Motzkin numbers; while Dyck paths without peaks of odd heights are counted by Riordan numbers. Two proofs, one bijective and the other algebraic, of the mentioned numerical results are given.

In Section 3, we consider the general case that involves either an arbitrary avoiding set A or an arbitrary restricted set R, and derive a reduction formula for the generating function. Furthermore, when A (or R) is finite, the reduction formula yields an explicit formula in the form of a continued fraction. We also discuss the case when A and R are congruence classes modulo k for some integer $k > 1$.

In Section 4, we introduce an equivalence relation among avoiding sets, as well as among restricted sets, according to the shift equivalence between sequences arising from these sets. Several interesting equivalence classes, including generalized Catalan numbers, Fine numbers, k-Fibonacci numbers, and bisection of Fibonacci numbers, are given as examples.

In the final section, we conclude with several directions of further research pertaining to the present work.

In what follows, $\mathbb{Z}$, $\mathbb{P}$, and $\mathbb{N}$ denote, as customary, the sets of integers, positive integers, and non-negative integers, respectively. Also let $|S|$ be the cardinality of a set S and $\|P\|$ the semilength (resp. length) of a Dyck (resp. Motzkin) path P.

2. Dyck paths without peaks of odd or even heights

Let $\mathbb{O}$ and $\mathbb{E}$ be the sets of odd and even positive integers, respectively. In this section we discuss the classes $\mathcal{D}_{\overline{\mathbb{O}}}$ and $\mathcal{D}_{\overline{\mathbb{E}}}$. We first note that $\mathcal{D}_{\overline{\mathbb{O}}} = \mathcal{D}_{\mathbb{E}}$ and $\mathcal{D}_{\overline{\mathbb{E}}} = \mathcal{D}_{\mathbb{O}}$. For convenience, we set $o_n = |\mathcal{D}_{n,\overline{\mathbb{O}}}|$ and $e_n = |\mathcal{D}_{n,\overline{\mathbb{E}}}|$. It is obvious that $o_0 = e_0 = 1$. As for the other o_n and e_n, we found their connections with two famous sequences—Motzkin and Riordan numbers.

THEOREM 1. *For $n \geq 1$, the number e_n of n-Dyck paths without peaks of even heights is equal to the "shifted" Motzkin number m_{n-1}. The number o_n of n-Dyck paths without peaks of odd heights is equal to the number of n-Motzkin paths without level steps on the x-axis.*

Bijective proof: To prove the first part of this theorem, it suffices to devise a bijection between $\mathcal{M}_{n-1}$ and $\mathcal{D}_{n,\overline{\mathbb{E}}}$. Let us define a map $\phi : \{U, L, D\} \to \{UU, DU, DD\}$ by $\phi(U) = UU$, $\phi(L) = DU$, and $\phi(D) = DD$. Given any $(n-1)$-Motzkin path $M = S_1 S_2 \dots S_{n-1}$, we define $\phi(M) = \phi(S_1)\phi(S_2)\dots \phi(S_{n-1})$. The corresponding n-Dyck path of M is then $P = U\phi(M)D$. For example, $M = LULUDD$ corresponds to $P = UDUUUDUUUDDDDD$. It is clear that P contains no peaks of even heights, and that such map $\mathcal{M}_{n-1} \to \mathcal{D}_{n,\overline{\mathbb{E}}}$ is bijective.

Now we prove the second part of the theorem. It is not hard to see that $\phi^{-1}(\mathcal{D}_{n,\overline{\mathbb{O}}})$, the inverse image of n-Dyck paths without peaks of odd heights, consists of n-Motzkin paths without level steps on the x-axis, and that ϕ is a bijection between these two sets. Thus the second statement follows. ■

Surprisingly, we found that o_n also equals the nth Riordan number r_n. So we unveil the connections of Riordan numbers with Dyck paths and with Motzkin paths. To verify this combinatorial equivalence, we need the following combinatorial bijection.

LEMMA 1. *The number of n-Motzkin paths with at least a level step on the x-axis is equal to the number of $(n+1)$-Motzkin paths without level steps on the x-axis.*

Proof: Let M be an n-path of the former type in the statement. Decompose M into M_1LM_2, where L is the first level step on the x-axis. The corresponding $(n+1)$-Motzkin path is defined as $M' = M_1UM_2D$, which obviously has no level steps on the x-axis. The inverse of the map $M \to M'$ is easily constructed by replacing the last U of M' rising from the x-axis with a level step L and then removing the last step of M', which is a fall step D of course. Clearly, such map is a bijection between these two types of Motzkin paths and the lemma follows. ■

In the next theorem, we innovate three representations for the Riordan number.

THEOREM 2. *The nth Riordan number r_n counts the following classes of paths*:

(i) *n-Motzkin paths without level steps on the x-axis*;
(ii) *$(n+1)$-Motzkin paths with level steps on the x-axis*;
(iii) *n-Dyck paths without peaks of odd heights*, *i.e.*, $r_n = o_n$.

Proof: By Lemma 1, (i) and (ii) are equivalent. By Theorem 1, (i) and (iii) are equivalent. Now we need only prove that r_n counts the class of paths in (i). The statement is obviously true when $n = 0$. To proceed inductively, we assume that it is true for n. Now let us refer to the identity

$$m_n = r_n + r_{n+1}, \tag{1}$$

given by Berhart [1]. By this identity and the induction hypothesis, it follows that r_{n+1} counts those n-Motzkin paths with at least a level step on the x-axis. Therefore, by Lemma 1, r_{n+1} counts those $(n+1)$-Motzkin paths without level steps on the x-axis. This completes the induction. ■

Those combinatorial structures enumerated by the Riordan numbers studied previously involve plane trees and partitions [1]. Theorem 2 enriches the list of objects counted by the Riordan numbers. It would be interesting to find bijections from either one of the three classes of paths to plane trees and to partition, etc. It is also worth mentioning that Theorems 1, 2, and Equation (1) together imply that

$$|\mathcal{D}_{n,\overline{\mathbb{E}}}| = |\mathcal{D}_{n-1,\overline{\mathbb{O}}}| + |\mathcal{D}_{n,\overline{\mathbb{O}}}|.$$

To prove Theorems 1 and 2 algebraically, we consider the generating functions $M(z)$ and $R(z)$ of m_n and r_n, respectively. It is known that $R(z) = \frac{1}{1+z} + zR(z)^2$ and explicitly

$$R(z) = \frac{1 + z - \sqrt{1 - 2z - 3z^2}}{2z(1 + z)}.$$

Algebraic proof of Theorems 1 and 2: Let $O(z)$ and $E(z)$ be the generating functions of o_n and e_n, respectively. For any $P \in \mathcal{D}_{\overline{\mathbb{O}}}$, either $P = \emptyset$ or $P = UP_1DP_2$, where $P_1 \in \mathcal{D}_{\overline{\mathbb{E}}}$ with the semilength $\|P_1\| > 0$ and $P_2 \in \mathcal{D}_{\overline{\mathbb{O}}}$. Thus, the generating functions $O(z)$ and $E(z)$ satisfy

$$O(z) = 1 + z(E(z) - 1)O(z).$$

Similarly, for any $P \in \mathcal{D}_{\overline{\mathbb{E}}}$, either $P = \emptyset$ or $P = UP_1DP_2$, where $P_1 \in \mathcal{D}_{\overline{\mathbb{O}}}$ and $P_2 \in \mathcal{D}_{\overline{\mathbb{E}}}$. Thus we have

$$E(z) = 1 + zO(z)E(z).$$

Solving the system of these two identities, we obtain $E(z) = \frac{1+z-\sqrt{1-2z-3z^2}}{2z} = 1 + zM(z)$ and $O(z) = \frac{1+z-\sqrt{1-2z-3z^2}}{2z(1+z)} = R(z)$; thus, the proof follows. ■

3. Dyck paths with heights of peaks avoiding/restricted to any set

In this section we discuss the generating functions $D_{\overline{A}}(z)$ and $D_R(z)$ of $|\mathcal{D}_{n,\overline{A}}|$ and $|\mathcal{D}_{n,R}|$, respectively, where A and R are arbitrary sets of positive integers. Clearly, $D_{\overline{\emptyset}}(z) = C(z)$ and $D_{\emptyset}(z) = 1$, where $C(z)$ is the Catalan generating function. Given $S \subseteq \mathbb{P}$ and $i \in \mathbb{P}$, we define $S + i := \{n + i \mid n \in S\}$ and $S - i := \mathbb{P} \cap \{n - i \mid n \in S\}$. Notice that the notation $S - \{i\}$ is the set obtained by removing i from S.

Given any Dyck path $P \in \mathcal{D}_{\overline{A}}$ with semi length $\|P\| > 1$, it can be decomposed into UP_1DP_2, where D is the first fall step back to the x-axis. Clearly, P_1 belongs to $\mathcal{D}_{\overline{A-1}}$ and P_2 is in $\mathcal{D}_{\overline{A}}$. Notice that when $1 \in A$ the semilength $\|P_1\|$ must be greater than zero; while $\|P_2\|$ is unrestricted in any condition. So one obtains $D_{\overline{A}}(z) = 1 + z[D_{\overline{A-1}}(z) - \beta(1 \in A)]D_{\overline{A}}(z)$, where β is the Boolean function such that $\beta(\text{true}) = 1$ and $\beta(\text{false}) = 0$, and derives a reduction formula

$$D_{\overline{A}}(z) = \frac{1}{1 + z\beta(1 \in A) - zD_{\overline{A-1}}(z)}. \tag{2}$$

In case A is a finite set with $m = \max A$, we derive an explicit formula in the form of a continued fraction with $(m + 1)$ layers:

$$D_{\overline{A}}(z) = \cfrac{1}{1+z\beta(1\in A) - \cfrac{z}{1+z\beta(2\in A) - \cfrac{z}{\ddots \cfrac{}{1+z\beta(m-1\in A) - \cfrac{z}{1+z-zC(z)}}}}},$$

where $C(z)$ is the Catalan generating function. For the combinatorial aspects of continued fractions, see [7].

Example 1. Let $A = \{2, 3\}$. We have

$$D_{\overline{\{2,3\}}}(z) = \cfrac{1}{1 - \cfrac{z}{1+z - \cfrac{z}{1+z-zC(z)}}}.$$

Because $C(z) = 1 + zC(z)^2$ or $1 - zC(z) = \frac{1}{C(z)}$, this continued fraction is simplified as $D_{\overline{\{2,3\}}}(z) = 1 + z + z^2C(z)$. Thus, for $n \geq 2$, the number of n-Dyck paths with heights of peaks avoiding $\{2, 3\}$ equals the shifted Catalan number c_{n-2}.

Example 1 is not coincidental. The fact that $|\mathcal{D}_{n,\overline{\{2\}}}| = c_{n-1}$ was shown by Peart and Woan [11]. Here we provide a generalization as follows.

THEOREM 3. *Let m and n be integers with $n \geq m \geq 2$. The number of n-Dyck paths with heights of peaks avoiding $\{2, 3, \ldots, m\}$ equals the shifted Catalan number*, c_{n-m+1}.

Proof: With $1 - zC(z) = \frac{1}{C(z)}$, it is easy to prove by induction that the m-layer continued fraction

$$\cfrac{z}{1+z - \cfrac{z}{1+z - \cfrac{z}{\ddots \cfrac{}{1+z - \cfrac{z}{1+z-zC(z)}}}}}$$

is equal to $\frac{z+z^2+\cdots+z^{m-2}+z^{m-1}C(z)}{1+z+z^2+\cdots+z^{m-2}+z^{m-1}C(z)}$.

Therefore

$$D_{\overline{\{2,3,\ldots,m\}}}(z) = \cfrac{1}{1 - \cfrac{z + z^2 + \cdots + z^{m-2} + z^{m-1}C(z)}{1 + z + z^2 + \cdots + z^{m-2} + z^{m-1}C(z)}}$$
$$= 1 + z + z^2 + \cdots + z^{m-2} + z^{m-1}C(z),$$

and the proof follows. ■

In the next section, we will give further generalizations of this theorem for both avoidance and restriction. Bijective proofs of the generalizations will also be provided.

In an analogous manner, we derive the following reduction formula for any restricted set R.

$$D_R(z) = \frac{1}{1 + z\beta(1 \notin A) - zD_{R-1}(z)}.$$

When R is finite with $m = \max R$, we obtain the following explicit formula:

$$D_R(z) = \cfrac{1}{1 + z\beta(1 \notin R) - \cfrac{z}{1 + z\beta(2 \notin R) - \cfrac{z}{\ddots \atop 1 + z\beta(m-1 \notin R) - \cfrac{z}{1-z}}}}$$

by using the fact $D_{\emptyset}(z) = 1$.

Example 2. Let us consider the restricted set R to be either $\{1, 2\}$ or $\{1, 3\}$. Through an easy calculation, we derive that

$$D_{\{1,2\}}(z) = \cfrac{1}{1 - \cfrac{z}{1-z}} = \frac{-1+z}{-1+2z}$$
$$= 1 + z + 2z^2 + 4z^3 + 8z^4 + 16z^5 + \cdots,$$
$$D_{\{1,3\}}(z) = \cfrac{1}{1 - \cfrac{z}{1 + z - \cfrac{z}{1-z}}} = \frac{-1+z+z^2}{-1+2z}$$
$$= 1 + z + z^2 + 2z^3 + 4z^4 + 8z^5 + \cdots.$$

So $|\mathcal{D}_{n+1,\{1,2\}}| = |\mathcal{D}_{n+2,\{1,3\}}| = 2^n$ for any $n \in \mathbb{N}$. The cardinality of $\mathcal{D}_{n+1,\{1,2\}}$ is easy to explain: because each path of $\mathcal{D}_{n+1,\{1,2\}}$ must be in the form of $UP_1P_2\ldots P_nD$, where $P_i \in \{UD, DU\}$. But the fact $|\mathcal{D}_{n+2,\{1,3\}}| = 2^n$ is not obvious. The combinatorial relationship between restricted sets $\{1, 2\}$ and $\{1, 3\}$ will be investigated in the next section.

Given any integer $k \geq 2$ and $I \subset [1, k]$, where $[1, k] := \{1, 2, \ldots, k\}$ we now consider A the congruent class $(k, I) := \{n \in \mathbb{P} \mid n \equiv i (\mathrm{mod}\, k)$ for some $i \in I\}$. Notice that having heights restricted to (k, I) is the same as having height avoiding $(k, [1, k] - I)$. The cases considered in Section 2 are special cases of this general setting. We conclude the result of such general case as follows.

THEOREM 4. *The generating function $D_{\overline{(k,I)}}(z)$ satisfies the functional equation*

$$D_{\overline{(k,I)}}(z) = \cfrac{1}{1 + z\beta(1 \in I) - \cfrac{z}{1 + z\beta(2 \in I) - \cfrac{z}{\ddots \over 1 + z\beta(k \in I) - D_{\overline{(k,I)}}(z)}}}, \tag{3}$$

and $D_{\overline{(k,I)}}(z) = (a(z) + \sqrt{b(z)})/c(z)$ *for some polynomials* $a(z), b(z), c(z) \in \mathbb{Z}[z]$.

Proof: The equation of the above continued fraction directly follows the same discussion about $D_{\overline{A}}$ at the very beginning of this section. By this equation, $D_{\overline{(k,I)}}(z)$ satisfies a quadratic polynomial with coefficients in $\mathbb{Z}[z]$; therefore, $D_{\overline{(k,I)}}(z) = (a(z) + \sqrt{b(z)})/c(z)$ for some $a(z), b(z)$, $c(z) \in \mathbb{Z}[z]$. ■

Example 3. Let us consider $k = 3$. We find

$$D_{\overline{(3,\{1\})}}(z) = \cfrac{1}{1 + z - \cfrac{z}{1 - \cfrac{z}{1 - zD_{\overline{(3,\{1\})}}(z)}}},$$

$$D_{\overline{(3,\{2\})}}(z) = \cfrac{1}{1 - \cfrac{z}{1 + z - \cfrac{z}{1 - zD_{\overline{(3,\{2\})}}(z)}}}, \text{ and}$$

$$D_{\overline{(3,\{3\})}}(z) = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{z}{1 + z - zD_{\overline{(3,\{3\})}}(z)}}}.$$

Solving these equations we get the generating functions and the first few terms of them as follows:

$$D_{\overline{(3,\{1\})}}(z) = \frac{1 - z^2 - \sqrt{1 - 4z + 2z^2 + z^4}}{2z}$$
$$= 1 + z^2 + 2z^3 + 5z^4 + 13z^5 + 35z^6 + 97z^7 + 275z^8 + \cdots;$$
$$D_{\overline{(3,\{2\})}}(z) = \frac{1 + z^2 - \sqrt{1 - 4z + 2z^2 + z^4}}{2z}$$
$$= D_{\overline{(3,\{1\})}}(z) + z; \quad \text{and}$$
$$D_{\overline{(3,\{3\})}}(z) = \frac{1 - z^2 - \sqrt{1 - 4z + 2z^2 + z^4}}{2z(1 - z)}$$
$$= D_{\overline{(3,\{1\})}}(z)\frac{1}{1 - z}$$
$$= 1 + z + 2z^2 + 4z^3 + 9z^4 + 22z^5 + 57z^6 + 154z^7 + 429z^8 + \cdots.$$

The coefficients of $D_{\overline{(3,\{2\})}}(z)$, excluding the constant term, form a sequence called the generalized Catalan number, which is the sequence A025242 in [13]. Note that $|\mathcal{D}_{n,\overline{(3,\{1\})}}| = |\mathcal{D}_{n,\overline{(3,\{2\})}}|$ for $n \geq 2$. It would be interesting to have a bijective proof of this equation.

4. Equivalence classes of avoiding or restricted sets

Let us turn to the values $d_{n,\overline{A}} := |\mathcal{D}_{n,\overline{A}}|$ and $d_{n,R} := |\mathcal{D}_{n,R}|$. Clearly, $d_{n,\overline{\varnothing}} = c_n$, $d_{0,\varnothing} = 1$, and $d_{n,\varnothing} = 0$ for $n \geq 1$. Following the discussion in Examples 1 and 2, we demonstrate a general theory in this section.

Theorem 5. *Let $n \in \mathbb{N}$ and $A \subset \mathbb{P}$ with $1 \notin A$. Then*

$$d_{n,\overline{A}} = d_{n+1,\overline{(A+1)\cup\{2\}}}.$$

Conversely, given any $B \subset \mathbb{P}$ with $1 \notin B$ but $2 \in B$, then

$$d_{n+1,\overline{B}} = d_{n,\overline{(B-\{2\})-1}}.$$

Proof: The second statement is directly from the first one. In the following we show a bijection for proving the first statement, whose idea was originated from [11]. Given any $P \in \mathcal{D}_{n,\overline{A}}$, we construct an $(n+1)$-Dyck path P' by first lifting up P (i.e., UPD), followed by replacing every peak UD of height two in UPD with a valley DU. For example, if $P = UDUUDDUDUD$, then we have $P' = UDUUUDDDUDUD$. Clearly, the lifted path UPD avoids peaks of heights in $A + 1$, so does P'. Moreover, P' avoids peaks of height two because all such peaks in UPD turn to be valleys. In fact, all valleys of height zero arise in this way.

The inverse of this bijection is described as follows. Given any $P' \in \mathcal{D}_{n+1,\overline{A+1\cup\{2\}}}$, to obtain P we need only replace every valley DU of height zero in P' with a peak UD, and then remove the first U and the last D. Because some P' might have a valley of height zero, the corresponding P would have a peak of height one. Therefore, it is necessary that $1 \notin A$. ■

Iterating Theorem 5, we get the following general result.

COROLLARY 1. *Let $m, n \in \mathbb{N}$ and $A \subset \mathbb{P}$ with $1 \notin A$. Then*

$$d_{n,\overline{A}} = d_{n+m,\overline{(A+m)\cup[2,m+1]}},$$

where $[2, m+1] := \{2, 3, \ldots, m+1\}$.

By dualizing Corollary 1, we obtain the next result:

COROLLARY 2. *Let $m, n \in \mathbb{N}$ and $R \subseteq \mathbb{P}$ with $1 \in R$. Then*

$$d_{n,R} = d_{n+m,((R-\{1\})+m)\cup\{1\}}.$$

The above two corollaries suggest an equivalence relation among sequences. We say that two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ are *shift equivalent*, denoted by $\langle a_n \rangle \equiv_s \langle b_n \rangle$, if there exist non-negative integers p and q such that $a_{p+n} = b_{q+n}$ for all $n \in \mathbb{N}$. The shift equivalence of sequence induces an equivalence relation on avoiding sets as well as on restricted sets. We say two avoiding sets A and A' are equivalent if the corresponding sequences $\langle d_{n,\overline{A}} \rangle$ and $\langle d_{n,\overline{A'}} \rangle$ are shift equivalent. The equivalence of restricted sets are defined similarly. Because avoidance and restriction are dual concepts, the equivalence relations of avoiding sets and of restricted sets are two sides of one thing.

Example 4. We clearly have

$$\langle d_{n,\overline{\mathbb{P}}} \rangle = \langle d_{n,\emptyset} \rangle = \langle 1, 0, 0, 0, \ldots \rangle \quad \text{and}$$
$$\left\langle d_{n,\overline{[2,\infty)}} \right\rangle = \left\langle d_{n,\{1\}} \right\rangle = \langle 1, 1, 1, \ldots \rangle.$$

These two sequences are unique. The reader can easily check that each of these sequences corresponds to an equivalence class with only one avoiding set (or with only one restricted set.)

Corollary 1 suggests that an avoiding set A with $1, 2 \notin A$ is the "simplest" representative of the equivalence class containing it. If $B \subsetneq [2, \infty)$ with $2 \in B$, then there is a simplest A with both $1, 2 \notin A$ such that $\langle d_{n,\overline{A}} \rangle \equiv_s \langle d_{n,\overline{B}} \rangle$. The case $B = [2, \infty)$ is an exception, as shown in Example 4. As for restriction, any R with both $1, 2 \notin R$ is the simplest representative of its equivalence class by Corollary 2. For instance, we have found several interesting equivalence classes, with the first sequence in each class being the simplest, as follows.

$$\langle d_{n,\overline{\emptyset}}\rangle \equiv_s \left\langle d_{n,\overline{\{2\}}}\right\rangle \equiv_s \left\langle d_{n,\overline{\{2,3\}}}\right\rangle \equiv_s \left\langle d_{n,\overline{\{2,3,4\}}}\right\rangle \equiv_s \cdots \equiv_s$$
$$\langle \text{Catalan numbers}\rangle [11],$$
$$\left\langle d_{n,\overline{\mathbb{O}-\{1\}}}\right\rangle \equiv_s \langle d_{n,\overline{\mathbb{E}}}\rangle \equiv_s \left\langle d_{n,\overline{(\mathbb{E}+1)\cup\{2\}}}\right\rangle \equiv_s \left\langle d_{n,\overline{(\mathbb{E}+2)\cup\{2,3\}}}\right\rangle \equiv_s \cdots \equiv_s$$
$$\langle \text{Motzkin numbers}\rangle,$$
$$\left\langle d_{n,\{1,2\}}\right\rangle \equiv_s \left\langle d_{n,\{1,3\}}\right\rangle \equiv_s \left\langle d_{n,\{1,4\}}\right\rangle \equiv_s \left\langle d_{n,\{1,5\}}\right\rangle \equiv_s \cdots \equiv_s \langle 2^n\rangle,$$
$$\left\langle d_{n,\{1,2,3\}}\right\rangle \equiv_s \left\langle d_{n,\{1,3,4\}}\right\rangle \equiv_s \left\langle d_{n,\{1,4,5\}}\right\rangle \equiv_s \left\langle d_{n,\{1,5,6\}}\right\rangle \equiv_s \cdots \equiv_s$$
$$\langle \text{bisection of Fibonacci}\rangle,$$
$$\left\langle d_{n,\{1,2,3,4\}}\right\rangle \equiv_s \left\langle d_{n,\{1,3,4,5\}}\right\rangle \equiv_s \left\langle d_{n,\{1,4,5,6\}}\right\rangle \equiv_s \left\langle d_{n,\{1,5,6,7\}}\right\rangle \equiv_s \cdots \equiv_s$$
$$\langle (3^n+1)/2\rangle,$$

where the sequence of the bisection of Fibonacci is defined recursively by $a_n = 3a_{n-1} - a_{n-2}$ with the initial values $a_0 = 1$ and $a_1 = 1$ [4].

Besides the above examples, Example 3 also shows that

$$\left\langle d_{n,\overline{(3,\{1\})}}\right\rangle \equiv_s \left\langle d_{n,\overline{(3,\{2\})}}\right\rangle \equiv_s \left\langle d_{n,\overline{((3,\{2\})+j)\cup[2,j+1]}}\right\rangle \equiv_s$$
$$\langle \text{generalized Catalan Numbers}\rangle$$

for $j \in \mathbb{P}$. The first equivalence relation is trivial in view of algebra, but cannot be explained combinatorially by the theory developed in this work. Generally speaking, when $1 \in A$ (i.e., $1 \notin R$), we still know little about their equivalence classes. Among many examples we studied, the case when $1 \in A$ seems to form an equivalence class that contains only one avoiding set, namely A. For instance, $\langle d_{n,\overline{\mathbb{P}}}\rangle$ in Example 4 and the following cases:

$$\left\langle d_{n,\overline{\{1\}}}\right\rangle \equiv_s \langle \text{Fine numbers}\rangle [4],$$
$$\langle d_{n,\overline{\mathbb{O}}}\rangle \equiv_s \langle \text{Riordan numbers}\rangle,$$
$$\left\langle d_{n,\{2\}}\right\rangle \equiv_s \langle \text{Fibonacci numbers}\rangle,$$
$$\left\langle d_{n,\{k\}}\right\rangle \equiv_s \langle k\text{-Fibonacci numbers}\rangle,$$

where the sequence of k-Fibonacci numbers is defined by $a_n = a_{n-1} + a_{n-2} + \cdots + a_{n-k}$ with the initial values $a_0 = 1$ and $a_1 = a_2 = \cdots = a_k - 1 = 0$. The k-Fibonacci class above can be easily derived by the continued fraction form of $D_{\{k\}}(z)$.

5. Concluding remarks

In this paper, certain classes of Dyck paths and Motzkin paths are shown to be enumerated by certain integer sequences, some well known and some new. The idea of shift equivalence of sequences and the corresponding equivalence on sets helps classify various cases of height avoidance and restriction. However, there are questions, which await answers, for instance, under what circumstances will the equivalence class of sets contain a single element?

The concept of height avoidance and restriction considered here has led to new research directions for lattice path enumeration: the enumeration of Dyck paths with the heights of, valleys, or both peaks and valleys simultaneously, avoiding or restricted to certain subsets of $\mathbb{P}$, for instance. It is also of interest to consider the Motzkin analogue of the present work.

Acknowledgments

The authors appreciate C.-O. Chow and C. D. Fuh for their careful proof-reading of the manuscript. S.-C. Liu and Y.-N. Yeh were partially supported by grants NSC 91-2115-M-163-001 and NSC 91-2115-M-001-022 respectively.

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(Received February 28, 2003)