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Restricted Motzkin permutations, Motzkin paths, continued fractions, and Chebyshev polynomials

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Abstract

We say that a permutation π is a *Motzkin permutation* if it avoids 132 and there do not exist $a < b$ such that $\pi_a < \pi_b < \pi_{b+1}$. We study the distribution of several statistics on Motzkin permutations, including the length of the longest increasing and decreasing subsequences and the number of rises and descents. We also enumerate Motzkin permutations with additional restrictions and study the distribution of occurrences of fairly general patterns in this class of permutations.

Résumé

On dit qu'une permutation π est une *permutation de Motzkin* si elle évite le motif 132 et s'il n'existe pas $a < b$ tels que $\pi_a < \pi_b < \pi_{b+1}$. Nous étudions la distribution de plusieurs statistiques sur permutations de Motzkin, entre autres la longueur des sous-suites croissantes et décroissantes les plus longues et le nombre de montées et descentes. Nous énumérons aussi des permutations de Motzkin avec des contraintes supplémentaires et nous étudions la distribution du nombre d'occurrences de motifs assez généraux dans cette classe de permutations.

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1. Introduction

1.1. Background. Let $\alpha \in S_n$ and $\tau \in S_k$ be two permutations. We say that α *contains* τ if there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called a *pattern*. We say that α *avoids* τ, or is τ*-avoiding*, if such a subsequence does not exist. The set of all τ-avoiding permutations in S_n is denoted $S_n(\tau)$. For an arbitrary finite collection of patterns T, we say that α avoids T if α avoids any $\tau \in T$; the corresponding subset of S_n is denoted $S_n(T)$.

While the case of permutations avoiding a single pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of patterns τ_1, τ_2. This problem was solved completely for $\tau_1, \tau_2 \in S_3$ (see [**24**]) and for $\tau_1 \in S_3$ and $\tau_2 \in S_4$ (see [**25**]). Several recent papers [**5, 15, 11, 16, 17, 18**] deal with the case $\tau_1 \in S_3$, $\tau_2 \in S_k$ for various pairs τ_1, τ_2. Another natural question is to study permutations avoiding τ_1 and containing τ_2 exactly r times. Such a problem for certain $\tau_1, \tau_2 \in S_3$ and $r = 1$ was investigated in [**20**], and for certain $\tau_1 \in S_3$, $\tau_2 \in S_k$ in [**22, 15, 11**]. The tools involved in these papers include Catalan numbers, Chebyshev polynomials, and continued fractions.

In [**1**] Babson and Steingrímsson introduced *generalized patterns* that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In this context, we write a classical pattern with dashes between any two adjacent letters of the pattern (for example, 1423 as 1-4-2-3). If we omit the dash between two letters, we mean that for it to be an occurrence in a permutation π, the corresponding letters of π have to be adjacent. For example, in an occurrence of the pattern 12-3-4 in a permutation π, the letters in π that correspond to 1 and 2 are adjacent. For instance, the permutation $\pi = 3542617$ has only one occurrence of the pattern 12-3-4, namely the subsequence 3567, whereas π has two occurrences of the pattern 1-2-3-4, namely the subsequences 3567 and 3467. Claesson [**3**] presented a complete solution for the number of permutations avoiding any single 3-letter generalized pattern with exactly one adjacent pair of letters. Elizalde and Noy [**8**] studied some cases of avoidance of patterns where all letters have to occur in consecutive positions. Claesson and Mansour [**4**] (see also [**12, 13, 14**]) presented a complete solution for the number of permutations avoiding any pair of 3-letter generalized patterns with exactly one adjacent pair of letters. Besides, Kitaev [**9**] investigated simultaneous avoidance of two or more 3-letter generalized patterns without internal dashes.

A remark about notation: throughout the paper, a pattern represented with no dashes will always denote a classical pattern (i.e., with no requirement about elements being consecutive). All the generalized patterns that we will consider will have at least one dash.

1.2. Basic tools. *Catalan numbers* are defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$ for all $n \geq 0$. The generating function for the Catalan numbers is given by $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.

Chebyshev polynomials of the second kind (in what follows just Chebyshev polynomials) are defined by $U_r(\cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta}$ for $r \geq 0$. Clearly, $U_r(t)$ is a polynomial of degree r in t with integer coefficients, and the following recurrence holds:

$$U_0(t) = 1,\ U_1(t) = 2t,\ \text{and } U_r(t) = 2tU_{r-1}(t) - U_{r-2}(t) \text{ for all } r \geq 2. \tag{1}$$

The same recurrence is used to define $U_r(t)$ for $r < 0$ (for example, $U_{-1}(t) = 0$ and $U_{-2}(t) = -1$). Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, combinatorics, and number theory (see [**21**]). Apparently, the relation between restricted permutations and Chebyshev polynomials was discovered for the first time by Chow and West in [**5**], and later by Mansour and Vainshtein [**15, 16, 17, 18**], Krattenthaler [**11**].

Recall that a *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$ between $(0,0)$ and $(2n,0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ which never goes below the x-axis. Denote by $\mathcal{D}_n$ the set of Dyck paths of length $2n$, and by $\mathcal{D} = \bigcup_{n\geq 0}\mathcal{D}_n$ the class of all Dyck paths. If $D \in \mathcal{D}_n$, we will write $|D| = n$. Recall that a *Motzkin path* of length n is a lattice path in $\mathbb{Z}^2$ between $(0,0)$ and $(n,0)$ consisting of up-steps $(1,1)$, down-steps $(1,-1)$ and horizontal steps $(1,0)$ which never goes below the x-axis. Denote by $\mathcal{M}_n$ the set of *Motzkin paths* with n steps, and let $\mathcal{M} = \bigcup_{n\geq 0}\mathcal{M}_n$. We will write $|M| = n$ if $M \in \mathcal{M}_n$. Sometimes it will be convenient to encode each up-step by a letter u, each down-step by d, and each horizontal step by h. Denote by $M_n = |\mathcal{M}_n|$ the n-th *Motzkin number*. The generating function for these numbers is $M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$.

Define a *Motzkin permutation* π to be a 132-avoiding permutation in which there do not exist indices $a < b$ such that $\pi_a < \pi_b < \pi_{b+1}$. In such a context, π_a, π_b, π_{b+1} is called an occurrence of the pattern 1-23 (for instance, see [**3**]). For example, there are exactly 4 Motzkin permutations of length 3, namely, 213, 231, 312, and 321. The set of all Motzkin permutations in S_n we denote by $\mathfrak{M}_n$. The main reason for the term "Motzkin permutation" is that $|\mathfrak{M}_n| = M_n$, as we will see in Section 2.

It follows from the definition that the set $\mathfrak{M}_n$ is the same as the set of 132-avoiding permutations $\pi \in S_n$ where there is no a such that $\pi_a < \pi_{a+1} < \pi_{a+2}$. Indeed, assume that $\pi \in S_n(132)$ has an occurrence of 1-23, say $\pi_a < \pi_b < \pi_{b+1}$ with $a < b$. Now, if $\pi_{b-1} > \pi_b$, then π would have an occurrence of 132, namely $\pi_a\pi_{b-1}\pi_{b+1}$. Therefore, $\pi_{b-1} < \pi_b < \pi_{b+1}$, so π has three consecutive increasing elements.

For any subset $A \subseteq S_n$ and any pattern α, define $A(\alpha) := A \cap S_n(\alpha)$. For example, $\mathfrak{M}_n(\alpha)$ denotes the set of Motzkin permutations of length n that avoid α.

1.3. Organization of the paper. In Section 2 we exhibit a bijection between the set of Motzkin permutations and the set of Motzkin paths. Then we use it to obtain generating functions of Motzkin permutations with respect to the length of the longest decreasing and increasing subsequences together with the number of rises. The section ends with another application of the bijection, to the enumeration of fixed points in permutations avoiding simultaneously 231 and 32-1.

In Section 3 we consider additional restrictions on Motzkin permutations. Using a block decomposition, we enumerate Motzkin permutations avoiding the pattern $12\dots k$, and we find the distribution of occurrences of this pattern in Motzkin permutations. Then we obtain generating functions for Motzkin permutations avoiding patterns of more general shape. We conclude the section considering two classes of generalized patterns (as described above), and we study its distribution in Motzkin permutations.

2. The bijection $\Theta : \mathfrak{M}_n \longrightarrow \mathcal{M}_n$

In this section we establish a bijection Θ between Motzkin permutations and Motzkin paths. This bijection allows us to give the distribution of some interesting statistics on the set of Motzkin permutations.

2.1. The bijection Θ. We can give a bijection Θ between $\mathfrak{M}_n$ and $\mathcal{M}_n$. For that we use first the following bijection φ from $S_n(132)$ to $\mathcal{D}_n$, which is essentially due to Krattenthaler [**11**]. Consider $\pi \in S_n(132)$ given as an $n \times n$ array with crosses in the squares (i, π_i). Take the path with *up* and *right* steps that goes from the lower-left corner to the upper-right corner, leaving all the crosses to the right, and staying always as close to the diagonal connecting these two corners as possible. Then $\varphi(\pi)$ is the Dyck path obtained from this path by reading an up-step every time the path goes up and a down-step every time it goes right. Figure 1 shows an example when $\pi = 67435281$.

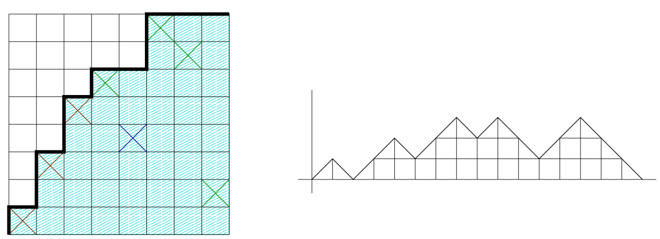

FIGURE 1. The bijection φ.

There is an easy way to recover π from $\varphi(\pi)$. Assume we are given the path from the lower-left corner to the upper-right corner or the array. Row by row, put a cross in the leftmost square to the right of this path such that there is exactly one cross in each column. This gives us π back.

One can see that $\pi \in S_n(132)$ avoids 1-23 if and only if the Dyck path $\varphi(\pi)$ does not contain three consecutive up-steps (a *triple rise*). Indeed, assume that $\varphi(\pi)$ has three consecutive up-steps. Then, the path from the lower-left corner to the upper-right corner of the array has three consecutive vertical steps. The crosses in the corresponding three rows give three consecutive increasing elements in π (this follows from the definition of the inverse of φ), and hence an occurrence of 1-23.

Reciprocally, assume now that π has an occurrence of 1-23. The path from the lower-left to the upper-right corner of the array of π must have two consecutive vertical steps in the rows of the crosses corresponding to '2' and '3'. But if $\varphi(\pi)$ has no triple rise, the next step of this path must be horizontal, and the cross corresponding to '2' must be right below it. But then all the crosses above this cross are to the right of it, which contradicts the fact that this was an occurrence of 1-23.

Denote by $\mathcal{E}_n$ the set of Dyck paths of length $2n$ with no triple rise. We have given a bijection between $\mathfrak{M}_n$ and $\mathcal{E}_n$. The second step is to exhibit a bijection between $\mathcal{E}_n$ and $\mathcal{M}_n$, so that Θ will be defined as the composition of the two bijections. Given $D \in \mathcal{E}_n$, divide it in n blocks, splitting after each down-step. Since D has no triple rises, each block is of one of these three forms: uud, ud, d. From left to right, transform the blocks according to the rule

$$uud \to u, \qquad ud \to h, \qquad d \to d. \tag{2}$$

We obtain a Motzkin path of length n. This step is clearly a bijection.

Up to reflection of the Motkin path over a vertical line, Θ is essentially the same bijection that was given by Claesson [**3**] between $\mathfrak{M}_n$ and $\mathcal{M}_n$, using a recursive definition.

2.2. Statistics in $\mathfrak{M}_n$. Here we show applications of the bijection Θ to give generating functions for several statistics in Motzkin permutations. For a permutation π, denote by $\mathrm{lis}(\pi)$ and $\mathrm{lds}(\pi)$ respectively the length of the longest increasing subsequence and the length of the longest decreasing subsequence of π. The following lemma follows from the definitions of the bijections and from the properties of φ (see [**11**]).

Lemma 1. *Let $\pi \in \mathfrak{M}_n$, let $D = \varphi(\pi) \in \mathcal{D}_n$, and let $M = \Theta(\pi) \in \mathcal{M}_n$. We have*

(1) $\mathrm{lds}(\pi) = \#\{$*peaks of* $D\} = \#\{$*steps* u *in* $M\} + \#\{$*steps* h *in* $M\}$,
(2) $\mathrm{lis}(\pi) =$ *height of* D = *height of* $M + 1$,
(3) $\#\{$*rises of* $\pi\} = \#\{$*double rises of* $D\} = \#\{$*steps* u *in* $M\}$.

Theorem 2. *The generating function for Motzkin permutations with respect to the length of the longest decreasing subsequence and to the number of rises is*

$$A(v,y,x) := \sum\nolimits_{n\geq 0} \sum\nolimits_{\pi\in\mathfrak{M}_n} v^{\mathrm{lds}(\pi)} y^{\#\{\text{rises of } \pi\}} x^n = \frac{1 - vx - \sqrt{1-2vx+(v^2-4vy)x^2}}{2vyx^2}.$$

Moreover,

$$A(v,y,x) = \sum\nolimits_{n\geq 0} \sum\nolimits_{m\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{m+2n}{2n} x^{m+2n} v^{m+n} y^n.$$

Proof. By Lemma 1, we can express A as

$$A(v,y,x) = \sum\nolimits_{M\in\mathcal{M}} v^{\#\{\text{steps } u \text{ in } M\}+\#\{\text{steps } h \text{ in } M\}} y^{\#\{\text{steps } u \text{ in } M\}} x^{|M|}.$$

Using the standard decomposition of Motzkin paths, we obtain the following equation for the generating function A.

$$A(v,y,x) = 1 + vxA(v,y,x) + vyx^2A^2(v,y,x). \tag{3}$$

Indeed, any nonempty $M \in \mathcal{M}$ can be written uniquely in one of the following two forms: (1) $M = hM_1$ and (2) $M = uM_1dM_2$, where M_1, M_2, M_3 are arbitrary Motzkin paths. In the first case, the number of horizontal steps of hM_1 is one more than in M_1, the number of up steps is the same, and $|hM_1| = |M_1|+1$, so we get the term $vxA(v,y,x)$. Similarly, the second case gives the term $vyx^2A^2(v,y,x)$. Solving equation (3) we get the desired expression. $\square$

Theorem 3. *For $k > 0$, let $B_k(v,y,x) := \sum_{n\geq 0} \sum_{\pi\in\mathfrak{M}_n(12\ldots(k+1))} v^{\mathrm{lds}(\pi)} y^{\#\{\text{rises of } \pi\}} x^n$ be the generating function for Motzkin permutations avoiding $12\ldots(k+1)$ with respect to the length of the longest decreasing subsequence and to the number of rises. Then we have the recurrence*

$$B_k(v,y,x) = \frac{1}{1 - vx - vyx^2 B_{k-1}(v,y,x)},$$

with $B_1(v,y,x)=\frac{1}{1-vx}$. *Thus,* B_k *can be expressed as*

$$B_k(v,y,x)=\cfrac{1}{1-vx-\cfrac{vyx^2}{\cfrac{\ddots}{1-vx-\cfrac{vyx^2}{1-vx}}}},$$

where the fraction has k *levels, or in terms of Chebyshev polynomials of the second kind, as*

$$B_k(v,y,x)=\frac{U_{k-1}\left(\frac{1-vx}{2x\sqrt{vy}}\right)}{x\sqrt{vy}U_k\left(\frac{1-vx}{2x\sqrt{vy}}\right)}.$$

Proof. The condition that π avoids $12\ldots(k+1)$ is equivalent to the condition $\mathrm{lis}(\pi)\le k$. By Lemma 1, permutations in $\mathfrak{M}_n$ satisfying this condition are mapped by Θ to Motzkin paths of height strictly less than k. Thus, we can express B_k as

$$B_k(v,y,x)=\sum_{M\in\mathcal{M}\text{ of height}<k} v^{\#\{\text{steps } u \text{ in } M\}+\#\{\text{steps } h \text{ in } M\}}y^{\#\{\text{steps } u \text{ in } M\}}x^{|M|}.$$

For $k>1$, we use again the standard decomposition of Motzkin paths. In the first of the above cases, the height of hM_1 is the same as the height of M_1. However, in the second case, in order for the height of uM_2dM_3 to be less than k, the height of M_2 has to be less than $k-1$. So we obtain the equation

$$B_k(v,y,x)=1+vxB_k(v,y,x)+vyx^2B_{k-1}(v,y,x)B_k(v,y,x).$$

For $k=1$, the path can have only horizontal steps, so we get $B_1(v,y,x)=\frac{1}{1-vx}$. Now, using the above recurrence and Equation 1 we get the desired result. $\square$

2.3. Fixed points in the reversal of Motzkin permutations. Here we show another application of Θ. A slight modification of it will allow us to enumerate fixed points in another class of pattern-avoiding permutations closely related to Motzkin permutations. For any $\pi=\pi_1\pi_2\ldots\pi_n\in S_n$, denote its reversal by $\pi^R=\pi_n\ldots\pi_2\pi_1$. Let $\mathfrak{M}_n^R:=\{\pi\in S_n:\pi^R\in\mathfrak{M}_n\}$. In terms of pattern avoidance, $\mathfrak{M}_n^R$ is the set of permutations that avoid 231 and 32-1 simultaneously, that is, the set of 231-avoiding permutations $\pi\in S_n$ where there do not exist $a<b$ such that $\pi_{a-1}>\pi_a>\pi_b$. Recall that i is called a *fixed point* of π if $\pi_i=i$.

Theorem 4. *The generating function* $\sum_{n\ge0}\sum_{\pi\in\mathfrak{M}_n^R}w^{\mathrm{fp}(\pi)}x^n$ *for permutations avoiding simultaneously* 231 *and* 32-1 *with respect to to the number of fixed points is*

$$\cfrac{1}{1-wx-\cfrac{x^2}{1-x-M_0(w-1)x^2-\cfrac{x^2}{1-x-M_1(w-1)x^3-\cfrac{x^2}{1-x-M_2(w-1)x^4-\cfrac{x^2}{\ddots}}}}},\tag{4}$$

where after the second level, the coefficient of $(w-1)x^{n+2}$ *is the Motzkin number* M_n.

Proof. We have the following composition of bijections:

$$\begin{array}{ccccccc}\mathfrak{M}_n^R & \longleftrightarrow & \mathfrak{M}_n & \longleftrightarrow & \mathcal{E}_n & \longleftrightarrow & \mathcal{M}_n\\ \pi & \mapsto & \pi^R & \mapsto & \varphi(\pi^R) & \mapsto & \Theta(\pi^R)\end{array}$$

The idea of the proof is to look at how the fixed points of π are transformed by each of these bijections.

For this we use the definition of tunnel of a Dyck path given in [**6**], and generalize it to Motzkin paths. A *tunnel* of $M \in \mathcal{M}$ (resp. $D \in \mathcal{D}$) is a horizontal segment between two lattice points of the path that intersects M (resp. D) only in these two points, and stays always below the path. Tunnels are in obvious one-to-one correspondence with decompositions of the path as $M = XuYdZ$ (resp. $D = XuYdZ$), where $Y \in \mathcal{M}$ (resp. $Y \in \mathcal{D}$). In the decomposition, the tunnel is the segment that goes from the beginning of the u to the end of the d. Clearly such a decomposition can be given for each up-step u, so the number of tunnels of a path equals its number of up-steps. The *length* of a tunnel is just its length as a segment, and the *height* is its y-coordinate.

Fixed points of π are mapped by the reversal operation to elements j such that $\pi^R_j = n+1-j$, which in the array of π^R correspond to crosses on the diagonal between the bottom-left and top-right corners. Each cross in this array naturally corresponds to a tunnel of the Dyck path $\varphi(\pi^R)$, namely the one determined by the vertical step in the same row as the cross and the horizontal step in the same column as the cross. It is not hard to see (and is also shown in [**7**]) that crosses on the diagonal between the bottom-left and top-right corners correspond in the Dyck path to tunnels T satisfying the condition $\mathrm{height}(T)+1 = \frac{1}{2}\mathrm{length}(T)$.

The next step is to see how these tunnels are transformed by the bijection from $\mathcal{E}_n$ to $\mathcal{M}_n$. Tunnels of height 0 and length 2 in the Dyck path $D := \varphi(\pi^R)$ are just hills ud landing on the x-axis. By the rule (3) they are mapped to horizontal steps at height 0 in the Motzkin path $M := \Theta(\pi^R)$. Assume now that $k \geq 1$. A tunnel T of height k and length $2(k+1)$ in D corresponds to a decomposition $D = XuYdZ$ where X ends at height k and $Y \in \mathcal{D}_{2k}$. Note that Y has to begin with an up-step (since it is a nonempty Dyck path) followed by a down-step, otherwise D would have a triple rise. Thus, we can write $D = XuudY'dZ$ where $Y' \in \mathcal{D}_{2(k-1)}$. When we apply to D the bijection given by rule (3), X is mapped to an initial segment $\widetilde{X}$ of a Motzkin path ending at height k, uud is mapped to u, Y' is mapped to a Motzkin path $\widetilde{Y'} \in \mathcal{M}_{k-1}$ of length $k-1$, the d following Y' is mapped to d (since it is preceded by another d), and Z is mapped to a final segment $\widetilde{Z}$ of a Motzkin path going from height k to the x-axis. Thus, we have that $M = \widetilde{X}u\widetilde{Y'}d\widetilde{Z}$. It follows that tunnels T of D satisfying $\mathrm{height}(T)+1 = \frac{1}{2}\mathrm{length}(T)$ are transformed by the bijection into tunnels $\widetilde{T}$ of M satisfying $\mathrm{height}(\widetilde{T})+1 = \mathrm{length}(\widetilde{T})$. We will call *good* tunnels the tunnels of M satisfying this last condition. It remains to show that the generating function for Motzkin paths where w marks the number of good tunnels plus the number of horizontal steps at height 0, and x marks the length of the path, is given by (4).

To do this we imitate the technique used in [**7**] to enumerate fixed points in 231-avoiding permutations. We will separate good tunnels according to their height. It is important to notice that if a good tunnel of M corresponds to a decomposition $M = XuYdZ$, then M has no good tunnels inside the part given by Y. In other words, the orthogonal projections on the x-axis of all the good tunnels of a given Motzkin path are disjoint. Clearly, they are also disjoint from horizontal steps at height 0. This observation allows us to give a continued fraction expression for our generating function.

For every $k \geq 1$, let $\mathrm{gt}_k(M)$ be the number of tunnels of M of height k and length $k+1$. Let $\mathrm{hor}(M)$ be the number of horizontal steps at height 0. We have seen that for $\pi \in \mathfrak{M}^R_n$, $\mathrm{fp}(\pi) = \mathrm{hor}(\Theta(\pi^R)) + \sum_{k\geq 1} \mathrm{gt}_k(\Theta(\pi^R))$. We will show now that for every $k \geq 1$, the generating function for Motzkin paths where w marks the statistic $\mathrm{hor}(M) + \mathrm{gt}_1(M) + \cdots + \mathrm{gt}_{k-1}(M)$ is given by the continued fraction (4) truncated at level k, with the $(k+1)$-st level replaced with $M(x)$.

A Motzkin path M can be written uniquely as a sequence of horizontal steps h and elevated Motzkin paths $uM'd$, where $M' \in \mathcal{M}$. In terms of the generating function $M(x) = \sum_{M\in\mathcal{M}} x^{|M|}$, this translates into the equation $M(x) = \frac{1}{1-x-x^2M(x)}$. The generating function where w marks horizontal steps at height 0 is just

$$\sum_{M\in\mathcal{M}} w^{\mathrm{hor}(M)} x^{|M|} = \frac{1}{1-wx-x^2M(x)}.$$

If we want w to mark also good tunnels at height 1, each M' from the elevated paths above has to be decomposed as a sequence of horizontal steps and elevated Motzkin paths $uM''d$. In this decomposition, a tunnel of height 1 and length 2 is produced by each empty M'', so we have

$$\sum_{M\in\mathcal{M}} w^{\mathrm{hor}(M)+\mathrm{gt}_1(M)}x^{|M|} = \cfrac{1}{1-wx-\cfrac{x^2}{1-x-x^2[w-1+M(x)]}}. \tag{5}$$

Indeed, the $M_0(=1)$ possible empty paths M'' have to be accounted as w, not as 1.

Let us now enumerate simultaneously horizontal steps at height 0 and good tunnels at heights 1 and 2. We can rewrite (5) as

$$\cfrac{1}{1-wx-\cfrac{x^2}{1-x-x^2\left[w-1+\cfrac{1}{1-x-x^2M(x)}\right]}}.$$

Combinatorially, this corresponds to expressing each M'' as a sequence of horizontal steps and elevated paths $uM'''d$, where $M'''\in\mathcal{M}$. Notice that since $uM'''d$ starts at height 2, a tunnel of height 2 and length 3 is created whenever $M'''\in\mathcal{M}_1$. Thus, if we want w to mark also these tunnels, such an M''' has to be accounted as wx, not x. The corresponding generating function is

$$\sum_{M\in\mathcal{M}} w^{\mathrm{hor}(M)+\mathrm{gt}_1(M)+\mathrm{gt}_2(M)}x^{|M|} = \cfrac{1}{1-wx-\cfrac{x^2}{1-x-x^2\left[w-1+\cfrac{1}{1-x-x^2[(w-1)x+M(x)]}\right]}}.$$

Now it is clear how iterating this process indefinitely we obtain the continued fraction (4). From the generating function where w marks $\mathrm{hor}(M)+\mathrm{gt}_1(M)+\cdots+\mathrm{gt}_k(M)$, we can obtain the one where w marks $\mathrm{hor}(M)+\mathrm{gt}_1(M)+\cdots+\mathrm{gt}_{k+1}(M)$ by replacing the $M(x)$ at the lowest level with

$$\frac{1}{1-x-x^2[M_k(w-1)x^k+M(x)]},$$

to account for tunnels of height k and length $k+1$, which in the decomposition correspond to elevated Motzkin paths at height k. $\square$

3. Restricted Motzkin permutations

In this section we consider those Motzkin permutations in $\mathfrak{M}_n$ that avoid an another pattern τ. More generally, we enumerate Motzkin permutations according to the number of occurrences of τ. Subsection 3.2 deals with the increasing pattern $\tau = 12\dots k$. In Subsection 3.3 we show that if τ has a certain form, we can express the generating function for τ-avoiding Motzkin permutations in terms of the the corresponding generating functions for some subpatterns of τ. Finally, Subsection 3.4 studies the case of the generalized patterns 12-3-…-k and 21-3-…-k.

We begin by setting some notation. Let $M_\tau(n)$ be the number of Motzkin permutations in $\mathfrak{M}_n(\tau)$, and let $N_\tau(x)=\sum_{n\geq 0} M_\tau(n)x^n$ be the corresponding generating function. Let $\pi\in\mathfrak{M}_n$. Using the block decomposition approach (see [**18**]), we have two possible block decompositions of π. These decompositions are described in Lemma 5, which is the basis for all the results in this section.

Lemma 5. *Let $\pi\in\mathfrak{M}_n$. Then one of the following holds:*
(i) $\pi=(n,\beta)$ *where* $\beta\in\mathfrak{M}_{n-1}$,
(ii) *there exists* t, $2\leq t\leq n$, *such that* $\pi=(\alpha,n-t+1,n,\beta)$, *where*

$$(\alpha_1-(n-t+1),\dots,\alpha_{t-2}-(n-t+1))\in\mathfrak{M}_{t-2} \text{ and } \beta\in\mathfrak{M}_{n-t}.$$

Proof. Given $\pi \in \mathfrak{M}_n$, take j so that $\pi_j = n$. Then $\pi = (\pi', n, \pi'')$, and the condition that π avoids 132 is equivalent to π' being a permutation of the numbers $n-j+1, n-j+2, \dots, n-1$, π'' being a permutation of the numbers $1, 2, \dots, n-j$, and both π' and π'' being 132-avoiding. On the other hand, it is easy to see that if π' is nonempty, then π avoids 1-23 if and only if the minimal entry of π' is adjacent to n, and both π' and π'' avoid 1-23. Therefore, π avoids 132 and 1-23 if and only if either (i) or (ii) hold. ☐

3.1. The pattern $\tau = \emptyset$. Here we show the simplest application of Lemma 5, to enumerate Motzkin permutations of a given length. This also follows from the bijection to Motzkin paths in Section 2.

Proposition 6. *The number of Motzkin permutations of length n is given by M_n, the n-th Motzkin number.*

Proof. As a consequence of Lemma 5, there are two possible block decompositions of an arbitrary Motzkin permutation $\pi \in \mathfrak{M}_n$. Let us write an equation for $N_\emptyset(x)$. The first (resp. second) of the block decompositions above contributes as $xN_\emptyset(x)$ (resp. $x^2N_\emptyset^2(x)$). Therefore $N_\emptyset(x) = 1 + xN_\emptyset(x) + x^2N_\emptyset^2(x)$, where 1 is the contribution of the empty Motzkin permutation. Hence, $N_\emptyset(x)$ is the generating function for the Motzkin numbers M_n, as claimed. ☐

3.2. The pattern $\tau = 12\dots k$. For the first values of k, we have from the definitions that $N_1(x) = 1$ and $N_{12}(x) = \frac{1}{1-x}$. Here we consider the case $\tau = 12\dots k$ for arbitrary k. From Theorem 3 we get the following expression for N_τ, for which we also give a direct derivation using the block decomposition of Motzkin permutations.

Theorem 7. *For all $k \geq 2$, $N_{12\dots k}(x) = \frac{U_{k-2}\left(\frac{1-x}{2x}\right)}{xU_{k-1}\left(\frac{1-x}{2x}\right)}$.*

Proof. By Lemma 5, we have two possibilities for the block decomposition of an arbitrary Motzkin permutation $\pi \in \mathfrak{M}_n$. Let us write an equation for $N_{12\dots k}(x)$. The contribution of the first (resp. second) block decomposition is $xN_{12\dots k}(x)$ (resp. $x^2N_{12\dots(k-1)}(x)N_{12\dots k}(x)$). Therefore,

$$N_{12\dots k}(x) = 1 + xN_{12\dots k}(x) + x^2N_{12\dots k}(x)N_{12\dots(k-1)}(x),$$

where 1 comes from the empty Motzkin permutation. Now, using induction on k and the recursion (1) we get the desired result. ☐

This theorem can be generalized as follows. Let $N(x_1, x_2, \dots)$ be the generating function

$$\sum_{n\geq 0}\sum_{\pi\in\mathfrak{M}_n}\prod_{j\geq 1} x_j^{12\dots j(\pi)},$$

where $12\dots j(\pi)$ is the number of occurrences of the pattern $12\dots j$ in π.

Theorem 8. *The generating function $\sum_{n\geq 0}\sum_{\pi\in\mathfrak{M}_n}\prod_{j\geq 1} x_j^{12\dots j(\pi)}$ is given by the following continued fraction:*

$$\cfrac{1}{1 - x_1 - \cfrac{x_1^2x_2}{1 - x_1x_2 - \cfrac{x_1^2x_2^3x_3}{1 - x_1x_2^2x_3 - \cfrac{x_1^2x_2^5x_3^4x_4}{\ddots}}}},$$

in which the n-th numerator is $\prod_{i=1}^{n+1} x_i^{\binom{n}{i-1}+\binom{n-1}{i-1}}$ and the monomial in the n-th denominator is $\prod_{i=1}^{n} x_i^{\binom{n-1}{i-1}}$.

Proof. By Lemma 5, we have two possibilities for the block decomposition of an arbitrary Motzkin permutation $\pi \in \mathfrak{M}_n$. Let us write an equation for $N(x_1, x_2, \dots)$. The contribution of the first decomposition is $x_1N(x_1, x_2, \dots)$, and the second decomposition gives $x_1^2x_2N(x_1x_2, x_2x_3, \dots)N(x_1, x_2, \dots)$. Therefore,

$$N(x_1, x_2, \dots) = 1 + x_1N(x_1, x_2, \dots) + x_1^2x_2N(x_1x_2, x_2x_3, \dots)N(x_1, x_2, \dots),$$

where 1 is the contribution of the empty Motzkin permutation. The theorem follows now by induction. □

3.2.1. *Counting occurrences of the pattern* $12\ldots k$ *in a Motzkin permutation.* Using Theorem 8 we can enumerate occurrences of the pattern $12\ldots k$ in Motzkin permutations.

THEOREM 9. *Fix* $k \geq 2$. *The generating function for the number of Motzkin permutations which contain* $12\ldots k$ *exactly* r, $r = 1, 2, \ldots, k$, *times is given by*

$$\frac{\left(U_{k-2}\left(\frac{1-x}{2x}\right) - xU_{k-3}\left(\frac{1-x}{2x}\right)\right)^{r-1}}{U_{k-1}^{r+1}\left(\frac{1-x}{2x}\right)}.$$

PROOF. Let $x_1 = x$, $x_k = y$, and $x_j = 1$ for all $j \neq 1, k$. Let $G_k(x, y)$ be the function obtained from $N(x_1, x_2, \ldots)$ after this substitution. Theorem 8 gives

$$G_k(x,y) = \cfrac{1}{1 - x - \cfrac{x^2}{\ddots - \cfrac{\ddots}{1 - x - \cfrac{x^2 y}{1 - xy - \cfrac{x^2 y^{k+1}}{\ddots}}}}}.$$

So, $G_k(x, y)$ can be expressed as follows. For all $k \geq 2$,

$$G_k(x,y) = \frac{1}{1 - x - x^2 G_{k-1}(x,y)},$$

and there exists a continued fraction $H(x, y)$ such that $G_1(x, y) = \frac{y}{1-xy-y^{k+1}H(x,y)}$. Now, using induction on k together with (1) we get that there exists a formal power series $J(x, y)$ such that

$$G_k(x,y) = \frac{U_{k-2}\left(\frac{1-x}{2x}\right) - \left(U_{k-3}\left(\frac{1-x}{2x}\right) - xU_{k-4}\left(\frac{1-x}{2x}\right)\right)y}{xU_{k-1}\left(\frac{1-x}{2x}\right) - x\left(U_{k-2}\left(\frac{1-x}{2x}\right) - xU_{k-3}\left(\frac{1-x}{2x}\right)\right)y} + y^{k+1}J(x,y).$$

The series expansion of $G_k(x, y)$ about the point $y = 0$ gives

$$G_k(x,y) = \left[U_{k-2}\left(\tfrac{1-x}{2x}\right) - \left(U_{k-3}\left(\tfrac{1-x}{2x}\right) - xU_{k-4}\left(\tfrac{1-x}{2x}\right)\right)y\right] \cdot \sum_{r\geq 0} \frac{\left(U_{k-2}\left(\frac{1-x}{2x}\right) - xU_{k-3}\left(\frac{1-x}{2x}\right)\right)^r}{xU_{k-1}^{r+1}\left(\frac{1-x}{2x}\right)} y^r + y^{k+1}J(x,y).$$

Hence, by using the identities $U_k^2(t) - U_{k-1}(t)U_{k+1}(t) = 1$ and $U_k(t)U_{k-1}(t) - U_{k-2}(t)U_{k+1}(t) = 2t$ we get the desired result. □

3.2.2. *More statistics on Motzkin permutations.* We can use the above theorem to find the generating function for the number of Motzkin permutations with respect to various statistics.

For another application of Theorem 8, recall that i is a *free rise* of π if there exists j such that $\pi_i < \pi_j$. We denote the number of free rises of π by $fr(\pi)$. Using Theorem 8 for $x_1 = x$, $x_2 = q$, and $x_j = 1$ for $j \geq 3$, we get the following result.

COROLLARY 10. *The generating function* $\sum_{n\geq 0} \sum_{\pi \in \mathfrak{M}_n} x^n q^{fr(\pi)}$ *is given by the following continued fraction:*

$$\cfrac{1}{1 - x - \cfrac{x^2 q}{1 - xq - \cfrac{x^2 q^3}{1 - xq^2 - \cfrac{x^2 q^5}{\ddots}}}},$$

in which the n*-th numerator is* $x^2 q^{2n-1}$ *and the monomial in the* n*-th denominator is* xq^{n-1}.

For our next application, recall that π_j is a *right-to-left maximum* of a permutation π if $\pi_i < \pi_j$ for all $i > j$. We denote the number of right-to-left maxima of π by $rlm(\pi)$.

Corollary 11. *The generating function $\sum_{n\geq 0}\sum_{\pi\in\mathfrak{M}_n} x^n q^{rlm(\pi)}$ is given by the following continued fraction:*

$$\cfrac{1}{1-xq-\cfrac{x^2q}{1-x-\cfrac{x^2}{1-x-\cfrac{x^2}{\ddots}}}}.$$

Moreover, $\sum_{n\geq 0}\sum_{\pi\in\mathfrak{M}_n} x^n q^{lrm(\pi)} = \sum_{m\geq 0} x^m(1+xM(x))^m q^m$.

Proof. Using Theorem 8 for $x_1 = xq$, and $x_{2j} = x_{2j+1}^{-1} = q^{-1}$ for $j \geq 1$, together with [**2**, Proposition 5] we get the first equation as claimed. The second equation follows from the fact that the continued fraction

$$\cfrac{1}{1-x-\cfrac{x^2}{1-x-\cfrac{x^2}{\ddots}}}$$

is given by the generating function for the Motzkin numbers, namely $M(x)$. □

3.3. General restriction. Let us find the generating function for those Motzkin permutations which avoid τ in terms of the generating function for Motzkin permutations avoiding ρ, where ρ is a permutation obtained by removing some entries from τ.

Theorem 12. *Let $k \geq 4$, $\tau = (\rho', 1, k) \in \mathfrak{M}_k$, and let $\rho \in \mathfrak{M}_{k-2}$ be the permutation obtained by decreasing each entry of ρ' by 1. Then*

$$N_\tau(x) = \frac{1}{1-x-x^2N_\rho(x)}.$$

Proof. By Lemma 5, we have two possibilities for block decomposition of a nonempty Motzkin permutation in $\mathfrak{M}_n$. Let us write an equation for $N_\tau(x)$. The contribution of the first decomposition is $xN_\tau(x)$, and from the second decomposition we get $x^2N_\rho(x)N_\tau(x)$. Hence $N_\tau(x) = 1 + xN_\tau(x) + x^2N_\rho(x)N_\tau(x)$, where 1 corresponds to the empty Motzkin permutation. Solving the above equation we get the desired result. □

For example, using Theorem 12 for $\tau = 23\ldots(k-1)1k$ ($\rho = 12\ldots(k-2)$) we have

$$N_{23\ldots(k-1)1k}(x) = \frac{1}{1-x-x^2N_{12\ldots(k-2)}(x)}.$$

Hence, by Theorem 7 together with (1) we get

$$N_{23\ldots(k-1)1k}(x) = \frac{U_{k-3}\left(\frac{1-x}{2x}\right)}{xU_{k-2}\left(\frac{1-x}{2x}\right)}.$$

Corollary 13. *For all $k \geq 1$,*

$$N_{k(k+1)(k-1)(k+2)(k-2)(k+3)\ldots 1(2k)}(x) = \frac{U_{k-1}\left(\frac{1-x}{2x}\right)}{xU_k\left(\frac{1-x}{2x}\right)},$$

and

$$N_{(k+1)k(k+2)(k-1)(k+3)\ldots 1(2k+1)}(x) = \frac{U_k\left(\frac{1-x}{2x}\right)+U_{k-1}\left(\frac{1-x}{2x}\right)}{x\left(U_{k+1}\left(\frac{1-x}{2x}\right)+U_k\left(\frac{1-x}{2x}\right)\right)}.$$

PROOF. Theorem 12 for $\tau = k(k+1)(k-1)(k+2)(k-2)(k+3)\dots 1(2k)$ gives

$$N_\tau(x) = \frac{1}{1 - x - x^2 N_{(k-1)k(k-2)(k+1)(k-3)(k+2)\dots 1(2k-2)}(x)}.$$

Now we argue by induction on k, using (1) and the fact that $N_{12}(x) = \frac{1}{1-x}$. Similarly, we get the explicit formula for $N_{(k+1)k(k+2)(k-1)(k+3)\dots 1(2k+1)}(x)$. □

Theorem 7 and Corollary 13 suggest that there should exist a bijection between the sets $\mathfrak{M}_n(12\dots(k+1))$ and $\mathfrak{M}_n(k(k+1)(k-1)(k+2)(k-2)(k+3)\dots 1(2k))$. Finding it remains an interesting open question.

THEOREM 14. *Let $\tau = (\rho', t, k, \theta', 1, t-1) \in \mathfrak{M}_k$ such that $\rho'_a > t > \theta'_b$ for all a, b. Let ρ and θ be the permutations obtained by decreasing each entry of ρ' by t and decreasing each entry of θ' by 1, respectively. Then*

$$N_\tau(x) = \frac{1 - x^2 N_\rho(x) \widetilde{N}_\theta(x)}{1 - x - x^2(N_\rho(x) + \widetilde{N}_\theta(x))},$$

where $\widetilde{N}_\theta(x) = \frac{1}{1-x-x^2 N_\theta(x)}$.

PROOF. By Lemma 5, we have two possibilities for block decomposition of a nonempty Motzkin permutation $\pi \in \mathfrak{M}_n$. Let us write an equation for $N_\tau(x)$. The contribution of the first decomposition is $xN_\tau(x)$. The second decomposition contributes $x^2 N_\rho(x) N_\tau(x)$ if α avoids ρ, and $x^2(N_\tau(x) - N_\rho(x))\widetilde{N}_\theta(x)$ if α contains ρ. This last case follows from Theorem 12, since if α contains ρ, β has to avoid $(\theta, 1, t-1)$. Hence,

$$N_\tau(x) = 1 + xN_\tau(x) + x^2 N_\rho(x) N_\tau(x) + x^2(N_\tau(x) - N_\rho(x))\widetilde{N}_\theta(x),$$

where 1 is the contribution of the empty Motzkin permutation. Solving the above equation we get the desired result. □

For example, for $\tau = 546213$ ($\tau = \rho 46 \theta 13$), Theorem 14 gives $N_\tau(x) = \frac{1-2x}{(1-x)(1-2x-x^2)}$.

The last two theorems can be generalized as follows.

THEOREM 15. *Let $\tau = (\tau^1, t_1+1, t_0, \tau^2, t_2+1, t_1, \dots, \tau^m, t_m+1, t_{m-1})$ where $t_{j-1} > \tau^j_a > t_j$ for all a and j. We define $\sigma^j = (\tau^1, t_1+1, t_0, \dots, \tau^j)$ for $j = 2, \dots, m$, $\sigma^0 = \emptyset$, and $\theta^j = (\tau^j, t_j+1, t_{j-1}, \dots, \tau^m, t_m+1, t_{m-1})$ for $j = 1, 2, \dots, m$. Then*

$$N_\tau(x) = 1 + xN_\tau(x) + x^2 \sum_{j=1}^{m} (N_{\sigma^j}(x) - N_{\sigma^{j-1}}) N_{\theta^j}(x).$$

(By convention, if ρ is a permutation of $\{i+1, i+2, \dots, i+l\}$, then N_ρ is defined as $N_{\rho'}$, where ρ' is obtained from ρ decreasing each entry by i.)

PROOF. By Lemma 5, we have two possibilities for block decomposition of a nonempty Motzkin permutation $\pi \in \mathfrak{M}_n$. Let us write an equation for $N_\tau(x)$. The contribution of the first decomposition is $xN_\tau(x)$. The second decomposition contributes $x^2(N_{\sigma^j}(x) - N_{\sigma^{j-1}}(x))N_{\theta^j}(x)$ if α avoids σ^j and contains σ^{j-1} (which happens exactly for one value of j), because in this case β must avoid θ^j. Therefore, adding all the possibilities of contributions with the contribution 1 for the empty Motzkin permutation we get the desired result. □

For example, this theorem can be used together with Theorem 7 to give the following result.

COROLLARY 16. (i) *For all $k \geq 3$, $N_{(k-1)k12\dots(k-2)}(x) = \frac{U_{k-3}\left(\frac{1-x}{2x}\right)}{xU_{k-2}\left(\frac{1-x}{2x}\right)}$;*

(ii) *For all $k \geq 4$, $N_{(k-1)(k-2)k12\dots(k-3)}(x) = \frac{U_{k-4}\left(\frac{1-x}{2x}\right) - xU_{k-5}\left(\frac{1-x}{2x}\right)}{x\left(U_{k-3}\left(\frac{1-x}{2x}\right) - xU_{k-4}\left(\frac{1-x}{2x}\right)\right)}$;*

(iii) *For all* $1 \le t \le k-3$, $N_{(t+2)(t+3)\dots(k-1)(t+1)k12\dots t}(x) = \frac{U_{k-4}\left(\frac{1-x}{2x}\right)}{xU_{k-3}\left(\frac{1-x}{2x}\right)}$.

3.4. Generalized patterns. In this section we consider the case of generalized patterns (see Subsection 1.1), and we study some statistics on Motzkin permutations.

3.4.1. *Counting occurrences of the generalized patterns* 12-3-…-k *and* 21-3-…-k. We denote by $F(t,X,Y) = F(t,x_2,x_3,\dots,y_2,y_3,\dots)$ the generating function $\sum_{n\ge0}\sum_{\pi\in\mathfrak{M}_n} t^n \prod_{j\ge2} x_j^{12\text{-}3\text{-}\dots\text{-}j(\pi)} y_j^{21\text{-}3\text{-}\dots\text{-}j(\pi)}$, where 12-3-…-$j(\pi)$ and 21-3-…-$j(\pi)$ are the number of occurrences of the pattern 12-3-…-j and 21-3-…-j in π, respectively.

Theorem 17. *We have*

$$F(t,X,Y) = 1 - \cfrac{t}{ty_2 - \cfrac{1}{1+tx_2(1-y_2y_3)+tx_2y_2y_3F(t,X',Y')}},$$

where $X' = (x_2x_3, x_3x_4, \dots)$ *and* $Y' = (y_2y_3, y_3y_4, \dots)$. *In other words, the generating function* $F(t,x_2,x_3,\dots,y_2,y_3,\dots)$ *is given by the continued fraction*

$$1 - \cfrac{t}{ty_2 - \cfrac{1}{1+tx_2 - \cfrac{t^2x_2y_2y_3}{ty_2y_3 - \cfrac{1}{1+tx_2x_3 - \cfrac{t^2x_2x_3y_2y_3^2y_4}{ty_2y_3^2y_4 - \cfrac{1}{1+tx_2x_3^2x_4 - \cfrac{t^2x_2x_3^2x_4y_2y_3^3y_4^3y_5}{\ddots}}}}}}}.$$

Proof. As usual, we consider the two possible block decompositions of a nonempty Motzkin permutation $\pi\in\mathfrak{M}_n$ (see Lemma 5). Let us write an equation for $F(t,X,Y)$. The contribution of the first decomposition is $t+ty_2(F(t,X,Y)-1)$. The contribution of the second decomposition gives t^2x_2, $t^2x_2y_2(F(t,X,Y)-1)$, $t^2x_2y_2y_3(F(t,X',Y')-1)$, and $t^2x_2y_2^2y_3(F(t,X,Y)-1)(F(t,X',Y')-1)$ for the four possibilities $\alpha=\beta=\emptyset$, $\alpha=\emptyset\ne\beta$, $\beta=\emptyset\ne\alpha$, and $\beta,\alpha\ne\emptyset$, respectively. Hence,

$$\begin{aligned}F(t,X,Y) = 1+t+ty_2(F(t,X,Y)-1)+t^2x_2+t^2x_2y_2y_3(F(t,X'Y')-1)\\ +t^2x_2y_2(F(t,X,Y)-1)+t^2x_2y_2^2y_3(F(t,X,Y)-1)(F(t,X',Y')-1),\end{aligned}$$

where 1 is as usual the contribution of the empty Motzkin permutation. Simplifying the above equation we get

$$F(t,X,Y) = 1 - \cfrac{t}{ty_2 - \cfrac{1}{1+tx_2(1-y_2y_3)+tx_2y_2y_3F(t,X',Y')}}.$$

The second part of the theorem now follows by induction. □

As a corollary of Theorem 17 we recover the distribution of the number of rises and number of descents on the set of Motzkin permutations, which also follows easily from Theorem 2.

Corollary 18. *We have*

$$\sum_{n\ge0}\sum_{\pi\in\mathfrak{M}_n} t^n p^{\#\{\text{rises in }\pi\}} q^{\#\{\text{descents in }\pi\}} = \frac{1-qt-2pq(1-q)t^2-\sqrt{(1-qt)^2-4pqt^2}}{2pq^2t^2}.$$

As an application of Theorem 17 let us consider the case of Motzkin permutations which contain either 12-3-…-k or 21-3-…-k exactly r times. Using the same arguments as in the proof of Theorem 9, we can apply Theorem 17 to obtain the following result.

THEOREM 19. *Fix $k \geq 2$. Let $N_\tau(x;r)$ be the generating function for the number of Motzkin permutations which contain τ exactly r times. Then*

$$N_{12\text{-}3\text{-}\dots\text{-}k}(x;0) = \frac{U_{k-1}\left(\frac{1-x}{2x}\right)}{xU_k\left(\frac{1-x}{2x}\right)}, \qquad N_{21\text{-}3\text{-}\dots\text{-}k}(x;0) = \frac{U_{k-3}\left(\frac{1-x}{2x}\right) - xU_{k-4}\left(\frac{1-x}{2x}\right)}{x\left(U_{k-2}\left(\frac{1-x}{2x}\right) - xU_{k-3}\left(\frac{1-x}{2x}\right)\right)},$$

and for all $r = 1, 2, \dots, k-1$,

$$N_{12\text{-}3\text{-}\dots\text{-}k}(x;r) = \frac{x^{r-1}U_{k-2}^{r-1}\left(\frac{1-x}{2x}\right)}{(1-x)^r U_{k-1}^{r+1}\left(\frac{1-x}{2x}\right)}, \qquad N_{21\text{-}3\text{-}\dots\text{-}k}(x;r) = \frac{x^r(1+x)^r U_{k-2}^{r-1}\left(\frac{1-x}{2x}\right)}{\left(U_{k-2}\left(\frac{1-x}{2x}\right) - xU_{k-3}\left(\frac{1-x}{2x}\right)\right)^{r+1}}.$$

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