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From Motzkin to Catalan permutations

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Abstract

For every integer $j \geqslant 1$, we define a class of permutations in terms of certain forbidden subsequences. For $j = 1$, the corresponding permutations are counted by the Motzkin numbers, and for $j = \infty$ (defined in the text), they are counted by the Catalan numbers. Each value of $j > 1$ gives rise to a counting sequence that lies between the Motzkin and the Catalan numbers. We compute the generating function associated to these permutations according to several parameters. For every $j \geqslant 1$, we show that only this generating function is algebraic according to the length of the permutations. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

Permutations with forbidden subsequences have been widely studied in the last few years. They are of interest in both Computer Science and Combinatorics because the permutations with forbidden subsequences are related to the characterization of words without any regularities or to the analysis of some regularities in words [7,19]. They are found in some sorting [21,24,25] and pattern matching [8] problems, and they codify a large number of nontrivial combinatorial objects [10,12,13,15–17]. Several classical sequences of numbers in Combinatorics arise in the problem of enumerating permutations with forbidden subsequences. For example, we refer to binomial coefficients and the Pell, Fibonacci, Motzkin and Schröder numbers [23]. Some of these results were obtained by West [22,24,26,27] and Gire [15]. Knuth studied the permutations sortable through a stack [18] and showed that they are the permutations with the forbidden subsequence 231. All the permutations with a forbidden subsequence of length three are enumerated by Catalan numbers. West [25] proved that the permutations sortable twice through a stack avoid the subsequences 2341 and $3\bar{5}241$. A permutation π avoids the barred pattern $3\bar{5}241$ if every subsequence of type 3241 is contained in a subsequence of type 35241 in π. West conjectured that the number of two-stack sortable

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permutations of length n is $2(3n)!/((2n+1)!(n+1)!)$. This conjecture was proved by Zeilberger [28]. Even though permutations with forbidden subsequences have been widely studied (see [16], for a survey), there are still many interesting problems to be solved on the subject. For instance, many efforts have been made to enumerate the permutations with the forbidden subsequence $12..(k+1)$ and which are related to pairs of standard Young tableaux having the same shape and length at most k. The results obtained refer to the number of permutations of length n avoiding the patterns 123 [22] and 1234 [14]. If k is larger than three, the enumeration problem is still open. In this case, Regev [20] obtained quite an interesting result, that is: the number of permutations of length n avoiding the pattern $1\ldots(k+1)$ is asymptotically equal to $c(k-1)^{2n}/n^{(k^2-2k)/2}$, where c is a constant.

In this paper, we characterize the permutations avoiding the patterns 321 and $(j+2)\bar{1}(j+3)2\ldots(j+1)$ and we denote this class by $\mathscr{M}(j)$. The enumeration of $\mathscr{M}(j)$ permutations produces sequences of numbers and provides a kind of 'discrete continuity' between the well-known Motzkin and Catalan number sequences. In Section 2 some definitions regarding permutations with forbidden subsequences are recalled. In Section 3, we describe the basic idea of the ECO method [6]. It is used for Enumerating Combinatorial Objects and allows us to obtain all the objects of size $(n+1)$ from the objects of size n by means of a 'local expansion'. In Section 4, we apply this method to the class $\mathscr{M}(j)$ and determine a recursive construction of the class which can be translated into a functional equation verified by the generating function of $\mathscr{M}(j)$ permutations according to their length, number of active sites, inversions and right minima. The definitions of these parameters are given in the following section (see Definitions 2.4 and 2.5). We also prove that this generating function is algebraic only according to the length of the permutations. The following well-known cases have algebraic generating functions of degree two:

- $\mathscr{M}(1)$ permutations enumerated by Motzkin numbers;
- $\mathscr{M}(2)$ permutations enumerated by the numbers of the left factors in Motzkin words;
- $\mathscr{M}(\infty)$ permutations enumerated by Catalan numbers.

Finally, in Section 5, we propose some perspectives for future research on this subject.

2. Notations and definitions

In this section, we recall the basic definitions used in this paper. A permutation $\pi=\pi(1)\pi(2)\ldots\pi(n)$ on $[n]=\{1,2,\ldots,n\}$ is a bijection between $[n]$ and $[n]$. Let $\mathscr{S}_n$ be the set of permutations on $[n]$.

Definition 2.1. A permutation $\pi\in\mathscr{S}_n$ contains a subsequence of type $\tau\in\mathscr{S}_k$ if a sequence of indexes $1\leqslant i_{\tau(1)}<i_{\tau(2)}<\cdots<i_{\tau(k)}\leqslant n$ exists such that $\pi(i_1)<\pi(i_2)<\cdots<\pi(i_k)$. We denote the set of permutations of $\mathscr{S}_n$ not containing any subsequences of type τ by $\mathscr{S}_n(\tau)$.

Example 2.1. The permutation $\pi = 5\ 1\ 6\ 2\ 7\ 3\ 8\ 4\ 9$ belongs to $\mathscr{S}_9(321)$ because all its subsequences of length 3 are not of type 321. This permutation does not belong to $\mathscr{S}_9(3412)$ because it contains subsequences of type 3412:

$$\pi(1)\pi(3)\pi(4)\pi(6) = 5623,$$

$$\pi(1)\pi(3)\pi(4)\pi(8) = 5624,$$

$$\pi(1)\pi(3)\pi(6)\pi(8) = 5634,$$

$$\pi(1)\pi(5)\pi(6)\pi(8) = 5734,$$

$$\pi(3)\pi(5)\pi(6)\pi(8) = 6734.$$

Definition 2.2. A *barred* permutation $\bar{\tau}$ of $[k]$ is a permutation of $\mathscr{S}_k$ having a bar over one of its elements. Let τ be a permutation on $[k]$ identical to $\bar{\tau}$ but unbarred: $\hat{\tau}$ is the permutation of $[k-1]$ made up of the $k-1$ unbarred elements of $\bar{\tau}$, rearranged as a permutation on $[k-1]$.

Definition 2.3. A permutation $\pi \in \mathscr{S}_n$ contains a type $\bar{\tau}$ subsequence if π contains a type $\hat{\tau}$ subsequence and which, in turn, is not a type τ subsequence. We denote the set of permutations in $\mathscr{S}_n$ not containing any type $\bar{\tau}$ subsequences by $\mathscr{S}_n(\bar{\tau})$.

Example 2.2. If $\bar{\tau} = 4\bar{1}523$, we have $\tau = 41523$ and $\hat{\tau} = 3412$. The permutation $\pi = 5\ 1\ 6\ 2\ 7\ 3\ 8\ 4\ 9$ belongs to $\mathscr{S}_9(\bar{\tau})$ because the subsequences of type $\hat{\tau}$ (see Example 2.1) are subsequences of:

$$\pi(1)\pi(2)\pi(3)\pi(4)\pi(6) = 51623,$$

$$\pi(1)\pi(2)\pi(3)\pi(4)\pi(8) = 51624,$$

$$\pi(1)\pi(2)\pi(3)\pi(6)\pi(8) = 51634,$$

$$\pi(1)\pi(2)\pi(5)\pi(6)\pi(8) = 51734,$$

$$\pi(3)\pi(4)\pi(5)\pi(6)\pi(8) = 62734,$$

which are of type τ.

If we have a set $\tau_1 \in \mathscr{S}_{k_1}, \ldots, \tau_p \in \mathscr{S}_{k_p}$ of barred or unbarred permutations, we denote the set $\mathscr{S}_n(\tau_1) \cap \cdots \cap \mathscr{S}_n(\tau_p)$ by $\mathscr{S}_n(\tau_1, \ldots, \tau_p)$. We call the family $F = \{\tau_1, \ldots, \tau_p\}$ a *family of forbidden subsequences* and the set $\mathscr{S}_n(F)$, a *family of permutations with forbidden subsequences*.

Example 2.3. Since the permutation $\pi = 5\ 1\ 6\ 2\ 7\ 3\ 8\ 4\ 9$ avoids the patterns 321 and $4\bar{1}523$, we have that π belongs to $\mathscr{S}_9(321, 4\bar{1}523)$.

Let us now define some permutation parameters. Let $\pi \in \mathscr{S}_n$: we denote the position lying to the left of $\pi(1)$ by s_0, the position between $\pi(i)$,$\pi(i+1)$, $1 \leqslant i \leqslant n-1$, by s_i

and the position to the right of $\pi(n)$ by s_n. The positions $s_0, s_1, \ldots, s_{n-1}, s_n$ are the *sites* of π.

Definition 2.4. Let $F = \{\tau_1, \ldots, \tau_p\}$: a site s_i, $0 \leqslant i \leqslant n$, of a permutation $\pi \in \mathscr{S}_n(F)$ is *active* if the insertion of $(n+1)$ into s_i produces a permutation belonging to the set $\mathscr{S}_{n+1}(F)$; otherwise, it is said to be *inactive*. We denote the set of active sites of π by $\mathscr{A}(\pi)$.

Definition 2.5. Let $\pi \in \mathscr{S}_n$. The pair (i,j), $i < j$ is an inversion if $\pi(i) > \pi(j)$. An element $\pi(i)$ is a *right minimum* if $\pi(i) < \pi(j)$, $\forall j \in [i+1, n]$.

Given a permutation π, we denote its length by $n(\pi)$, the number of its active sites by $a(\pi)$, the number of its right minima by $m(\pi)$ and the number of its inversions by $i(\pi)$.

Example 2.4. The permutation π=5 1 6 2 7 3 8 4 9 has 10 inversions: $(1,2)(1,4)(1,6)$ $(1,8)(3,4)(3,6)(3,8)(5,6)(5,8)(7,8)$ and 4 right minima: $\pi(2)=1$, $\pi(4)=2$, $\pi(6)=3$ and $\pi(8)=4$.

In this paper, we study the $\mathscr{S}_n(321, (j+2)\bar{1}(j+3)2\ldots(j+1))$ family and denote it by $\mathscr{M}_n(j)$. Moreover, we indicate the $S_n(321)$ family by $\mathscr{M}_n(\infty)$ and $\bigcup_{n\geqslant 1} \mathscr{M}_n(j)$ by $\mathscr{M}(j)$.

3. The ECO method

In this section, we review the basic ideas of the ECO method [6] and refer to [1–4] for some further applications and examples of it. Let $\mathcal{O}$ be a class of combinatorial objects and p a parameter of enumeration on $\mathcal{O}$ taking values in $\mathbb{N}$: we denote $\mathcal{O}_n = \{s \in \mathcal{O}\colon p(s) = n\}$. An operator θ on $\mathcal{O}$ is a function from $\mathcal{O}_n$ to $2^{\mathcal{O}_{n+1}}$, where $2^{\mathcal{O}_{n+1}}$ is the power set of $\mathcal{O}_{n+1}$.

Proposition 3.1. *Let θ be an operator on $\mathcal{O}$. If θ satisfies the following conditions*:

(1) $\forall Y \in \mathcal{O}_{n+1}\ \exists X \in \mathcal{O}_n$ *such that* $Y \in \theta(X)$,
(2) *if* $X_1, X_2 \in \mathcal{O}_n$ *and* $X_1 \neq X_2$ *then* $\theta(X_1) \cap \theta(X_2) = \emptyset$,

then the family of sets $\mathscr{F}_{n+1} = \{\theta(X)\colon \forall X \in \mathcal{O}_n\}$ *is a partition of* $\mathcal{O}_{n+1}$.

Given a class $\mathcal{O}$ of combinatorial objects, if we are able to define an operator θ satisfying conditions (1) and (2), then Proposition 3.1 allows us to construct each object $Y \in \mathcal{O}_{n+1}$ from another object $X \in \mathcal{O}_n$ and each $Y \in \mathcal{O}_{n+1}$ is given by only one $X \in \mathcal{O}_n$. If we have an operator on $\mathcal{O}$ which satisfies conditions (1) and (2), we obtain

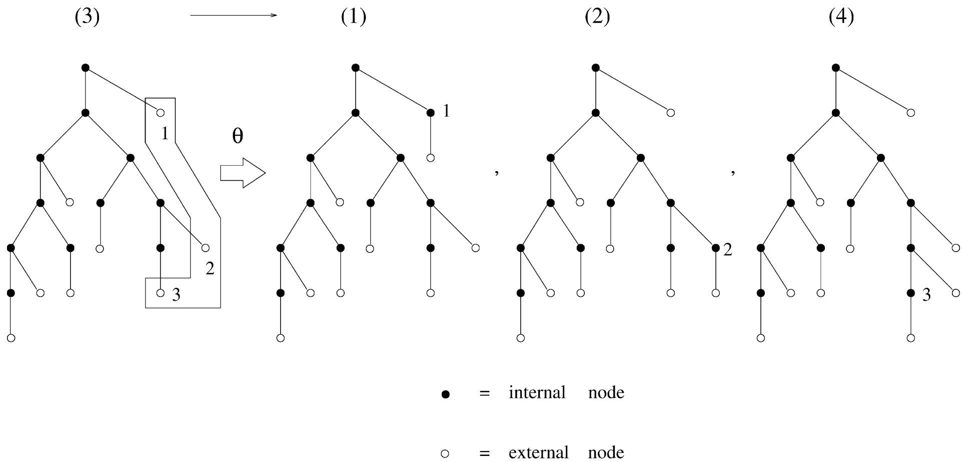

Fig. 1. The operator θ on a 1–2 tree.

a recursive description of the elements of $\mathcal{O}$. In some cases, this recursive description allows us to deduce a functional equation verified by the generating function of $\mathcal{O}$.

Example 3.1. A 1–2 tree is an ordered tree (in the sense used by Knuth [18, p. 305]) whose internal nodes all have degree 0, 1 or 2. Let $\mathcal{O}$ be the class of 1–2 trees and p the number of their internal nodes. Let $P \in \mathcal{O}$ and $\mathscr{A}(P)$ be the set of external nodes that follow the last internal node in the preorder traversal. The operator θ replaces each external node in $\mathscr{A}(P)$ by an internal one (see Fig. 1). It is easy to prove that θ satisfies Proposition 3.1. For some detailed definitions and proofs see [4].

4. Permutations with one forbidden subsequence of increasing length

In this section, we define an operator θ on the class $\mathscr{M}(j)$, $j \geqslant 1$, which satisfies Proposition 3.1. We obtain a recursive description of the objects in $\mathscr{M}(j)$ and deduce the set of functional equations verified by its generating function according to the permutations' length, number of right minima and inversions.

Proposition 4.1. *Let π be a permutation in $\mathscr{M}(j)$, $j \geqslant 1$. If s is an active site of π, each site to its right is also active.*

Proof. Let s be an active site of π and t be a site on its right. If we assume that t is not active, by inserting $(n+1)$ into t, we obtain a permutation containing 321 or $(j+2)\bar{1}(j+3)2\ldots(j+1)$.

If the insertion of $(n+1)$ into t produces a subsequence $(n+1)\pi(i_1)\pi(i_2)$ of type 321, the insertion of $(n+1)$ into s produces the same subsequence, and so s is not active.

$$\begin{array}{cccccccc} & & a(\pi) & & a(\pi)-1\ \ldots\ 3 & & 2 & 1 \\ \pi = \pi(1)\ldots\ldots\pi(n-a(\pi)+1) & - & \pi(n-a(\pi)+2) & - \ldots\ldots - & \pi(n-1) & - & \pi(n) & - \\ & s_{n-a(\pi)+1} & & s_{n-a(\pi)+2}\ \cdots\ s_{n-2} & & s_{n-1} & & s_n \end{array}$$

Fig. 2. A permutation $\pi \in \mathscr{M}_n(j)$ with $a(\pi)$ active sites.

If the insertion of $(n+1)$ into t gives us a permutation containing $(j+2)\bar{1}(j+3)2\ldots(j+1)$, we obtain a subsequence $\pi(i_1)(n+1)\pi(i_2)\ldots\pi(i_{j+1})$ of type $(j+2)(j+3)2\ldots(j+1)$, and there is no element in π between $\pi(i_1)$ and $(n+1)$ corresponding to the 1 of $(j+2)1(j+3)2\ldots(j+1)$. Moreover, $\pi(i_1)$ is on the left of s: as a matter of fact, if we assume that $\pi(i_1)$ is on the right of s, by inserting $(n+1)$ into s, we obtain the subsequence $(n+1)\pi(i_1)\pi(i_2)$ which is of type 321. Consequently, the insertion of $(n+1)$ into s produces a sequence of type $(j+2)\bar{1}(j+3)2\ldots(j+1)$. Therefore, s is not active and contradicts our hypothesis. $\square$

Let π be a permutation in $\mathscr{M}_n(j)$, $j\geqslant 1$. From Proposition 4.1, it follows that the set $\mathscr{A}(\pi)$ of active sites of π is $\{s_{n-a(\pi)+1},\ldots,s_n\}$ (see Fig. 2). We define the operator θ on π as follows:

$$\theta(\pi)=\{\sigma\in\mathscr{M}_{n+1}(j)\,|\,\sigma=\pi(1)\cdots\pi(n-a(\pi)+i)(n+1)\cdots\pi(n),\\ \forall i=1,\ldots,a(\pi)\}.$$

We can easily prove that the operator θ satisfies conditions (1) and (2) of Proposition 3.1.

The number of active sites in π is equal to the number of permutations obtained by performing the operator θ on π (i.e., $|\mathscr{A}(\pi)|=|\theta(\pi)|$). We number the sites in $\mathscr{A}(\pi)$ from right to left in increasing order (see Fig. 2). If θ inserts $(n+1)$ into the ith active site of $\pi\in\mathscr{M}_n(j)$, $1\leqslant i\leqslant a(\pi)$, we obtain $\pi'\in\mathscr{M}_{n+1}(j)$, and its set of active sites is such that:

(a) if $i=1$, then $\mathscr{A}(\pi')=\{s_{n(\pi)-(a(\pi)-1)},\ldots,s_{n(\pi)},s_{n(\pi)+1}\}$ and $|\mathscr{A}(\pi')|=a(\pi)+1$;
(b) if $2\leqslant i\leqslant\min\{a(\pi),j\}$, then $\mathscr{A}(\pi')=\{s_{n(\pi)-(i-2)},\ldots,s_{n(\pi)},s_{n(\pi)+1}\}$ and $|\mathscr{A}(\pi')|=i$;
(c) if $\min\{a(\pi),j\}<i\leqslant a(\pi)$, then $\mathscr{A}(\pi')=\{s_{n(\pi)-(i-3)},\ldots,s_{n(\pi)},s_{n(\pi)+1}\}$ and $|\mathscr{A}(\pi')|=i-1$.

Therefore, if π has k active sites, then θ constructs k permutations $\sigma_1,\sigma_2,\ldots,\sigma_k$ whose number of active sites is:

- $k+1,2,\ldots,k$, for $a(\pi)\leqslant j$,
- $k+1,2,3,\ldots,j,j,\ldots,k-1$, for $a(\pi)>j$.

From this recursive description, we deduce a functional equation verified by the generating function of the $\mathscr{M}(j)$ permutations according to their length, number of active sites, inversions and right minima:

$$M^{(j)}(x,y,s,q)=\sum_{\pi\in\mathcal{M}(j)} x^{n(\pi)}y^{m(\pi)}s^{a(\pi)}q^{i(\pi)}.$$

If θ inserts the subsequent element into the ith active site of π, $1\leqslant i\leqslant a(\pi)$, we obtain a permutation $\pi'\in\mathcal{M}(j)$ and the parameters change as follows:

(a) if $i=1$, then

$$n(\pi')=n(\pi)+1,\quad m(\pi')=m(\pi)+1,\quad a(\pi')=a(\pi)+1,\quad i(\pi')=i(\pi);$$

(b) if $2\leqslant i\leqslant \min\{a(\pi),j\}$, then

$$n(\pi')=n(\pi)+1,\quad m(\pi')=m(\pi),\quad a(\pi')=i,\quad i(\pi')=i(\pi)+i-1;$$

(c) if $\min\{a(\pi),j\}+1\leqslant i\leqslant a(\pi)$, then

$$n(\pi')=n(\pi)+1,\quad m(\pi')=m(\pi),\quad a(\pi')=i-1,\quad i(\pi')=i(\pi)+i-1.$$

At this point, we can translate this recursive construction into a functional equation verified by $M^{(j)}(x,y,s,q)$.

Let us note that:

- if $\boldsymbol{j}=1$ then point (b) does not hold. The permutation of length one has 2 active sites and is represented by xys^2. From the recursive construction, we obtain the following functional equation:

$$M^{(1)}(x,y,s,q)=\frac{xys^2}{(1-xys)}+\frac{xsq\,M^{(1)}(x,y,1,q)}{(1-xys)(1-sq)}-\frac{x\,M^{(1)}(x,y,sq,q)}{(1-xys)(1-sq)};\tag{1}$$

- if $\boldsymbol{j}=\infty$ (i.e., $\mathcal{M}_n(\infty)=S_n(321)$), then point (c) does not hold and we obtain the following functional equation:

$$M^{(\infty)}(x,y,s,q)=\frac{xys^2}{(1-xys)}+\frac{xs^2q\,M^{(\infty)}(x,y,1,q)}{(1-xys)(1-sq)}-\frac{xs\,M^{(\infty)}(x,y,sq,q)}{(1-xys)(1-sq)}.\tag{2}$$

We refer to [1] for the solution of these equations and only wish to point out that:

$$M^{(1)}(x,1,1,1)=\frac{1-x-2x^2-\sqrt{-3x^2-2x+1}}{2x^2},$$

$$M^{(\infty)}(x,1,1,1)=\frac{1-2x-\sqrt{1-4x}}{2x},$$

which means that, the number of n-length permutations in $\mathcal{M}(1)$ is equal to the nth Motzkin number while the nth Catalan number counts the n-length permutations in $\mathcal{M}(\infty)$. Therefore, the sequence of numbers enumerating the permutations in $\mathcal{M}(j)$, $j\geqslant 2$, according to their length, lies between the sequences of Motzkin and Catalan numbers.

Let us now take the case of $j\geqslant 2$ into consideration. By translating the construction into formulae, we obtain a set of functional equations. We partition the set $\mathcal{M}(j)$ into j subsets:

$$\mathcal{M}(j,2),\mathcal{M}(j,3),\dots,\mathcal{M}(j,j),\mathcal{M}(j,>),$$

where

$$\mathcal{M}(j,i)=\{\pi\in\mathcal{M}(j)\,|\,a(\pi)=i\}\quad\text{and}\quad\mathcal{M}(j,>)=\{\pi\in\mathcal{M}(j)\,|\,a(\pi)>j\}.$$

We get the following proposition:

Proposition 4.2. *The generating function* $M^{(j)}(x,y,s,q)$ *of the* $\mathcal{M}(j)$ *permutations,* $j\geqslant 2$, *is such that*

$$M^{(j)}(x,y,s,q)=\sum_{i=2}^{j}M^{(j,i)}(x,y,1,q)s^i+M^{(j,>)}(x,y,s,q), \tag{3}$$

where the $M^{(j,i)}(x,y,s,q)$ *satisfy*:

$$M^{(j,2)}(x,y,s,q)=xs^2[y+qM^j(x,y,1,q)],\quad j>2,$$

$$M^{(j,i)}(x,y,s,q)=\frac{xys^i}{1-xq^{i-1}}M^{(j,i-1)}(x,y,1,q)+\frac{xq^{i-1}s^i}{1-xq^{i-1}}(M^{(j,i+1)}(x,y,1,q)$$
$$+\cdots+M^{(j,>)}(x,y,1,q)),\quad 3\leqslant i\leqslant j-1,$$

$$M^{(j,j)}(x,y,s,q)=\frac{xys^j}{1-xq^{j-1}}M^{(j,j-1)}(x,y,1,q)+\frac{xq^{j-1}s^j}{1-xq^{j-1}}(1+q)M^{(j,>)}(x,y,1,q),$$

$$M^{(j,>)}(x,y,s,q)=\frac{xys}{1-xys}M^{(j,j)}(x,y,s,q)+\frac{x(sq)^{j+1}}{(1-xys)(1-sq)}M^{(j,>)}(x,y,1,q)$$
$$-\frac{x}{(1-xys)(1-sq)}M^{(j,>)}(x,y,sq,q).$$

We set $M^{(j,1)}(x,y,1,q)=1$ in the case of $j=2$.

Proof. Eq. (3) immediately follows from the definition of $\mathcal{M}(j,i)$. Let $\pi\in\mathcal{M}(j)$ and $\mathcal{A}(\pi)=\{s_{n(\pi)-(a(\pi)-1)},\ldots,s_{n(\pi)-1},s_{n(\pi)}\}$: we obtain a permutation $\pi'\in\mathcal{M}(j)$ by performing the operator θ on π and obtain the following:

- $\pi'\in\mathcal{M}(j,2)$, $j>2$, is obtained from π by an insertion into $s_{n(\pi)-1}$;
- $\pi'\in\mathcal{M}(j,i)$, $3\leqslant i\leqslant j-1$, is obtained from:
  - ∘ $\pi\in\mathcal{M}(j,i-1)$ by an insertion into $s_{n(\pi)}$,
  - ∘ $\pi\in\mathcal{M}(j,i)\cup\cdots\cup\mathcal{M}(j,j)\cup\mathcal{M}(j,>)$ by an insertion into $s_{n(\pi)-(i-1)}$;
- $\pi'\in\mathcal{M}(j,j)$ is obtained from:
  - ∘ $\pi\in\mathcal{M}(j,j-1)$ by an insertion into $s_{n(\pi)}$,
  - ∘ $\pi\in\mathcal{M}(j,j)$ by an insertion into $s_{n(\pi)-(j-1)}$,
  - ∘ $\pi\in\mathcal{M}(j,>)$ by an insertion into both $s_{n(\pi)-(j-1)}$ and $s_{n(\pi)-j}$;
- $\pi'\in\mathcal{M}(j,>)$ is obtained from:
  - ∘ $\pi\in\mathcal{M}(j,j)\cup\mathcal{M}(j,>)$ by an insertion into $s_{n(\pi)}$,
  - ∘ $\pi\in\mathcal{M}(j,>)$ by an insertion into $s_{n(\pi)-(i-1)}$, $j+2\leqslant i\leqslant a(\pi)$.

The permutation of length one belongs to $\mathscr{M}(j,2)$ and is represented by xys^2. The set of equations:

$$M^{(j,2)}(x,y,s,q)=s^2[xy+xqM^{(j)}(x,y,1,q)], \quad j>2,$$

$$M^{(j,i)}(x,y,s,q)=s^i[xyM^{(j,i-1)}(x,y,1,q)+xq^{i-1}(M^{(j,i)}(x,y,1,q) + \cdots + M^{(j,>)}(x,y,1,q))], \quad 3\leqslant i\leqslant j-1,$$

$$M^{(j,j)}(x,y,s,q)=s^j[xyM^{(j,j-1)}(x,y,1,q)+xq^{j-1}M^{(j,j)}(x,y,1,q) + xq^{j-1}(1+q)M^{(j,>)}(x,y,1,q)]$$

$$M^{(j,>)}(x,y,s,q)=xysM^{(j,j)}(x,y,s,q)+\sum_{\pi\in M^{(j,>)}} x^{n(\pi)+1}y^{m(\pi)+1}s^{a(\pi)+1}q^{i(\pi)} + \sum_{\pi\in M^{(j,>)}}\sum_{i=j+2}^{a(\pi)} x^{n(\pi)+1}y^{m(\pi)}s^{i-1}q^{i(\pi)+i-1}$$

follows from the previous discussion and so the proposition is proved. □

The fourth equation in Proposition 4.2 can be solved by Bousquet-Mélou's lemma [9]:

Lemma 4.3. *Let $\mathscr{R}=\mathbb{R}[[x,y,s,q]]$ be the algebra of formal power series in variables x,y,s and q with real coefficients, and let $\mathscr{S}$ be a sub-algebra of $\mathscr{R}$ such that the series converge for $s=1$. Let $A(s)=A(x,y,s,q)$ be a formal power series in $\mathscr{S}$. We assume that*

$$A(s)=xe(s)+xf(s)A(1)+xg(s)A(sq),$$

where $e(s)$, $f(s)$ and $g(s)$ are some given power series in $\mathscr{A}$. Then

$$A(s)=\frac{J_1(s)+J_1(1)J_0(s)-J_1(s)J_0(1)}{1-J_0(1)},$$

where $J_1(s)=\sum_{n\geqslant 0}x^{n+1}g(s)g(sq)\dots g(sq^{n-1})e(sq^n)$ and $J_0(s)=\sum_{n\geqslant 0}x^{n+1}g(s)g(sq)\dots g(sq^{n-1})f(sq^n)$.

By means of Lemma 4.3 and the fourth equation in Proposition 4.2, we get the following:

Proposition 4.4. *The generating function $M^{(j,>)}(x,y,s,q)$ is given by*

$$M^{(j,>)}(x,y,s,q)=\frac{J_1(s)J_0(1)-J_1(1)J_0(s)+J_1(1)}{J_0(1)},$$

where

$$J_1(x,y,s,q)=\sum_{n\geqslant 0}\frac{(-1)^n x^{n+1} y s^{j+1} q^{n(j+1)}}{(xys,q)_{n+1}(sq,q)_n} M^{(j,j)}(x,y,1,q),$$

$$J_0(x,y,s,q)=1+\sum_{n\geqslant 0}\frac{(-1)^{n+1} x^{n+1} s^{j+1} q^{(n+1)(j+1)}}{(xys,q)_{n+1}(sq,q)_{n+1}},$$

and

$$(a,q)_n=\prod_{k=0}^{n-1}(1-aq^k).$$

Let us now take the generating function $M^{(j,>)}(x,y,1,q)$ into consideration. We denote the functions $M^{(j,>)}(x,y,1,q)$, $M^{(j,j)}(x,y,1,q)$, $J_0(x,y,1,q)$ and $J_1(x,y,1,q)$ by $M^{(j,>)}(x,y,q)$, $M^{(j,j)}(x,y,q)$, $J_0(x,y,q)$ and $J_1(x,y,q)$, respectively. From Proposition 4.4, it follows that

$$M^{(j,>)}(x,y,q)=f(x,y,q)M^{(j,j)}(x,y,q), \tag{4}$$

where

$$f(x,y,q)=y\frac{\sum_{n\geqslant 0}\frac{(-1)^n x^{n+1} q^{n(j+1)}}{(xy,q)_{n+1}(q,q)_n}}{\sum_{n\geqslant 0}\frac{(-1)^n x^n q^{n(j+1)}}{(xy,q)_n(q,q)_n}}.$$

Theorem 4.5. *The generating function $M^{(j)}(x,y,q)$ of the $\mathscr{M}(j)$ permutations is such that*:

- $M^{(2)}(x,y,q)=\dfrac{xy(1+f(x,y,q))}{1-xq-xq(1+q)f(x,y,q)}$;
- $M^{(j)}(x,y,q)=\dfrac{xy(1-xq^2)\Delta_j(x,y,q)}{(1-xq)(1-xq^2)\Delta_j(x,y,q)+xy\Delta_{j-1}(x,y,q)},\quad j\geqslant 3,$

  where $\Delta_j(x,y,q)=c_1(x,y,q)\lambda_1^j(x,y,q)+c_2(x,y,q)\lambda_2^j(x,y,q)$ *being*

$$\lambda_1(x,y,q)=\frac{1}{2}\left[-\left(1+\frac{xy}{1-xq^2}\right)+\sqrt{\left(1+\frac{xy}{1-xq^2}\right)^2-\frac{4xy}{(1-xq^2)(1-xq^3)}}\right]$$

and

$$\lambda_2(x,y,q)=\frac{1}{2}\left[-\left(1+\frac{xy}{1-xq^2}\right)-\sqrt{\left(1+\frac{xy}{1-xq^2}\right)^2-\frac{4xy}{(1-xq^2)(1-xq^3)}}\right].$$

The functions $c_1(x,y,q)$, $c_2(x,y,q)$ *satisfy*:

$$\begin{aligned} c_1(x,y,q)\lambda_1^2(x,y,q)+c_2(x,y,q)\lambda_2^2(x,y,q) &= 1+f(x,y,q),\\ c_1(x,y,q)\lambda_1^3(x,y,q)+c_2(x,y,q)\lambda_2^3(x,y,q) &= f(x,y,q)\frac{xq^{j-1}(1+q)-xy}{1-xq^{j-1}}\\ &\quad -\frac{1+xy-xq^{j-1}}{1-xq^{j-1}}. \end{aligned}$$

Proof.

- The generating function $M^{(2)}(x,y,s,q)$ satisfies the following equations:

$$\begin{aligned}
M^{(2,2)}(x,y,s,q) &= \frac{xys^2}{1-xq} + \frac{xs^2q}{1-xq}(1+q)M^{(2,>)}(x,y,1,q),\\
M^{(2,>)}(x,y,s,q) &= \frac{xys}{1-xys}M^{(2,2)}(x,y,s,q) + \frac{x(sq)^3}{(1-xys)(1-sq)}\\
&\quad \times M^{(2,>)}(x,y,1,q) - \frac{x}{(1-xys)(1-sq)}M^{(2,>)}(x,y,sq,q),\\
M^{(2)}(x,y,s,q) &= M^{(2,2)}(x,y,s,q) + M^{(2,>)}(x,y,s,q).
\end{aligned} \tag{5}$$

  The thesis is obtained by some easy substitutions involving the solution of this set of equations and Eq. (4).
- Let $A_{j-i}(x,y,q)$ be an array of dimension $(j-i)\times(j-i)$ defined by:

$$A_{j-i}(x,y,q)$$
$$= \begin{pmatrix}
1 & 1 & 1 & \ldots.. & 1 & 1 & 1 & 1\\
\frac{xy}{1-xq^{i+2}} & -1 & \frac{xq^{i+2}}{1-xq^{i+2}} & \cdots & \frac{xq^{i+2}}{1-xq^{i+2}} & \frac{xq^{i+2}}{1-xq^{i+2}} & \frac{xq^{i+2}}{1-xq^{i+2}} & \frac{xq^{i+2}}{1-xq^{i+2}}\\
 & \ddots & & & \vdots & \vdots & \vdots & \vdots\\
 & & \ddots & & \vdots & \vdots & \vdots & \vdots\\
 & & & \ddots & \frac{xy}{1-xq^{j-2}} & -1 & \frac{xq^{j-2}}{1-xq^{j-2}} & \frac{xq^{j-2}}{1-xq^{j-2}}\\
 & & & & & \frac{xy}{1-xq^{j-1}} & -1 & \frac{xq^{j-1}(1+q)}{1-xq^{j-1}}\\
 & & & & & & -f(x,y,q) & 1
\end{pmatrix},$$

$$i \geqslant 0. \tag{6}$$

Its first and last row always appear because they do not depend on $(j-i)$; on the contrary, the $((j-i)-k)$th row is in $A_{j-i}(x,y,q)$ if and only if $(j-i)-2 \geqslant k \geqslant 1$. This means that

$$A_2(x,y,q) = \begin{pmatrix} 1 & 1\\ -f(x,y,q) & 1 \end{pmatrix}$$

and

$$A_3(x,y,q) = \begin{pmatrix} 1 & 1 & 1\\ \frac{xy}{1-xq^{j-1}} & -1 & \frac{xq^{j-1}(1+q)}{1-xq^{j-1}}\\ 0 & -f(x,y,q) & 1 \end{pmatrix}.$$

The set of equations given by Proposition 4.2 and Eq. (4) is represented by the matrix expression

$$A_j(x,y,q)X_j(x,y,q) = Y_j(x,y,q), \tag{7}$$

where

$$X_j(x,y,q)=\begin{pmatrix} M^{(j,2)}(x,y,q)\\ M^{(j,3)}(x,y,q)\\ \vdots\\ M^{(j,j)}(x,y,q)\\ M^{(j,>)}(x,y,q)\end{pmatrix},\qquad Y_j(x,y,q)=\begin{pmatrix} M^{(j)}(x,y,q)\\ 0\\ \vdots\\ 0\\ 0\end{pmatrix}$$

and $A_j(x,y,q)$ is defined by (6) for $i=0$. It is easy to verify that the determinant $\Delta_{j-i}(x,y,q)$ of (6) satisfies the recursive relation:

$$\begin{aligned}&\Delta_{j-i}(x,y,q)+\left(1+\frac{xy}{1-xq^{i+2}}\right)\Delta_{j-(i+1)}(x,y,q)+\frac{xy}{(1-xq^{i+2})(1-xq^{i+3})}\\&\quad\times\Delta_{j-(i+2)}(x,y,q)=0,\quad 0\leqslant i\leqslant j-4,\\&\Delta_2(x,y,q)=1+f(x,y,q)\quad(i=j+2),\\&\Delta_3(x,y,q)=f(x,y,q)\frac{xq^{j-1}(1+q)-xy}{1-xq^{j-1}}-\frac{1+xy-xq^{j-1}}{1-xq^{j-1}}\quad(i=j+3).\end{aligned}\tag{8}$$

By using standard solution techniques, we get

$$\Delta_j(x,y,q)=c_1(x,y,q)\lambda_1^j(x,y,q)+c_2(x,y,q)\lambda_2^j(x,y,q)$$

for $i=0$; where $\lambda_1(x,y,q)$ and $\lambda_2(x,y,q)$ are the solutions of the equation:

$$\lambda^2+\left(1+\frac{xy}{1-xq^2}\right)\lambda+\left(\frac{xy}{(1-xq^2)(1-xq^3)}\right)=0$$

and $c_1(x,y,q)$, $c_2(x,y,q)$ the terms given by the initial conditions of the recursive relation (8).

The solution of (7) gives

$$M^{(j,2)}(x,y,q)=M^{(j)}(x,y,q)\left(1+\frac{xy}{1-xq^2}\frac{\Delta_{j-1}(x,y,q)}{\Delta_j(x,y,q)}\right),\tag{9}$$

moreover, from Proposition 4.2, $M^{(j,2)}(x,y,q)$ yields

$$M^{(j,2)}(x,y,q)=x[y+qM^{(j)}(x,y,q)],$$

and so

$$M^{(j)}(x,y,q)=\frac{xy}{(1-xq)+\frac{xy}{q-xq^2}\frac{1}{\frac{\Delta_j(x,y,q)}{\Delta_{j-1}(x,y,q)}}},$$

and therefore our thesis is proved. $\square$

Remark 4.1.

- Let us note that

$$\frac{\Delta_{j-i}(x,y,q)}{\Delta_{j-(i+1)}(x,y,q)} = -\frac{1+xy-xq^{i+2}}{1-xq^{i+2}} - \frac{xy}{(1-xq^{i+2})(1-xq^{i+3})\frac{\Delta_{j-(i+1)}(x,y,q)}{\Delta_{j-(i+2)}(x,y,q)}},$$

  thus we have

$$\lim_{j\to\infty} M^{(j)}(x,y,q) = \cfrac{xy}{1-xq-\cfrac{xy}{1+xy-xq^2-\cfrac{xy}{1+xy-xq^3-\cfrac{xy}{\vdots}}}}.$$

  If $y=q=1$, we obtain the continued fraction representing the generating function of Catalan numbers less 1, that is $(1-\sqrt{1-4x}/2x)-1=C(x)-1=\sum_{n\geqslant 1} S_n(321)x^n$.
- This continued fraction represents the generating function of Catalan permutations enumerated according to their length, number of right minima and inversions and is an alternative to the one obtained by developing the functional equation that appears in [1]. Consequently, we obtain the following identities:

$$\frac{\sum_{n\geqslant 0} \frac{(-1)^n x^{n+1} q^{\frac{n(n+3)}{2}}}{(q;q)_n (xy;q)_{n+1}}}{\sum_{n\geqslant 0} \frac{(-1)^n x^n q^{\frac{n(n+1)}{2}}}{(q;q)_n (xy;q)_n}} = \cfrac{xy}{1-xq-\cfrac{xy}{1+xy-xq^2-\cfrac{xy}{1+xy-xq^3-\cfrac{xy}{\vdots}}}}$$

$$= \cfrac{xy}{1-xq-xy-\cfrac{x^2yq^2}{1-xq^2-xyq-\cfrac{x^2yq^4}{1-xq^3-xyq^2-\cfrac{x^2yq^6}{\vdots}}}}.$$

Theorem 4.5 gives us the generating function for $\mathscr{M}(j)$ permutations according to various parameters. At the moment, we wish to treat the enumeration of $\mathscr{M}(j)$ permutations only according to their length. The generating function $M^{(j)}(x,1,1)$ is obtained from Theorem 4.5 by setting $y=q=1$ and by substituting $f(x,y,q)$ with $\tilde{f}(x)=(1-x-\sqrt{1-2x-3x^2})/2x$. The function $\tilde{f}(x)$ is obtained by the following steps:

(1) from Proposition 4.4, we obtain the following equalities by some computations:

$$\frac{J_1(x,y,q)}{M^{(j,j)}(x,y,q)} - xyJ_0(x,y,q) = \frac{x^2y^2}{1-xy}J_0(xq,y,q),$$

$$(x^2yq^{j+1}+xy-1)\frac{J_1(x,y,q)}{M^{(j,j)}(x,y,q)} + xyJ_0(x,y,q) = -\frac{x^3y^2q^{j+1}}{1-xy}\frac{J_1(xq,y,q)}{M^{(j,j)}(xq,y,q)};$$

(2) from Proposition 4.4, we deduce that

$$xq^{j+1}M^{(j,>)}(xq,y,q) = \frac{(1-xy-x^2yq^{j+1})\frac{M^{(j,>)}(x,y,q)}{M^{(j,j)}(x,y,q)} - xy}{\frac{M^{(j,>)}(x,y,q)}{M^{(j,j)}(x,y,q)} - xy} M^{(j,j)}(xq,y,q); \quad (10)$$

(3) by setting $y=q=1$ in Eq. (10), we obtain

$$x(M^{(j,>)}(x,1,1))^2 - M^{(j,j)}(x,1,1)(1-x)M^{(j,>)}(x,1,1) + x(M^{(j,j)}(x,1,1))^2 = 0; \tag{11}$$

(4) by solving Eq. (11), we obtain

$$M^{(j,>)}(x,1,1) = \tilde{f}(x)M^{(j,j)}(x,1,1), \quad \text{with } \tilde{f}(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x}.$$

In order to simplify the computations, we use the generating function $\bar{M}^{(j)}(x)$ of the $\mathscr{M}(j)$ permutations according to their length, including the empty permutation's length; consequently $M^{(j)}(x) = M^{(j)}(x,1,1) + 1$. The generating function $M^{(j)}(x,1,1)$ which can be deduced from Theorem 4.5 allows us to verify the equality:

$$\bar{M}^{(j)}(x) = \frac{1}{1 - x\bar{M}^{(j-1)}(x)}. \tag{12}$$

- If $j=\infty$, then (12) reduces to

$$\bar{M}^{(\infty)}(x) = \frac{1}{1 - x\bar{M}^{(\infty)}(x)},$$

  which is the functional equation verified by the generating function of Catalan numbers.
- Otherwise, we assume that $\bar{M}^{(j-1)}(x)$ satisfies the functional equation:

$$\bar{M}^{(j-1)}(x) = c_{j-1}(x) + b_{j-1}(x)\bar{M}^{(j-1)}(x) + a_{j-1}(x)(\bar{M}^{(j-1)}(x))^2 \tag{13}$$

  and we look for the expression of $c_j(x)$, $b_j(x)$ and $a_j(x)$ satisfying

$$\bar{M}^{(j)}(x) = c_j(x) + b_j(x)\bar{M}^{(j)}(x) + a_j(x)(\bar{M}^{(j)}(x))^2 \tag{14}$$

  in a recursive way.

  Let us note that $c_1(x)=1$, $b_1(x)=x$ and $a_1(x)=x^2$ because $\bar{M}^{(1)}(x)$ verifies

$$\bar{M}^{(1)}(x) = 1 + x\bar{M}^{(1)}(x) + x^2(\bar{M}^{(1)}(x))^2.$$

If we substitute $\bar{M}^{(j-1)}(x) = (\bar{M}^{(j)}(x)-1)/x\bar{M}^{(j)}(x)$ in (13), we obtain a functional equation satisfied by $\bar{M}^{(j)}(x)$. We want to make this equation exactly the same as (14) in order to obtain the recursive definition of the terms $c_j(x)$, $b_j(x)$ and $a_j(x)$. After some computations we get:

$$a_j(x) = \frac{x\bigl(-x^2(C(x))^2(1-C(x))(x^2(C(x))^4)^j-(1-3x-2x^2)(C(x))^2(x(C(x))^2)^j+x\bigr)}{(1-4x)(C(x))^2(1-C(x))^j}, \quad j>2,$$

$$b_j(x) = \frac{-2x^3(C(x))^2(x^2(C(x))^4)^j+(1-3x-2x^2)C(x)(x(C(x))^2)^j-2x^2}{(1-4x)C(x)(1-C(x))^j}, \quad j>2,$$

$$c_j(x) = \frac{(x^2(x^2(C(x))^4)^j-(1-3x-2x^2)(x(C(x))^2)^j+x^2}{(1-4x)(1-C(x))^j}, \quad j>2,$$

$$a_1(x) = x^2,$$

Class of Permutations	Forbidden Subsequences	Number Sequences
$\mathcal{M}(1)$	$321, 3\bar{1}42$	1 2 4 9 21 . . .
$\mathcal{M}(2)$	$321, 4\bar{1}523$	1 2 5 13 35 . . .
$\mathcal{M}(3)$	$321, 5\bar{1}6234$	1 2 5 14 41 . . .
$\mathcal{M}(4)$	$321, 6\bar{1}72345$	1 2 5 14 42 . . .
⋮	⋮	⋮
$\mathcal{M}(\infty)$	321	1 2 5 14 42 . . . $\frac{1}{n+1}\binom{2n}{n}$ . . .

Fig. 3. First numbers of the sequences enumerating the permutations in $\mathscr{M}(j)$.

$$b_1(x) = x,$$

$$c_1(x) = 1,$$

in which $C(x) = M^{(\infty)}(x) = (1 - \sqrt{1-4x})/2x$.

The final result follows from Eq. (14):

$$\bar{M}^{(j)}(x) = \frac{1 - b_j(x) + (-1)^j\sqrt{(1-b_j(x))^2 - 4a_j(x)c_j(x)}}{2a_j(x)}.$$

5. Conclusions

Our main aim in this work is to give an exhaustive description of some classes of permutations with a barred forbidden subsequence that orderly increases in length. The classes of permutations described in this paper are enumerated by numbers lying between the Motzkin and the Catalan numbers. We view the number sequences obtained as providing a ‘discrete continuity’ between the Motzkin and the Catalan sequences (see Fig. 3): we find the well-known numbers of the left factors in Motzkin words as a special case. We are presently working on the construction of some particular mesh lattices on which the n-area underdiagonal directed animals are in bijection with the $\mathscr{M}_n(j)$ family of permutations. This is suggested by the fact that the Motzkin permutations are in bijection with the underdiagonal directed animals on the square lattice and the Catalan permutations are in bijection with the underdiagonal directed animals on the triangular lattice [5]. A further step consists in translating the classical parameters of permutation enumeration into some parameters of directed animals and vice-versa. Since it is well known that the asymptotic values for $\mathscr{M}_n(1)$, $\mathscr{M}_n(2)$ and

$\mathscr{M}_n(\infty)$ are $(3^{n+1}/2n)\sqrt{3/\pi n}$, $3^{n+1}\sqrt{3/\pi(n+2)}$ and $[4^n/(n+1)]1/\sqrt{\pi n}$, respectively; it would be interesting to discover the asymptotic value for $|\mathscr{M}_n(j)|$, depending on the parameter j.

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