Gončarov Polynomials and Parking Functions

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Abstract

Let $\mathbf{u}$ be a sequence of non-decreasing positive integers. A $\mathbf{u}$-parking function of length n is a sequence $(x_1, x_2, \ldots, x_n)$ whose order statistics (the sequence $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ obtained by rearranging the original sequence in non-decreasing order) satisfy $x_{(i)} \leq u_i$. The Gončarov polynomials $g_n(x; a_0, a_1, \ldots, a_{n-1})$ are polynomials defined by the biorthogonality relation:

$$\varepsilon(a_i) D^i g_n(x; a_0, a_1, \ldots, a_{n-1}) = n! \delta_{in},$$

where $\varepsilon(a)$ is evaluation at a. Gončarov polynomials form a "natural basis" of polynomials for working with $\mathbf{u}$-parking functions. For example, the number of $\mathbf{u}$-parking functions of length n is $(-1)^n g_n(0; u_1, u_2, \ldots, u_n)$. Gončarov polynomials also satisfy a linear recursion obtained by expanding x^n as a linear combination of Gončarov polynomials. The combinatorial structure underlying this recursion is a decomposition of an arbitrary sequence of positive integers into two subsequences: a "maximum" $\mathbf{u}$-parking function and a subsequence consisting of terms of higher values. From this combinatorial decomposition, we derive linear recursions for sum enumerators, expected sums of $\mathbf{u}$-parking functions, and higher moments of sums of $\mathbf{u}$-parking functions. These recursions yield explicit formulas for these quantities in terms of Gončarov polynomials.

Key Words: Gončarov polynomials, parking functions, linear recurrence, sum enumerators, factorial moments

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1 Introduction

We shall think of finite sequences $(x_1, x_2, \ldots, x_n)$ interchangably as sequences and functions with domain $\{1, 2, \ldots, n\}$. If $(x_1, x_2, \ldots, x_n)$ is a sequence of real numbers of length n, then the sequence $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ of *order statistics* is obtained by rearranging the original sequence $(x_1, x_2, \ldots, x_n)$ in non-decreasing order. Let $\mathbf{u}$ be a non-decreasing sequence $(u_1, u_2, u_3, \ldots)$ of positive integers. A $\mathbf{u}$-*parking function* of length n is a sequence $(x_1, x_2, \ldots, x_n)$ of length n whose sequence of order statistics satisfies $x_{(i)} \leq u_i$.

We shall call $(1, 2, 3, \ldots)$-parking functions *ordinary* parking functions. Intuitively, an ordinary parking function can take on as many smaller values as one wishes, but it cannot take on too many larger values. Ordinary parking functions originated in the theory of hashing and searching in computer science (see [11, 9]). They have been extensively studied. In particular, it is known that the number of ordinary parking functions of length n is

$$(n+1)^{n-1},$$

a formula which is closely related to Cayley's formula for the number of labelled trees. This relation with trees had motivated much work in this area, particularly in finding bijections between ordinary parking functions and labelled trees. Less obvious, perhaps, is the observation that the formula is (up to a sign) an evaluation of an Abel polynomial. It is this observation which led us to Gončarov polynomials.

Gončarov polynomials (see [1, 2, 7]) arose in the following special case of Hermite interpolation in numerical analysis.

Gončarov Interpolation. Given two sequences of real or complex numbers $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$, find a polynomial $p(x)$ of degree n such that for each $i, 0 \leq i \leq n$, the ith derivative $p^{(i)}(x)$ evaluated at a_i equals b_i.

The natural basis of polynomials for this interpolation problem is the sequence of Gončarov polynomials defined in Section 3. A special case of this is *Abel interpolation*, where the point a_i is the integer i. The Gončarov polynomials for this case are the Abel polynomials.

The appearance of Abel polynomials in both the enumeration of parking functions and Abel interpolation was one of the motivations behind this paper. We shall show that the enumerative theory of ordinary parking functions can be generalized to $\mathbf{u}$-parking functions using Gončarov polynomials. We hope that it will become evident that the Gončarov polynomials are the natural basis of polynomials for working with parking functions, even in the ordinary case. In particular, we shall give explicit linear recursions which would allow one to compute any specific moment of the sum of a random $\mathbf{u}$-parking function of length n.

The approach in this paper is to apply results about Gončarov polynomials to parking functions. We start with a discussion of a general theory of biorthogonal polynomials in Section 2 and specialize this theory to Gončarov polynomials in Section 3. In Section 4, we present a combinatorial description of the coefficients of Gončarov polynomials in terms of rankings on ordered partitions. The key tool in this paper, a decomposition of an arbitrary sequence of positive integers into two subsequences, a "maximum" $\mathbf{u}$-parking functions of length m and a subsequence all of whose terms are strictly larger than u_m, is given in Section 5. An immediate application yields formulas for the number of parking functions (Section 5). This decomposition also yields results about sum enumerators (Section 6), expected sums (Section 7), and higher moments of sums of parking functions (Sections 11 and 12). In Section 10, we discuss the conjecture that the expected sum is an increasing function of the "gaps" $u_{i+1} - u_i$ in the sequence $\mathbf{u}$. We also derive formulas for the expected sum in the "classical" case when the sequence $\mathbf{u}$ is an arithmetic progression. Two methods are used. The first, involving Abel identities, is presented in Section 8. The second, using a matrix inverse relation, is presented in Section 9. With substantially more work, the matrix method can also be used to obtain formulas for higher moments of sums of classical parking functions. We shall present this in [14]. We end this paper with a brief discussion of variants of parking functions (Section 13) and some historical remarks (Section 14).

We shall use the following notation. If a and b are integers with $a \leq b$, then the *discrete interval* $[a, b]$ is the set $\{a, a+1, a+2, \ldots, b\}$.

2 Sequences of biorthogonal polynomials

We shall need several results about Gončarov polynomials in this paper. Many of these results are special cases of a general algebraic, that is to say, non-analytic, theory of sequences of polynomials biorthogonal to a sequence of linear functionals. Although this theory must be well-known (for some examples, see [1] or [2]), we have not been able to find an explicit description in the literature.

Consider the vector space $\mathcal{P}$ of all polynomials in the variable x over a field F of characteristic zero. Let $D : \mathcal{P} \to \mathcal{P}$ be the differentiation operator. For a scalar a in the field F, let

$$\varepsilon(a) : \mathcal{P} \to F,\ p(x) \mapsto p(a)$$

be the linear functional which evaluates $p(x)$ at a.

Let $\varphi_s(D), s = 0, 1, 2, \ldots$ be a sequence of linear operators on $\mathcal{P}$ of the form

$$\varphi_s(D) = D^s \sum_{r=0}^{\infty} b_{sr} D^r,$$

where the coefficients b_{s0} are assumed to be non-zero. Note that, although $\varphi_s(D)$ are infinite formal sums, they become finite sums when applied to a specific polynomial. Then there exists a unique sequence $p_n(x), n = 0, 1, 2, \ldots$ of polynomials such that $p_n(x)$ has degree n and

$$\varepsilon(0)\varphi_s(D)p_n(x) = n!\delta_{sn}, \tag{2.1}$$

where δ_{sn} is the Kronecker delta. To see this, let

$$p_n(x) = \sum_{k=0}^{n} c_{nk} x^k.$$

Then, for a given index n, the orthogonality relations are equivalent to the following upper triangular system of linear equations in the unknowns $c_{n,0}, c_{n,1}, c_{n,2}, \ldots, c_{n,n}$:

$$\begin{aligned}
b_{00}c_{n0} + b_{01}c_{n1} + 2!b_{02}c_{n2} + 3!b_{03}c_{n3} + \ldots + \quad n!b_{0n}c_{nn} &= 0 \\
b_{10}c_{n1} + 2!b_{11}c_{n2} + 3!b_{12}c_{n3} + \ldots + n!b_{1,n-1}c_{nn} &= 0 \\
2!b_{20}c_{n2} + 3!b_{21}c_{n3} + \ldots + n!b_{2,n-2}c_{nn} &= 0 \\
\ldots \\
n!b_{n0}c_{nn} &= n!.
\end{aligned}$$

This system of linear equations can be solved uniquely for every index n. Hence, the polynomials $p_n(x)$ exist and they are uniquely determined by the orthogonality relations (2.1). Note also that $p_n(x)$ depends only on the operators $\varphi_0(D), \varphi_1(D), \ldots, \varphi_{n-1}(D)$. When solving this system, we need only divide by the diagonal entries b_{s0}. Hence, if we put on the extra assumption that the entries b_{s0} all equal 1, then $p_n(x)$ is monic and the coefficients of $p_n(x)$ are polynomials in the entries b_{sr}.

The polynomial sequence $p_n(x)$ is said to be *biorthogonal* to the sequence $\varphi_s(D)$ of operators, or, as some would prefer, the sequence $\varepsilon(0)\varphi_s(D)$ of linear functionals. Using Cramer's rule to solve the linear system and Laplace's expansion to group the results, we obtain the following *determinantal formula*:

$$p_n(x) = \frac{n!}{b_{00}b_{10}\cdots b_{n0}} \begin{vmatrix} b_{00} & b_{01} & b_{02} & \ldots & b_{0,n-1} & b_{0n} \\ 0 & b_{10} & b_{11} & \ldots & b_{1,n-2} & b_{1,n-1} \\ 0 & 0 & b_{20} & \ldots & b_{2,n-3} & b_{2,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b_{n-1,0} & b_{n-1,1} \\ 1 & x & x^2/2! & \ldots & x^{n-1}/(n-1)! & x^n/n! \end{vmatrix}. \tag{2.2}$$

Another important consequence of the fact that the initial segment $\varphi_s(D), s = 0, 1, 2, \ldots, n$ gives a non-singular upper triangular system of linear equations is that if $p(x)$ is a degree-n polynomial, then the conditions

$$\varepsilon(0)\varphi_i(D)p(x) = 0 \quad \text{for} \quad 0 \le i \le n$$

imply that $p(x)$ is identically zero. In particular, if $p(x)$ has degree n, then

$$p(x) = \sum_{i=0}^{n} \frac{\varepsilon(0)\varphi_i(D)p(x)}{i!} p_i(x). \tag{2.3}$$

This gives an *expansion formula*. Furthermore, the unique solution to the interpolation problem, *given numbers* $d_0, d_1, \ldots, d_n$, *find a degree-*n *polynomial* $p(x)$ *such that for* $i = 0, 1, \ldots, n$,

$$\varepsilon(0)\varphi_i(D)p(x) = d_i,$$

is given by the formula

$$p(x) = \sum_{i=0}^{n} \frac{d_i p_i(x)}{i!}. \tag{2.4}$$

Since

$$\varepsilon(0)\varphi_i(D)x^n = n! b_{i,n-i},$$

a special case of equation (2.3) or equation (2.4) is

$$x^n = \sum_{i=0}^{n} \frac{n! b_{i,n-i} p_i(x)}{i!}. \tag{2.5}$$

Equation (2.5) gives a *linear recursion* for $p_n(x)$. These linear recursions are perhaps the most efficient way to calculate the sequence $p_n(x)$ explicitly on a computer. Multiplying these equations by $t^n/n!$, summing over all non-negative integers n, and rearranging the right-hand side into products, we obtain the following formal power series equation (which is an instance of what one might called an *Appell relation*):

$$e^{xt} = \sum_{n=0}^{\infty} \frac{p_n(x)\varphi_n(t)}{n!}. \tag{2.6}$$

Another way to prove the Appell relation (2.6) is to observe that when one applies $\varphi_s(D)$ to both sides, one obtains the same result. Observe also that when restricted to the subspace $\mathcal{P}_m$ of all polynomials of degree less than or equal to m in $\mathcal{P}$, the operators D^s are expressible as linear combinations of the operators $\varphi_t(D), t = 0, 1, 2, \ldots, m$. Hence, one also obtains the same result when D^s is applied to both sides of the Appell relation, that is, the coefficient of x^s are the same on both sides.

We end with a matrix version of the linear recursion. We can rewrite the first $n+1$ instances of equation (2.5) as the matrix equation

$$\overrightarrow{x^i} = \mathcal{B}\,\overrightarrow{p_i(x)},$$

where

$$\overrightarrow{x^i} = [1, x, x^2, \ldots, x^n]^T,$$

$$\overrightarrow{p_i(x)} = [p_0(x), p_1(x), p_2(x), \ldots, p_n(x)]^T,$$

and $\mathcal{B}$ is the $(n+1) \times (n+1)$ lower triangular matrix

$$\left[\binom{i}{j}(i-j)! b_{j,i-j}\right]_{0 \le i,j \le n}.$$

We use the convention that the binomial coefficient $\binom{i}{j}$ is zero if $j > i$. For example, when $n = 3$, we have

$$\begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b_{01} & 1 & 0 & 0 \\ 2b_{02} & 2b_{11} & 1 & 0 \\ 6b_{03} & 6b_{12} & 3b_{21} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ p_1(x) \\ p_2(x) \\ p_3(x) \end{bmatrix}$$

However, we also have

$$\overrightarrow{p_i(x)} = \mathcal{C}\, \overrightarrow{x^i},$$

where $\mathcal{C}$ is the $(n+1) \times (n+1)$ lower triangular *coefficient matrix*

$$[c_{ij}]_{0 \le i,j \le n}$$

whose entries c_{ij} are coefficients of the polynomials $p_i(x)$. We use the convention that c_{ij} is zero when $j > i$. Hence, we conclude that the two lower triangular matrices $\mathcal{B}$ and $\mathcal{C}$ are inverses of each other. In particular,

$$\overrightarrow{p_i(x)} = \mathcal{B}^{-1}\, \overrightarrow{x^i}. \tag{2.7}$$

This gives a determinantal formula for $p_n(x)$ which is row and column reducible to equation (2.2).

Summarizing, we have shown that the biorthogonality relations, the linear recursions, the Appell relation, and the matrix form of the linear recursions all define the same sequence $p_n(x)$ of polynomials.

Sequences of polynomials of binomial type are special cases of sequences of biorthogonal polynomials. We shall use a description of polynomials of binomial type given in the classic paper of Mullin and Rota [16]. Recall that a sequence $p_n(x)$ of polynomials is of binomial type if and only if

$$\sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!} = e^{xf(t)}, \tag{2.8}$$

for some formal power series $f(t)$ such that $f(0) = 0$ and $Df(0) \neq 0$. These conditions are equivalent to the condition that $f(t)$ have a compositional inverse in the ring of formal power series. Let $g(t)$ be the compositional inverse of $f(t)$. Then, substituting $g(t)$ for t in equation (2.8), we obtain the Appell relation

$$e^{xt} = \sum_{n=0}^{\infty} p_n(x) \frac{[g(t)]^n}{n!}.$$

From this, we conclude that sequences of polynomials of binomial type are precisely sequences of polynomials biorthogonal to operator sequences of the form

$$\varphi_s(D) = [g(D)]^s,$$

where $g(t)$ is a formal power series with $g(0) = 0$ and $Dg(0) \neq 0$.

3 Algebraic properties of Gončarov polynomials

Let $(a_0, a_1, a_3, \ldots)$ be a sequence of numbers or variables called *nodes*. The sequence of *Gončarov polynomials*

$$g_n(x; a_0, a_1, \ldots, a_{n-1}),\ n = 0, 1, 2, \ldots$$

is the sequence of polynomials biorthogonal to the operators

$$E^{a_s} D^s,$$

where for any number or variable a, the operator E^a is the shift by a, that is,

$$E^a p(x) = p(x + a).$$

Because $\varepsilon(0)E^a = \varepsilon(a)$, the sequence of Gončarov polynomials $g_n(x; a_0, a_1, \ldots, a_{n-1})$ are defined by the orthogonality relations

$$\varepsilon(a_s)D^s g_n(x; a_0, a_1, \ldots, a_{n-1}) = n!\delta_{sn}.$$

Since

$$E^a = \sum_{r=0}^{\infty} \frac{a^r D^r}{r!} = e^{a_i D},$$

the sequence of Gončarov polynomials is biorthogonal to the sequence

$$D^s \sum_{r=0}^{\infty} \frac{a_s^r D^r}{r!}.$$

As indicated by the notation, $g_n(x; a_0, a_1, \ldots, a_{n-1})$ depends only on the nodes $a_0, a_1, \ldots, a_{n-1}$. Indeed, from equation (2.2), we have the *determinantal formula,*

$$g_n(x; a_0, a_1, \ldots, a_{n-1}) = n! \begin{vmatrix} 1 & a_0 & \frac{a_0^2}{2!} & \frac{a_0^3}{3!} & \cdots & \frac{a_0^{n-1}}{(n-1)!} & \frac{a_0^n}{n!} \\ 0 & 1 & a_1 & \frac{a_1^2}{2!} & \cdots & \frac{a_1^{n-2}}{(n-2)!} & \frac{a_1^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & a_2 & \cdots & \frac{a_2^{n-3}}{(n-3)!} & \frac{a_2^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 1 & a_{n-1} \\ 1 & x & \frac{x^2}{2!} & \frac{x^3}{3!} & \cdots & \frac{x^{n-1}}{(n-1)!} & \frac{x^n}{n!} \end{vmatrix}.$$

From equations (2.5) and (2.6), we have the *linear recursion*

$$x^n = \sum_{i=0}^{n} \binom{n}{i} a_i^{n-i} g_i(x; a_0, a_1, \ldots, a_{i-1})$$

and the *Appell relation*

$$e^{xt} = \sum_{n=0}^{\infty} g_n(x; a_0, a_1, \ldots, a_{n-1}) \frac{t^n e^{a_n t}}{n!}.$$

Finally, from equation (2.3), we have the *expansion formula.* If $p(x)$ is a polynomial of degree n, then

$$p(x) = \sum_{i=0}^{n} \frac{\varepsilon(a_i)D^i p(x)}{i!} g_i(x; a_0, a_1, \ldots, a_{i-1}).$$

We turn now to properties specific to the sequence of Gončarov polynomials. The Gončarov polynomials can be equivalently defined by the *differential relations*

$$Dg_n(x; a_0, a_1, \ldots, a_{n-1}) = ng_{n-1}(x; a_1, a_2, \ldots, a_{n-1}),$$

with initial conditions

$$g_n(a_0; a_0, a_1, \ldots, a_{n-1}) = \delta_{0n}.$$

(To see this, check that the orthogonality relations are satisfied.) Integrating the differential relations, we obtain the *integral relation*

$$g_n(x; a_0, a_1, \ldots, a_{n-1}) = n \int_{a_0}^{x} g_{n-1}(t; a_1, a_2, \ldots, a_{n-1}) dt.$$

Iterating this, we obtain the *integral formula*

$$g_n(x; a_0, a_1, \ldots, a_{n-1}) = n! \int_{a_0}^{x} dt_1 \int_{a_1}^{t_1} dt_2 \cdots \int_{a_{n-1}}^{t_{n-1}} dt_n.$$

The integral relation makes it clear (by induction) that $g_n(x; a_0, a_1, \ldots, a_{n-1})$ is a homogeneous polynomial with integer coefficients in the variables $x, a_0, a_1, \ldots, a_{n-1}$ of total degree n. It also gives a quick way to calculate Gončarov polynomials of low degree by hand. For example,

$$\begin{aligned}
g_0(x) &= 1, \\
g_1(x; a_0) &= x - a_0, \\
g_2(x; a_0, a_1) &= x^2 - 2a_1 x + 2a_0 a_1 - a_0^2, \\
g_3(x; a_0, a_1, a_2) &= x^3 - 3a_2 x^2 + (6a_1 a_2 - 3a_1^2)x - a_0^3 + 3a_0^2 a_2 - 6a_0 a_1 a_2 + 3a_0 a_1^2.
\end{aligned}$$

Using a change of variable, the integral relation and induction, or, observing that the differential operator is "shift-invariant" or commutes with shifts, one obtains the following useful *shift formula:*

$$g_n(x + \xi; a_0 + \xi, a_1 + \xi, \ldots, a_{n-1} + \xi) = g_n(x; a_0, a_1, \ldots, a_{n-1}).$$

The integral formula also suggests a formula which shows the effect of shifting or perturbing a single node. Using the identity

$$\int_{a_m}^{t} F(t)dt = \int_{a_m}^{a_m+b_m} F(t)dt + \int_{a_m+b_m}^{t} F(t)dt$$

at the mth integral in the integral formula, we obtain the *perturbation formula:*

$$\begin{aligned}
& g_n(x; a_0, \ldots, a_{m-1}, a_m + b_m, a_{m+1}, \ldots, a_{n-1}) = g_n(x; a_0, \ldots a_{m-1}, a_m, a_{m+1}, \ldots, a_{n-1}) \\
- & \binom{n}{m} g_{n-m}(a_m + b_m; a_m, a_{m+1}, \ldots, a_{n-1}) g_m(x; a_0, a_1, \ldots, a_{m-1}).
\end{aligned}$$

Applying the perturbation formula repeatedly, we can perturb any subset of nodes. For example, the following formula allows us to perturb an initial segment of length $n - m + 1$:

$$\begin{aligned}
& g_n(x; a_0 + b_0, a_1 + b_1, \ldots, a_{n-m} + b_{n-m}, a_{n-m+1}, \ldots, a_{n-1}) \\
= & g_n(x; a_0, a_1, \ldots, a_{n-m}, a_{n-m+1}, \ldots, a_{n-1}) \\
& - \sum_{i=0}^{n-m} \binom{n}{i} g_{n-i}(a_i + b_i; a_i, a_{i+1}, \ldots, a_{n-1}) g_i(x; a_0 + b_0, a_1 + b_1, \ldots, a_{i-1} + b_{i-1}).
\end{aligned}$$

In general, perturbation formulas can also be obtained by expanding the unperturbed polynomial $g_n(x; a_0, a_1, \ldots, a_{n-1})$ as a series in suitably perturbed Gončarov polynomials.

In general, there are no nice closed-form expressions for Gončarov polynomials. But such expressions exist for two special cases studied in analysis. The first is the case when all the nodes a_i equals a. In this case,

$$g_n(x; a, a, \ldots, a) = (x - a)^n$$

and Gončarov interpolation is just expansion as a power series at $x = a$. For this case, the linear recursion specializes to the binomial identity

$$x^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} (x - a)^i,$$

The second case (which includes the first as a special case) is when $a_0, a_1, a_2, \ldots$ form an arithmetic progression. This is the case of *Abel polynomials* and we have

$$g_n(x; y, y + b, y + 2b, \ldots, y + (n-1)b) = (x - y)(x - y - nb)^{n-1}. \tag{3.1}$$

In particular,

$$g_n(x; 0, 1, 2, \ldots, n - 1) = x(x - n)^{n-1}.$$

The linear recursion is

$$x^n = \sum_{i=0}^{n} \binom{n}{i} (y+ib)^{n-i}(x-y)(x-y-ib)^{i-1}.$$

Substituting $x+y$ for x in the second identity, we obtain *Abel's binomial theorem,*

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} (y+ib)^{n-i} x(x-ib)^{i-1}.$$

With the substitution $x+y+nb$ for x, $y+nb$ for y, and $-b$ for b, we obtain *Hurwitz's versions* of Abel's binomial theorem:

$$(x+y+nb)^n = \sum_{i=0}^{n} \binom{n}{i} (y+(n-i)b)^{n-i} x(x+ib)^{i-1},$$

or, changing indices from i to $n-i$,

$$(x+y+nb)^n = \sum_{i=0}^{n} \binom{n}{i} (y+ib)^{i} x(x+(n-i)b)^{n-i-1}. \tag{3.2}$$

Differentiating both sides of equation (3.2) with respect to y, we obtain

$$\begin{aligned} n(x+y+nb)^{n-1} &= \sum_{i=1}^{n} \binom{n}{i} i(y+ib)^{i-1} x(x+(n-i)b)^{n-i-1} \\ &= \sum_{i=1}^{n} \binom{n}{i-1} (n-i+1)(y+ib)^{i-1} x(x+(n-i)b)^{n-i-1}. \end{aligned}$$

Taking the case $n-1$ of this identity and setting $x=b$ and $y=a$, we obtain

$$(n-1)(a+nb)^{n-2} = \sum_{i=1}^{n-1} \binom{n-1}{i-1} b^{n-i-1}(n-i)^{n-i-1}(a+ib)^{i-1}. \tag{3.3}$$

We shall need identity (3.3) in Section 8.

Abel's binomial theorem is a member of a family of Abel identities studied by Hurwitz, Riordan and others (see [21], pp. 18 to 22). The following identity is a slightly modified version of the identity called $A_n(x,y;1,-1)$ from this family:

$$\sum_{i=0}^{n} \binom{n}{i} (x+ib)^{i+1} y(y+(n-i)b)^{n-i-1} = \sum_{i=0}^{n} \frac{n!}{i!}(x+y+nb)^i b^{n-i}(x+(n-i)b).$$

We shall use two special cases of this Abel identity in Section 8.

A proof of this identity can be found in [21], but it is part of a larger proof and difficult to extract. For this reason, we provide a simple self-contained proof. Observe that the left hand side is an expansion in terms of Abel polynomials $y(y+mb)^{m-1}$ in y with nodes at $-mb$. Hence, the identity follows from the following computation, where D_y is differentiation with respect to y :

$$\begin{aligned} &\varepsilon(-(n-i)b) D_y^{n-i} \left(\sum_{j=0}^{n} \frac{n!}{j!}(x+y+nb)^j b^{n-j}(x+(n-j)b) \right) \\ = \quad & n! \sum_{k=0}^{i} \frac{b^{i-k}(x+ib)^k(x+(i-k)b)}{k!}. \end{aligned}$$

The zeroth term in the sum is $b^i(x+ib)$. When $k>0$, we can rewrite the kth term as

$$\frac{b^{i-k}(x+ib)^{k+1}}{k!} - \frac{b^{i-(k-1)}(x+ib)^k}{(k-1)!}.$$

Hence, the sum telescopes and the right hand side equals

$$\frac{n!(x+ib)^{i+1}}{i!}.$$

The identity now follows from the expansion formula.

The first special case is obtained by setting $x = a + b$ and $y = b$ in the case $n - 2$ of the Abel identity. Doing so, we obtain

$$\sum_{i=0}^{n-2} \binom{n-2}{j} (a + (j+1)b)^{j+1} b^{n-j-3} (n-1-i)^{n-j-3}$$

$$= \sum_{i=0}^{n-2} \frac{(n-2)!}{j!} (a+nb)^j b^{n-j-3} (a + (n-j-1)b). \tag{3.4}$$

Setting $x = a$ and $y = 0$ on the left hand side, we obtain the second special case:

$$(a+nb)^{n+1} = \sum_{i=0}^{n} \frac{n!}{i!} (a+nb)^i b^{n-i} (a + (n-i)b),$$

or, changing indices from i to $n - i$,

$$(a+nb)^{n+1} = \sum_{i=0}^{n} \frac{n!}{(n-i)!} (a+nb)^{n-i} b^i (a+ib). \tag{3.5}$$

4 Coefficients of Rončarov polynomials

The main result in this section is a combinatorial interpretation of the coefficients of Gončarov polynomials. We first show that it suffices to consider only the constant terms.

Expanding $g_n(x + y; a_0, \ldots, a_{n-1})$ as a Taylor expansion in x and using the differential relations, we obtain

$$g_n(x+y; a_0, a_1, \ldots, a_{n-1}) = \sum_{i=0}^{n} \binom{n}{i} g_{n-i}(y; a_i, a_{i+1}, \ldots, a_{n-1}) x^i. \tag{4.1}$$

This is a shifted or parametrized analogue of a Sheffer relation, but *not* an actual Sheffer relation unless all the nodes a_i are equal. Thus, the Gončarov polynomials may be viewed as a "shifted" Sheffer sequence for the operator D (see [17]). The beginnings of a theory of "shifted" or "decentralized" umbral calculus has been developed in [22].

Setting $y = 0$ in equation (4.1), we obtain

$$g_n(x; a_0, a_1, \ldots, a_{n-1}) = \sum_{i=0}^{n} \binom{n}{i} g_{n-i}(0; a_i, a_{i+1}, \ldots, a_{n-1}) x^i. \tag{4.2}$$

Thus, coefficients of Gončarov polynomials are constant terms of (shifted) Gončarov polynomials. In particular, we have the following special case of equation (2.7).

(4.1) Lemma. Let $\mathcal{A}$ be the lower triangular matrix

$$\left[\binom{i}{j} a_j^{i-j} \right]_{0 \le i,j \le n}.$$

Then, its inverse $\mathcal{A}^{-1}$ is the lower triangular coefficient matrix

$$\left[\binom{i}{j} g_{i-j}(0; a_j, a_{j+1}, \ldots, a_{i-1})\right]_{0 \le i,j \le n}.$$

In particular,

$$\mathcal{A}^{-1}\overrightarrow{x^i} = \overrightarrow{g_i(x; a_0, a_1, \ldots, a_{n-1})}.$$

We shall now give a combinatorial interpretation of the constant terms of Gončarov polynomials. This interpretation is obtained by considering the number f_n of monomials in the constant term $g_n(0; a_0, a_1, \ldots, a_{n-1})$, counted with multiplicity. The sequence f_n starts $1, 1, 3, 13, 75, \ldots$. Using, say, the integral relation, it is easy to show that the numbers f_n satisfy the recurrence

$$f_n = \sum_{i=1}^{n} \binom{n}{i} f_{n-i}$$

and have exponential generating function

$$\sum_{n=0}^{\infty} \frac{f_n t^n}{n!} = \frac{1}{2 - e^t}.$$

From this, we see (from [23], say) that f_n is the number of *preferential arrangements*, or ordered partitions of the set with n elements. These observations suggest that there is an interpretation of the constant term $g_n(0; a_0, a_1, \ldots, a_{n-1})$ in terms of objects related to ordered partitions.

From an ordered partition $B_1, B_2, \ldots, B_m$ of a set $\{x_1, x_2, \ldots, x_n\}$ with n elements, one can associate a *ranking* $\rho : \{x_1, x_2, \ldots, x_n\} \to \{0, 1, 2, \ldots, n-1\}$ as follows: if an element x_i is in the jth block B_j, then defined

$$\rho(x_i) = \sum_{l<j} |B_l|.$$

In particular, $\rho(x_i) = 0$ whenever x_i is in the first block B_1. We define the order $|\rho|$ to be the size of the image of ρ, which is also the number of blocks in the ordered partition associated with ρ. For example, from the ordered partition $\{2, 4\}, \{5\}, \{1, 3\}$ of $\{1, 2, 3, 4, 5\}$, one obtains the ranking defined by $\rho(2) = \rho(4) = 0$, $\rho(5) = 2$, and $\rho(1) = \rho(3) = 3$. Rankings are characterized by the property: for every element x_i, there are exactly $\rho(x_i)$ elements x_j such that $\rho(x_j) < \rho(x_i)$.

(4.2) Theorem.

$$g_n(0; a_0, a_1, \ldots, a_{n-1}) = \sum_{\rho} (-1)^{|\rho|} \prod_{j=0}^{n-1} a_{\rho(j)},$$

where the sum ranges over all rankings ρ of $\{1, 2, \ldots, n\}$.

Proof. The theorem holds when $n = 0$. When $n > 0$, the constant terms of Gončarov polynomials satisfy the recursion

$$g_n(0; a_0, a_1, \ldots, a_{n-1}) = -\sum_{i=0}^{n-1} \binom{n}{i} a_i^{n-i} g_i(0; a_0, a_1, \ldots, a_{i-1})$$

obtained by setting $x = 0$ in the linear recursion. We shall show that the sum on the right hand side of the equation in Theorem 4.2 satisfies the same recursion. Let $\mathcal{R}[n]$ be the set of all rankings on $\{1, 2, \ldots, n\}$. Divide $\mathcal{R}[n]$ into groups $\mathcal{R}[n, i]$ according to the maximum value i taken by the ranking, so that

$$\mathcal{R}[n, i] = \{\rho : \max\{\rho(1), \rho(2), \ldots, \rho(n)\} = i\}.$$

If ρ is in $\mathcal{R}[n,i]$, then the inverse image $\rho^{-1}(i)$ must contain exactly $n-i$ numbers. Thus, there is a bijection between rankings ρ in $\mathcal{R}_i$ and pairs consisting of an i-element subset of $\{1,2,\ldots,n\}$ (the complement of $\rho^{-1}(i)$) and a ranking ρ' (having order $|\rho|-1$) on that i-element subset obtained by restricting ρ. Hence,

$$\sum_{\rho\in\mathcal{R}[n]}(-1)^{|\rho|}\prod_{j=1}^{n}a_{\rho(j)} = -\sum_{i=0}^{n-1}a_i^{n-i}\binom{n}{i}\left(\sum_{\rho\in\mathcal{R}[n,i]}(-1)^{|\rho'|}\prod_{j=0}^{i-1}a_{\rho(j)}\right).$$

Since both sides of the equation in Theorem 4.2 satisfy the same recursion and initial condition, they are equal by induction.

By Theorem 4.2 and the shift formula, we obtain the following formula for Gončarov polynomials.

$$\begin{aligned} g_n(x;a_0,a_1,\ldots,a_{n-1}) &= g_n(0;a_0-x,\ldots,a_{n-1}-x) \\ &= \sum_{\rho}(-1)^{|\rho|}\prod_{i=1}^{n}(a_{\rho(i)}-x). \end{aligned}$$

Abel polynomials are intimately related to the enumeration of trees. In particular, if one set $a_i=i$, then the constant term $(-1)^n g_n(0;a_0,a_1,\ldots,a_n)$ is the number of labelled trees on $n+1$ vertices. Is there an interpretation for $(-1)^n g_n(0;a_0,a_1,\ldots,a_n)$ in terms of labelled trees?

5 A decomposition for sequences of positive integers

In this section, we describe the combinatorial decomposition underlying the theory of parking functions. For us, this decomposition was motivated by the linear recursion for Gončarov polynomials. After discovering this decomposition, we found out from Julian Gilbey that the special case of this decomposition for ordinary parking functions was already used by Konheim and Weiss in the *first* paper [11] on the subject.

(5.1) Theorem. Let $(u_1,u_2,\ldots,u_n)$ be a sequence of non-decreasing positive integers and let x be a positive integer. Then, every sequence $(x_1,x_2,\ldots,x_n)$ of length n with terms x_i integers from the discrete interval $[1,x]$ can be decomposed into a pair of subsequences

$$(x_{i_1},x_{i_2},\ldots,x_{i_m}),\ (x_{j_1},x_{j_2},\ldots,x_{j_{n-m}})$$

such that the first subsequence $(x_{i_1},x_{i_2},\ldots,x_{i_m})$ is a $\mathbf{u}$-parking function of length m, and all the terms in the second subsequence, the complementary subsequence of length $n-m$ obtained by removing the terms in the first subsequence from $(x_1,x_2,\ldots,x_n)$ are in the discrete interval $[u_{m+1}+1,x]$. This decomposition provides a bijection between all sequences of length n with terms in $[1,x]$ and all pairs of complementary subsequences, the first a $\mathbf{u}$-parking function of length m and the second a sequence of length $n-m$ taking values in $[u_{m+1}+1,x]$.

Proof. Consider the sequence $(x_{(1)},x_{(2)},\ldots,x_{(n)})$ of order statistics. Let m be the maximum index such that

$$x_{(i)}\le u_i \quad \text{for} \quad i=1,2,\ldots,m. \tag{5.1}$$

Then, the subsequence $(x_{i_1},x_{i_2},\ldots,x_{i_m})$ from which the sequence $(x_{(1)},x_{(2)},\ldots,x_{(m)})$ was obtained by rearrangement is a $\mathbf{u}$-parking function of length m. Furthermore, m is the maximum index satisfying condition (5.1) if and only if

$$x_{(n)}\ge x_{(n-1)}\ge\ldots\ge x_{(m+1)}>u_{m+1},$$

or, equivalently, the *complementary* subsequence $(x_{j_1},x_{j_2},\ldots,x_{j_{n-m}})$, obtained by deleting the subsequence $(x_{i_1},x_{i_2},\ldots,x_{i_m})$ from the original sequence, takes values in the interval $[u_{m+1}+1,x]$. Since the maximum index m and hence, the set $\{i_1, i_2, \ldots, i_m\}$ are uniquely determined by the sequence $(x_1, x_2, \ldots, x_n)$, and any pair of subsequences satisfying the conditions in the theorem can be reassembled into a sequence in $[1, x]^n$, this decomposition yields a bijection.

It will be useful to state the decomposition more explicitly.

(5.2) Corollary. There is a bijection between the set $[1, x]^n$ of all length-n integer sequences with terms in the discrete interval $[1, x]$ and the disjoint union of cartesian products

$$\bigcup_{\{i_1, i_2, \ldots, i_m\}} \mathrm{Park}(i_1, i_2, \ldots, i_m) \times [u_{m+1} + 1, x]^{n-m},$$

where $\mathrm{Park}(i_1, i_2, \ldots, i_m)$ is the set of length-m $\mathbf{u}$-parking functions indexed by the set $\{i_1, i_2, \ldots, i_m\}$ and $[u_{m+1} + 1, x]^{n-m}$ is the set of length-$(n - m)$ integer sequences with terms in $[u_{m+1} + 1, x]$ indexed by the complement of $\{i_1, i_2, \ldots, i_m\}$.

Let $P_n(\mathbf{u})$ be the number of $\mathbf{u}$-parking functions of length m. Since $P_n(\mathbf{u})$ depends only on the first n terms of $\mathbf{u}$, we will often write $P_n(u_1, u_2, \ldots, u_n)$ instead of $P_n(\mathbf{u})$ to make explicit the parameters on which $P_n(\mathbf{u})$ is dependent. The decomposition in Theorem 5.1 yields the following identity.

(5.3) Corollary. Let x be an integer greater than or equal to u_n. Then

$$x^n = \sum_{m=0}^{n} \binom{n}{m} (x - u_{m+1})^{n-m} P_m(u_1, u_2, \ldots, u_m).$$

Comparing the recursion in Corollary 5.3 with the linear recursion for Gončarov polynomials given in Section 3, we obtain

$$P_n(u_1, u_2, \ldots, u_n) = g_n(x; x - u_1, x - u_2, \ldots, x - u_n).$$

By the shift formula, the Gončarov polynomial equals

$$g_n(0; -u_1, -u_2, \ldots, -u_n).$$

Since the Gončarov polynomial $g_n(x; a_0, a_1, \ldots, a_{n-1})$ is a homogeneous polynomial of total degree n in $x, a_0, a_1, \ldots, a_{n-1}$, we have

$$g_n(0; -u_1, -u_2, \ldots, -u_n) = (-1)^n g_n(0; u_1, u_2, \ldots, u_n).$$

All three forms of the formula for $P_n(\mathbf{u})$ are useful.

(5.4) Theorem.

$$\begin{aligned} P_n(u_1, u_2, \ldots, u_n) &= g_n(x; x - u_1, x - u_2, \ldots, x - u_n) \\ &= g_n(0; -u_1, -u_2, \ldots, -u_n) \\ &= (-1)^n g_n(0; u_1, u_2, \ldots, u_n). \end{aligned}$$

When $u_i = a + (i - 1)b$, we obtain the following special case.

(5.5) Corollary.

$$P_n(a, a + b, a + 2b, \ldots, a + (n - 1)b) = a(a + nb)^{n-1}.$$

In particular, we have rederived the classic formula for ordinary parking functions:

$$P_n(1, 2, 3, \ldots, n) = (n + 1)^{n-1}.$$

From the fact that Gončarov polynomials are homogeneous, we obtain another consequence of Theorem 5.4.

(5.6) Corollary.

$$P_n(bu_1, bu_2, \ldots, bu_n) = b^n P(u_1, u_2, \ldots, u_n).$$

Any reasonable formula for Gončarov polynomials yields a reasonable formula for parking functions. We give an example which is motivated by results in [17] and [31]. Consider the sequence $a_0, a_1, \ldots, a_{n-m}, c + (n-m+1)d, c+(n-m+2)d, \ldots, c+(n-1)d$ of n nodes. This sequence can be obtained by perturbing the arithmetic progression $c, c+d, \ldots, c+(n-1)d$ by $b_i = a_i - (c+id)$ for $i = 0, 1, \ldots, n-m$. Using the perturbation formula, we have

$$\begin{aligned}
& g_n(x; a_0, a_1, \ldots, a_{n-m}, c+(n-m+1)d, c+(n-m+2)d, \ldots, c+(n-1)d) \\
= \quad & (x-c)(x-c-nd)^{n-1} \\
& -\sum_{i=0}^{n-m} \binom{n}{i} (a_i - c - id)(a_i - c - nd)^{n-i-1} g_i(x; a_0, a_1, \ldots, a_{i-1}).
\end{aligned}$$

Using this and Theorem 5.4, we obtain the following result.

(5.7) Corollary. If $c + (n-m+1)d \geq a_{n-m}$, then

$$\begin{aligned}
& P_n(u_1, u_2, \ldots, u_{n-m+1}, c+(n-m+1)d, c+(n-m+2)d, \ldots, c+(n-1)d) \\
= \quad & c(c+nd)^{n-1} - \sum_{i=0}^{n-m} \binom{n}{i} (c+id-u_{i+1})(c+id-u_{i+1})^{n-i-1} P_i(u_1, u_2, \ldots, u_i).
\end{aligned}$$

Note that c need not be positive and some of the terms in the sum may be negative in Corollary 5.7.

By the determinantal formula for Gončarov polynomials in Section 3, we have the discrete analog of a result for real-valued parking functions usually attributed to Steck [28].

(5.8) Corollary. The number $P_n(u_1, u_2, \ldots, u_n)$ of $\mathbf{u}$-parking functions of length n equals $(-1)^n n! \det \mathcal{D}$, where $\mathcal{D}$ is the matrix with ijth entry equal to

$$\frac{u_i^{j-i+1}}{(j-i+1)!}$$

if $j - i + 1 \geq 0$ and 0 otherwise.

Note that Lemma 4.1 and Jacobi's formula for the inverse of a matrix yields another determinantal formula for $P_n(\mathbf{u})$. However, this formula can easily be derived from the formula in Corollary 5.8 by row and column operations.

6 Sum enumerators of parking functions

In this section, we extend several results for enumerators of trees and ordinary parking functions to $\mathbf{u}$-parking functions. Let $\mathbf{u}$ be a sequence of non-decreasing positive integers. The *sum enumerator* $S_n(q; \mathbf{u})$

for the set of **u**-parking functions is the polynomial in q defined by

$$S_n(q;\mathbf{u}) = \sum_{(a_1,a_2,\dots,a_n)} q^{a_1+a_2+\ldots+a_n-n}$$

where the sum ranges over all **u**-parking functions $(a_1, a_2, \dots, a_n)$. The sum enumerator may be regarded as a "q-analogue" of $P_n(\mathbf{u})$. The sum enumerator for a subset $\mathcal{S}$ of $[1, x]^n$ is defined analogously by summing over all sequences in $\mathcal{S}$. Sum enumerators are *multiplicative* in the following sense. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are two sets of subsequences on disjoint index sets. Then the sum enumerator of the cartesian product $\mathcal{S}_1 \times \mathcal{S}_2$ consisting of all sequences formed by combining a subsequence from $\mathcal{S}_1$ and a subsequence from $\mathcal{S}_1$ is the product of the sum enumerators of $\mathcal{S}_1$ and $\mathcal{S}_2$.

For a **u**-parking function, the maximum value of the ith order statistic $x_{(i)}$ is at most u_i and hence, $u_i - x_{(i)} \geq 0$. The *reversed sum enumerator* $R_n(q;\mathbf{u})$ is defined by

$$R_n(q;\mathbf{u}) = \sum_{(a_1,a_2,\dots,a_n)} q^{u_1+u_2+\ldots+u_n-(a_1+a_2+\ldots+a_n)},$$

where the sum ranges over all **u**-parking functions $(a_1, a_2, \dots, a_n)$. Equivalently,

$$R_n(q;\mathbf{u}) = q^{u_1+u_2+\ldots+u_n-n} S_n(1/q;\mathbf{u}). \tag{6.1}$$

The reversed sum enumerator is a polynomial in the variable q of degree $u_1 + u_2 + \ldots + u_n - n$.

(6.1) Lemma.

$$(1+q+q^2+\ldots+q^{x-1})^n = \sum_{m=0}^{n} \binom{n}{m} (q^{u_{m+1}} + q^{u_{m+1}+1} + \ldots + q^{x-1})^{n-m} S_m(q;\mathbf{u}).$$

Proof. Since sum enumerators are multiplicative, the sum enumerator of $[1, x]^n$ is

$$(1+q+q^2+\ldots+q^{x-1})^n.$$

For the same reason, the sum enumerator of functions which are decomposed into a **u**-parking function of length m and a sequence in $[u_{m+1}+1, x]^{n-m}$ is

$$(q^{u_{m+1}} + q^{u_{m+1}+1} + \ldots + q^{x-1})^{n-m} S_m(q,\mathbf{u}).$$

The recursion now follows.

Comparing this recursion with the linear recursion in Corollary 5.3, we obtain the following theorem.

(6.2) Theorem.

$$S_n(q;\mathbf{u}) = P_n(1+q+\ldots+q^{u_1-1}, 1+q+\ldots+q^{u_2-1}, \dots, 1+q+\ldots+q^{u_n-1}).$$

Theorem 6.2 can also be obtain directly using a decomposition for the set of **u**-parking functions due to Pitman and Stanley [19]. Given a **u**-parking function $(\beta_1, \beta_2, \dots, \beta_n)$, we can associate an ordinary parking function $(\alpha_1, \alpha_2, \dots, \alpha_n)$ by setting $\alpha_i = r$ if β_i is in the discrete interval $[u_{r-1}+1, u_r]$. Conversely, given an ordinary parking function $(\alpha_1, \alpha_2, \dots, \alpha_n)$, there are

$$(u_{\alpha_1} - u_{\alpha_1-1})(u_{\alpha_2} - u_{\alpha_2-1}) \cdots (u_{\alpha_n} - u_{\alpha_n-1})$$

u-parking functions associated with it. These are obtained by choosing a number from each discrete interval $[u_{\alpha_j-1}+1, u_{\alpha_j}]$. Here, we use the convention that $u_0 = 0$. Hence,

$$P_n(u_1, u_2, \dots, u_n) = \sum_{(\alpha_1,\alpha_2,\dots,\alpha_n)} (u_{\alpha_1} - u_{\alpha_1-1})(u_{\alpha_2} - u_{\alpha_2-1}) \cdots (u_{\alpha_n} - u_{\alpha_n-1}),$$

where the sum ranges over all ordinary parking functions of length n. Replacing the number of elements $u_{\alpha_j} - u_{\alpha_j - 1}$ in the discrete interval $[u_{\alpha_{j-1}} + 1, u_{\alpha_j}]$ by its sum enumerator and using the fact that sum enumerators are multiplicative, we obtain Theorem 6.2.

Using Theorem 5.3, Theorem 6.1, and the shift formula, we can express sum enumerators in terms of Gončarov polynomials:

$$S_n(q; \mathbf{u}) = g_n \left(\frac{1}{1-q}; \frac{q^{u_1}}{1-q}, \frac{q^{u_2}}{1-q}, \dots, \frac{q^{u_n}}{1-q} \right).$$

By homogeneity of Gončarov polynomials,

$$(1-q)^n S_n(q; \mathbf{u}) = g_n(1; q^{u_1}, q^{u_2}, \dots, q^{u_n}).$$

Hence, sum enumerators satisfy the simpler linear recursion

$$1 = \sum_{m=0}^{n} \binom{n}{m} q^{u_{m+1}(n-m)} (1-q)^m S_m(q; \mathbf{u}). \tag{6.2}$$

They also satisfy the following Appell relation

$$\exp(t) = \sum_{n=0}^{\infty} (1-q)^n S_n(q; \mathbf{u}) \exp(q^{u_{n+1}} t) \frac{t^n}{n!}.$$

In the case of ordinary parking functions, $u_i = i$ and we have

$$(1-q)^n S_n(q; 1, 2, \dots, n) = g_n(1; q, q^2, \dots, q^n).$$

For example,

$$\begin{aligned} (1-q)^2 S_2(q; 1, 2) &= 1 - 3q^2 + 2q^3 \\ (1-q)^3 S_3(q; 1, 2, 3) &= 1 - 4q^3 - 3q^4 + 12q^5 - 6q^6. \end{aligned}$$

One does not expect simple generating functions for sum enumerators in general. However, when u_i is an arithmetic progression, we can group terms together to obtain a recursion which yields a simple exponential generating function. We shall show how this can be done for reversed sum enumerators.

Substituting $1/q$ for q in equation (6.2) and using equation (6.1), we obtain

$$q^{u_1 + u_2 + \ldots + u_n} = \sum_{m=0}^{n} \binom{n}{m} (q-1)^m R_m(q; \mathbf{u}) q^{-(n-m)u_{m+1} + u_{m+1} + u_{m+2} + \ldots + u_n}.$$

If the exponent

$$-(n-m)u_{m+1} + u_{m+1} + u_{m+2} + \ldots + u_n$$

is a function $\tau(n-m)$ depending only on $n-m$, then we have

$$q^{u_1 + u_2 + \ldots + u_n} = \sum_{m=0}^{n} \binom{n}{m} (q-1)^m R_m(q; \mathbf{u}) q^{\tau(n-m)}.$$

Multiplying this by $t^n/n!$, summing over all non-negative integers n, and manipulating the resulting formal power series, we obtain

$$\sum_{n=0}^{\infty} (q-1)^n R_n(q; \mathbf{u}) \frac{t^n}{n!} = \frac{\displaystyle\sum_{n=0}^{\infty} q^{u_1 + u_2 + \ldots + u_n} \frac{t^n}{n!}}{\displaystyle\sum_{n=0}^{\infty} q^{\tau(1) + \tau(2) + \ldots + \tau(n)} \frac{t^n}{n!}}.$$

The condition that the exponent is a function $\tau(n-m)$ of $n-m$ is in fact very strong. Consider the case $n-m=2$. Then the condition implies that for all m, $-2u_{m+1}+u_{m+1}+u_{m+2}$ equals a number $\tau(2)$ independently of m, that is, $u_{m+2}-u_{m+1}$ is a constant b for all m. This in turn implies that $\mathbf{u}$ is an arithmetic progression with common difference b. Conversely, if $u_i = a+(i-1)b$, then

$$\sum_{j=1}^{n} u_j = an + b\binom{n}{2}$$

and

$$\sum_{j=1}^{n} \tau(j) = b\binom{n}{2}.$$

We have thus proved the following theorem, which is best possible.

(6.3) Theorem. Let $\mathbf{u}$ be the arithmetic progression $(a, a+b, a+2b, \dots)$. Then

$$\sum_{n=0}^{\infty}(q-1)^n R_n(q;\mathbf{u})\frac{t^n}{n!} = \frac{\displaystyle\sum_{n=0}^{\infty} q^{an+b\binom{n}{2}}\frac{t^n}{n!}}{\displaystyle\sum_{n=0}^{\infty} q^{b\binom{n}{2}}\frac{t^n}{n!}}.$$

The reversed sum enumerator $R_n(q;\mathbf{u})$ also enumerates the number of inversions for certain sequences of rooted b-forests. For more details about this and the relation between rooted b-forests and generalized parking functions, see [32]. In particular, the reversed sum enumerator $R_n(q;1,2,\dots,n)$ for ordinary parking functions equals the inversion enumerator $I_n(q)$ for labelled trees (see [15, 12, 26, 27]). Hence, we obtain, as a special case of Theorem 6.3, the following result of Stanley ([26, 27]):

$$\sum_{n=0}^{\infty}(q-1)^n I_n(q)\frac{t^n}{n!} = \frac{\displaystyle\sum_{n=0}^{\infty} q^{\binom{n+1}{2}}\frac{t^n}{n!}}{\displaystyle\sum_{n=0}^{\infty} q^{\binom{n}{2}}\frac{t^n}{n!}}.$$

The theory of sum enumerators suggests that Gončarov polynomials with nodes forming a geometric progression $1, q, q^2, \dots$ are worthy of study. For example,

$$\begin{aligned}
g_2(x;1,q) &= x^2 - 2qx + 2q - 1,\\
g_3(x;1,q,q^2) &= x^3 - 3q^2x^2 + (6q^3 - 3q^2)x + -6q^3 + 6q^2 - 1,\\
g_4(x;1,q,q^2,q^3) &= x^4 - 4q^3x^3 + (12q^5 - 6q^4)x^2 + (-24q^6 + 24q^5 - 4q^3)x\\
&\quad +24q^6 - 36q^5 + 6q^4 + 8q^3 - 1.
\end{aligned}$$

These Gončarov polynomials can be regarded as q-analogues of Abel polynomials.

7 Expected sums of parking functions

In the remainder of this paper, we shall use methods from elementary probability theory. A subset $\mathcal{S}$ of the set $[1,x]^n$ of length-n sequences with terms in the discrete interval $[1,x]$ can be made into a discrete probability space with a uniform probability measure by assigning a probability of $1/|\mathcal{S}|$ to each sequence in $\mathcal{S}$. When $\mathcal{S}$ is $[1, x]^n$, then each sequence has probability $1/x^n$. In this case, the probability measure can also be obtained by choosing each term x_i independently and randomly with uniform distribution from the discrete interval $[1, x]$.

Given a subset $\mathcal{S}$ of length-n sequences, we define the random variable S_n to be the sum $x_1 + x_2 + \ldots + x_n$ of a random sequence in $\mathcal{S}$. The *expected sum* of a random sequence from $\mathcal{S}$ is the expectation $E[S_n]$. Let $(x)_k$ be the *k-falling factorial,* that is,

$$(x)_k = x(x-1)\cdots(x-k+1).$$

The *kth (falling) factorial moment of the sum of a random sequence* in $\mathcal{S}$ is the expectation $E[(S_n)_k]$. More explicitly, $E[(S_n)_k]$ equals

$$\frac{1}{|\mathcal{S}|} \sum_{(x_1, x_2, \ldots, x_n) \in \mathcal{S}} (x_1 + x_2 + \ldots + x_n)_k.$$

In particular, let $E_k(n; \mathbf{u})$ be the *kth falling factorial moment of the sum of a random $\mathbf{u}$-parking function*, that is,

$$E_k(n; \mathbf{u}) = \frac{1}{P_n(\mathbf{u})} \sum_{(x_1, x_2, \ldots, x_n)} (x_1 + x_2 + \ldots + x_n)_k,$$

where the sum ranges over all $\mathbf{u}$-parking functions of length n.

The decomposition in Theorem 5.1 also gives recursions for expected values of moments of sums of parking functions. From these recursions, one can, with some difficulty, get explicit formulas for the moments. In this section, we shall show how this can be done for the first moment or the expected sum. We begin with the linear recursion.

(7.1) Theorem. The expected sums of $\mathbf{u}$-parking functions satisfy the following linear recursion:

$$\frac{n(x+1)}{2} = \sum_{m=0}^{n} \binom{n}{m} \frac{(x - u_{m+1})^{n-m} P_m(\mathbf{u})}{x^n} \left(E_1(m; \mathbf{u}) + \frac{(n-m)(x + u_{m+1} + 1)}{2} \right).$$

Proof. We derive the expected sum of a sequence $(x_1, x_2, \ldots, x_n)$ in $[1, x]^n$ in two different ways. Since the expected value of any term x_i is $(1+x)/2$, the expected sum of a random sequence in $[1, x]^n$ is $n(1+x)/2$, the left hand side of the recursion.

By Theorem 5.1, each sequence in $[1, x]^n$ decomposes into a $\mathbf{u}$-parking function of length m and a sequence in $[u_{m+1} + 1, x]^{n-m}$. For a fixed m-element subset $\{i_1, i_2, \ldots, i_m\}$ of $\{1, 2, \ldots, n\}$, consider the subset of sequences decomposing into a length-m $\mathbf{u}$-parking function indexed by $\{i_1, i_2, \ldots, i_m\}$ and a length-$(n-m)$ sequence in $[u_{m+1} + 1, x]^{n-m}$ indexed by the complement. The probability that a random sequence is in this set is

$$\frac{(x - u_{m+1})^{n-m} P_m(u_1, u_2, \ldots, u_m)}{x^n}$$

and the expected sum of such a sequence is

$$E_1(m; u_1, u_2, \ldots, u_m) + \frac{(n-m)(x + u_{m+1} + 1)}{2}.$$

The right hand side of the recursion can now be obtained by conditioning on the event that the maximal subsequence forming a $\mathbf{u}$-parking function is indexed by $\{i_1, i_2, \ldots, i_m\}$ and summing over subsets of the index set $\{1, 2, \ldots, n\}$. This completes the proof of Theorem 7.1.

The recursion in Theorem 7.1 gives an Appell relation for the expected sums. Let $\mathbf{a}$ be the sequence defined by

$$a_i = x - u_{i+1},$$

with $0 \leq i < \infty$. Then the expected sum $E_1(n; u_1, u_2, \ldots, u_n)$, as a function of $u_1, u_2, \ldots, u_n$, becomes a function of x and $a_0, a_1, \ldots, a_{n-1}$. Let

$$e_n^{(1)}(x; a_0, a_1, \ldots, a_{n-1}) = E_1(n; x - a_0, \ldots, x - a_{n-1}).$$

In terms of $e_n^{(1)}(x;\mathbf{a})$, the recursion in Theorem 7.1 becomes

$$\begin{aligned}\frac{nx^n(x+1)}{2} &= \sum_{m=0}^{n}\binom{n}{m}a_m^{n-m}g_m(x;\mathbf{a})e_m^{(1)}(x;\mathbf{a})\\ &+ \sum_{m=0}^{n}\binom{n}{m}a_m^{n-m}g_m(x;\mathbf{a})(n-m)\left(\frac{2x-a_m+1}{2}\right).\end{aligned}$$

From the recursion, we conclude that $g_n(x;\mathbf{a})e_n^{(1)}(x;\mathbf{a})$ is the sum of two homogeneous polynomials in x and $a_0, a_1, \ldots, a_{n-1}$, one having total degree $n+1$ and the other having total degree n. The sum $g_n(x;\mathbf{a})e_n^{(1)}(x;\mathbf{a})$ is easier to work with than the expected sum $e_n^{(1)}(x;\mathbf{a})$. We shall derive an Appell relation and an explicit formula for $g_n(x;\mathbf{a})e_n^{(1)}(x;\mathbf{a})$ in terms of Gončarov polynomials.

We begin with the Appell relation. Multiplying both sides by $t^n/n!$ and summing over n, we get

$$\frac{(1+x)xt}{2}e^{xt}$$

on the left hand side. For the first sum on the right hand side, we get

$$\sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\binom{n}{m}a_m^{n-m}g_m(x;\mathbf{a})e_m^{(1)}(x;\mathbf{a})\right]\frac{t^n}{n!} = \sum_{m=0}^{\infty}g_m(x;\mathbf{a})e_m^{(1)}(x;\mathbf{a})\frac{e^{a_m t}t^m}{m!}.$$

For the second sum, we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\binom{n}{m}a_m^{n-m}g_m(x;\mathbf{a})(n-m)\left(\frac{2x-a_m+1}{2}\right)\right]\frac{t^n}{n!}\\ =\;& t\sum_{n=1}^{\infty}\left[\sum_{m=0}^{n-1}\binom{n-1}{m}a_m^{(n-1)-m}g_m(x;\mathbf{a})\left(a_m\left(x+\frac{1}{2}\right)-\frac{1}{2}a_m^2\right)\right]\frac{t^{n-1}}{(n-1)!}\\ =\;& t\sum_{m=0}^{\infty}g_m(x;\mathbf{a})\left[\left(x+\frac{1}{2}\right)a_m-\frac{1}{2}a_m^2\right]\frac{e^{a_m t}t^m}{m!}.\end{aligned}$$

Therefore we obtain the following Appell relation.

(7.2) Theorem.

$$\begin{aligned}&\sum_{n=0}^{\infty}g_n(x;\mathbf{a})e_n^{(1)}(x;\mathbf{a})\frac{e^{a_n t}t^n}{n!}\\ =\;& \frac{(1+x)xt}{2}e^{xt} - \sum_{n=0}^{\infty}g_n(x;\mathbf{a})\left[\left(x+\frac{1}{2}\right)a_n t-\frac{1}{2}a_n^2 t\right]\frac{e^{a_n t}t^n}{n!}.\end{aligned}$$

Our next objective is to derive an expression for $g_n(x;\mathbf{a})e_n^{(1)}(x;\mathbf{a})$ as a linear combination of Gončarov polynomials. This gives an formula to compute $e_n^{(1)}(x;\mathbf{a})$ assuming that the Gončarov polynomials are already computed. We remark that from a computer algebra point of view, the linear recursion in Theorem 7.1 is a very efficient way to calculate a specific expected sum, but "explicit" formulas are also useful.

We shall use the following vector notation introduced in Section 2. If $f_i(x), i = 0, 1, 2, \ldots, n$, is a sequence of polynomials, then

$$\overrightarrow{f_i(x)} = (f_0(x), f_1(x), \ldots, f_n(x))^T.$$

In particular, the linear recursions for Gončarov polynomials can be rewritten as

$$\mathcal{A}\ \overrightarrow{g_i(x;\mathbf{a})} = \overrightarrow{x^i},$$

where $\mathcal{A}$ is the matrix defined in Lemma 4.1. Similarly, we can rewrite the linear recursion in Theorem 7.1 as

$$\frac{x(1+x)}{2}\overrightarrow{ix^{i-1}} = \mathcal{A}\ \overrightarrow{g_i(x;\mathbf{a})e_i^{(1)}(x;\mathbf{a})} + \mathcal{B}\ \overrightarrow{\left(\frac{2x-a_i+1}{2}\right)g_i(x;\mathbf{a})}.$$

where $\mathcal{B}$ is the $(n+1)\times(n+1)$ matrix

$$\left[i\binom{i-1}{j}a_j^{i-j}\right]_{0\le i,j,\le n}.$$

Note that, as always, we use the convention that the binomial coefficient $\binom{i}{j}$ is zero if $j>i$. Applying the inverse of $\mathcal{A}$ to both sides, we obtain

$$\frac{x(1+x)}{2}\mathcal{A}^{-1}\ \overrightarrow{ix^{i-1}} = \overrightarrow{g_i(x;\mathbf{a})e_i^{(1)}(x;\mathbf{a})} + \mathcal{A}^{-1}\mathcal{B}\ \overrightarrow{\left(\frac{2x-a_i+1}{2}\right)g_i(x;\mathbf{a})}. \tag{7.1}$$

By Lemma 4.1, the inverse of $\mathcal{A}$ is the coefficient matrix of the Gončarov polynomials. Hence, observing that ix^{i-1} is the derivative of x^i,

$$\mathcal{A}^{-1}\overrightarrow{ix^{i-1}} = \overrightarrow{Dg_i(x;\mathbf{a})}.$$

Using the differential relation for Gončarov polynomials, we conclude that the left hand side of equation (7.2) equals

$$\frac{x(1+x)}{2}\overrightarrow{ig_{i-1}(x;a_1,a_2,\dots,a_{i-1})},$$

where we use the convention (consistent with the differential relation) that Gončarov polynomials with negative indices are identically zero.

To simplify the right hand side, consider the matrix $\mathcal{A}^{-1}\mathcal{B}$. Since both $\mathcal{A}$ and $\mathcal{B}$ are lower triangular and the diagonal entries of $\mathcal{B}$ are zero, $\mathcal{A}^{-1}\mathcal{B}$ is lower triangular with zero diagonal. In particular, the ij-entry of $\mathcal{A}^{-1}\mathcal{B}$ is zero if $i\le j$. Suppose that $i>j$. Then by Lemma 4.1, the ij-th entry of $\mathcal{A}^{-1}\mathcal{B}$ equals

$$\begin{aligned} &\sum_{k=0}^{n}\binom{i}{k}g_{i-k}(0;a_k,\dots,a_{i-1})k\binom{k-1}{j}a_j^{k-j} \\ = \quad &(i-j)\binom{i}{j}a_j\sum_{t=0}^{n-j-1}\binom{i-j-1}{t}g_{i-j-1-t}(0;a_{j+1+t},\dots,a_{i-1})a_j^t. \end{aligned}$$

By equation (4.2),

$$g_{i-j}(x;a_j,\dots,a_{i-1}) = \sum_{t=0}^{i-j}\binom{i-j}{t}g_t(0;a_{i-t},\dots,a_{i-1})x^{i-j-t}.$$

Taking the derivative on both sides, we obtain

$$\begin{aligned} Dg_{i-j}(x;a_j,\dots,a_{i-1}) &= (i-j)\sum_{t=0}^{i-j-1}\binom{i-j-1}{t}g_t(0;a_{i-t},\dots,a_{i-1})x^{i-j-t-1} \\ &= (i-j)\sum_{t=0}^{i-j-1}\binom{i-j-1}{t}g_{i-j-1-t}(0;a_{j+1+t},\dots,a_{i-1})x^{t}. \end{aligned}$$

We conclude that the ijth entry of $\mathcal{A}^{-1}\mathcal{B}$ equals

$$\binom{i}{j} a_j Dg_{i-j}(a_j; a_j, a_{j+1}, \ldots, a_{i-1}).$$

By the differential relation,

$$Dg_{i-j}(x; a_j, a_{j+1}, \ldots, a_{i-1}) = (i-j)g_{i-j-1}(x; a_{j+1}, a_{j+2}, \ldots, a_{i-1})$$

Hence, an alternate way to write the ijth entry of $\mathcal{A}^{-1}\mathcal{B}$ is

$$i\binom{i-1}{j} a_j g_{i-j-1}(a_j; a_{j+1}, a_{j+2}, \ldots, a_{i-1}).$$

Putting all the above into equation (7.1), we obtain the following theorem.

(7.3) Theorem. The sum $g_n(x; \mathbf{a})e_n^{(1)}(x; \mathbf{a})$ equals

$$\begin{aligned} & \frac{nx(1+x)}{2} g_{n-1}(x; a_1, a_2, \ldots, a_{n-1}) \\ - \quad & \frac{n}{2}\sum_{i=0}^{n-1} \binom{n-1}{i} a_i(2x - a_i + 1) g_{n-i-1}(a_i; a_{i+1}, a_{i+2}, \ldots, a_{n-1}) g_i(x; a_0, a_1, \ldots, a_{i-1}). \end{aligned}$$

Setting $x = 0$ and $a_i = -u_{i+1}$ and using Theorem 5.4 and the shift formula, we obtain a formula for the expected sum in terms of the sequence $\mathbf{u}$.

(7.4) Theorem. The expected sum $E_1(n; \mathbf{u})$ equals

$$\frac{n}{2}\sum_{j=1}^{n} \binom{n-1}{j-1} u_j(u_j+1) \frac{P_{n-j}(u_{j+1}-u_j, u_{j+2}-u_j, \ldots, u_n - u_j) P_{j-1}(u_1, u_2, \ldots, u_{j-1})}{P_n(u_1, u_2, \ldots, u_n)}.$$

This formula, a sum of positive terms, should have an revealing combinatorial interpretation.

8 The classical case with Abel identities

In this section, we shall give several equivalent formulas for the expected sum $E_1(n; a, a+b, \ldots, a+(n-1)b)$. We shall often abbreviate our notation and write $E_1(n; a, b)$ instead of $E_1(n; a, a+b, \ldots, a+(n-1)b)$. Using Theorem 7.4 and Corollary 5.5, we obtain the following formula for the expected sum.

(8.1) Theorem.

$$E_1(n; a, b) = \frac{n}{2}\sum_{i=0}^{n-1} \binom{n-1}{i} (a + ib + 1) b^{n-i-1}(n-i)^{n-i-2} \frac{(a+ib)^i}{(a+nb)^{n-1}}.$$

In the remainder of this section, we shall use Theorem 7.3 to obtain other formulas for expected sums. There are many – almost too many – such formulas, due mainly to the existence of Abel identities discussed in Section 3. We shall use the following substitutions

$$x = a,\ a_0 = 0,\ a_1 = -b,\ a_2 = -2b,\ \ldots,\ a_n = -(n-1)b.$$

We begin by calculating explicitly several values of Gončarov polynomials. By equation (3.1),

$$\begin{aligned} g_n(a;0,-b,-2b,\ldots,-(n-1)b) &= P_n(a,a+b,a+2b,\ldots,a+(n-1)b) \\ &= a(a+nb)^{n-1}, \end{aligned}$$

$$g_{n-1}(a;-b,-2b,\ldots,-(n-1)b) = (a+b)(a+nb)^{n-2},$$

$$\begin{aligned} g_{n-i-1}(-ib;-(i+1)b,\ldots,-(n-1)b) &= b[(n-i)b]^{n-i-2} \\ &= b^{n-i-1}(n-i)^{n-i-2}. \end{aligned}$$

Substituting these values into the formula in Theorem 7.3, we obtain

$$\begin{aligned} & a(a+bn)^{n-1}e_n^{(1)}(a;0,-b,-2b,\ldots,-(n-1)b) \\ = \; & \frac{na(a+1)}{2}(a+b)(a+bn)^{n-2} \\ & +n\sum_{i=0}^{n-1}\binom{n-1}{i}ib^{n-i}(n-i)^{n-i-2}\left(\frac{2a+ib+1}{2}\right)a(a+ib)^{i-1}. \end{aligned}$$

The sum in this expression can be simplified slightly (by manipulating binomial coefficients) to

$$n\sum_{i=1}^{n-1}\binom{n-1}{i-1}b^{n-i}(n-i)^{n-i-1}\left(\frac{2a+ib+1}{2}\right)a(a+ib)^{i-1}.$$

We break up this sum into two parts by writing

$$\frac{2a+ib+1}{2} = \frac{a+1}{2} + \frac{a+ib}{2}.$$

The first part is the following sum

$$\frac{nab(a+1)}{2}\sum_{i=1}^{n-1}\binom{n-1}{i-1}b^{n-i-1}(n-i)^{n-i-1}(a+ib)^{i-1}.$$

By equation (3.3), the sum equals $(n-1)(a+nb)^{n-2}$. Regrouping terms, we conclude that the first part equals

$$\binom{n}{2}ab(a+1)(a+nb)^{n-2}.$$

When this quantity is added to $na(a+1)(a+b)(a+nb)^{n-2}/2$, we get the refreshingly simple result $na(a+1)(a+nb)^{n-1}/2$. The second part,

$$\frac{na}{2}\sum_{i=1}^{n-1}\binom{n-1}{i-1}b^{n-i}(n-i)^{n-i-1}(a+ib)^{i}, \tag{8.1}$$

does not simplify into a single term. Hence, the following theorem gives a reasonable formula for the expected sum.

(8.2) Theorem. The expected sum $E_1(n;a,b)$ equals

$$\frac{n(a+1)}{2} + \frac{n}{2}\sum_{i=1}^{n-1}\binom{n-1}{i-1}b^{n-i}(n-i)^{n-i-1}\frac{(a+ib)^{i}}{(a+nb)^{n-1}}.$$

For ordinary parking function, this formula specializes to

$$E_1(n;1,1) = n + \frac{n}{2}\sum_{i=1}^{n-1}\binom{n-1}{i-1}(n-i)^{n-i-1}\frac{(i+1)^i}{(n+1)^{n-1}}. \tag{8.2}$$

This formula does not have the same form (and is not obviously the same) as the formula obtained by Gessel and Sagan [5] or Knuth [10] for ordinary parking functions, which states

$$E_1(n;1,1) = \binom{n+1}{2} - \frac{1}{2}\sum_{i=2}^{n}\binom{n}{i}i!(n+1)^{1-i}. \tag{8.3}$$

This formula suggests a third formula for $E_1(n;a,b)$ which specializes to equation (8.3) when both a and b are set equal to 1.

(8.3) Theorem.

$$E_1(n;a,b) = \frac{n(a+1)}{2} + b\binom{n}{2} - \frac{1}{2}\sum_{i=2}^{n}\binom{n}{i}\frac{i!b^i}{(a+nb)^{i-1}}.$$

There are two ways to prove Theorem 8.3. The first way to show that the expressions in Theorems 8.2 and 8.3 are equal. This can be done using a computer algebra program (see [18]) or by traditional methods. We will leave the computer algebra method to our silicon-based friends. The traditional method requires using two Abel identities. It is not particularly illuminating *per se*, but an intermediate form turns out to be useful later. Thus, it is worthwhile to show this method in some detail. The main step is to transform the sum (8.1) into a suitable form.

Using equation (3.4), the sum (8.1) equals

$$\frac{ab^2n(n-1)}{2}\sum_{j=0}^{n-2}\frac{(n-2)!}{j!}(a+nb)^j b^{n-j-3}(a+(n-j-1)b).$$

Changing indices from j to $n-j$ and regrouping terms, this becomes

$$\frac{a}{2}\sum_{j=2}^{n}\frac{n!}{(n-j)!}(a+nb)^{n-j}b^{j-1}(a+(j-1)b).$$

Hence, we have another formula for the expected sum. This formula will be used in Section 10.

(8.4) Theorem. The expected sum $E_1(n;a,a+b,\ldots,a+(n-1)b)$ equals

$$\frac{n(a+1)}{2} + \frac{1}{2}\sum_{j=2}^{n}\frac{n!}{(n-j)!}\frac{b^{j-1}(a+(j-1)b)}{(a+nb)^{j-1}}.$$

This is not yet Theorem 8.3 and we need identity (3.5). Extracting the first two terms in the sum, moving them to the left, simplifying the left hand side, finding that there is a factor of b on the left hand side, and dividing by b, we obtain :

$$n(n-1)b(a+nb)^{n-1} = \sum_{j=2}^{n}\frac{n!}{(n-j)!}(a+nb)^{n-j}b^{j-1}(a+jb).$$

Applying the last identity, we reach the required form for the sum (8.1). Dividing by $a(a+nb)^{n-1}$, we arrive finally at the equation in Theorem 8.3.

9 The classical case with an inverse relation

The second way to prove Theorem 8.3 is to proceed directly from the linear recursion. This method yields an interesting and simpler special case of the linear recursion. Consider the linear recursion in Theorem 7.1 with $u_{i+1} = a + ib$. Multiplying both sides by x^n, we obtain

$$\begin{aligned} &\frac{nx^n(1+x)}{2} \\ = \quad &\sum_{i=0}^{n} \binom{n}{i} (x-a-ib)^{n-i} a(a+ib)^{i-1} \left(E_1(i;a,b) + \frac{(n-i)(x+a+ib+1)}{2} \right). \end{aligned}$$

This identity holds for all integers x greater than or equal to $a + (n-1)b$. Hence, it is a polynomial identity in x and holds for all real numbers x. Setting $x = 0$ and rearranging terms, we have

$$\begin{aligned} &\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a(a+ib)^{n-1} E_1(i;a,b) \\ = \quad &-\frac{1}{2} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (n-i) \left[a(a+ib)^{n-1} + a(a+ib)^n \right]. \end{aligned} \tag{9.1}$$

When $n = 0$, equation (9.1) says that $E_1(0;a,b) = 0$, as expected. When $n \geq 1$, the right hand side of equation (9.1) can be simplified slightly to

$$\frac{n}{2} \sum_{i=0}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} \left[a(a+ib)^{n-1} + a(a+ib)^n \right].$$

In this form, it can be written as a single term by using the following lemma.

(9.1) Lemma.

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a(a+ib)^m = ab^n n! \sum_{r=0}^{m-n} \binom{m}{n+r} a^{m-n-r} b^r S(n+r,n).$$

where $S(m,n)$ is a Stirling number of the second kind and equals the number of partitions of an m-element set into n non-empty blocks.

Proof. Expand $(a+bi)^m$ with the binomial theorem, use the identity of Stirling ([29]; see, for example, [25], page 34):

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i^m = n! S(m,n),$$

and observe that $S(m,n) = 0$ if $m < n$.

The sum on the right hand side in Lemma 9.1 is empty if $m \leq n-1$. Hence, if $m \leq n-1$,

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a(a+ib)^m = 0.$$

Two other useful cases of Lemma 9.1 are

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a(a+ib)^n = n! ab^n$$

and

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}a(a+ib)^{n+1}=n!ab^n\left[b\binom{n+1}{2}+a(n+1)\right].$$

Using Lemma 9.1 (for the case $n-1$), the right hand side of equation (9.1) can be written as a single term and we obtain the following simpler linear recursion for the expected sum when $n\geq 1$:

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}a(a+ib)^{n-1}E_1(i;a,b)=\frac{ab^{n-1}n!}{2}\left[b\binom{n}{2}+an+1\right]. \tag{9.2}$$

With a reasonable linear recursion in hand, we have two ways of proving Theorem 8.3. The first and somewhat unsatisfactory way is to check that the formula for $E_1(n;a,b)$ given in Theorem 8.3 yields $E_1(0)=0$ and satisfies the linear recursion (9.2). The checking can be done easily by hand (using Lemma 9.1) or by a computer algebra program. Either way, we obtain Theorem 8.3.

The second is to "discover" the solution in a systematic way. This method will be necessary for finding formulas for the higher moments for which we have no reasonable guess. See [14]. We begin by transforming the recursion into a matrix equation.

Let $\mathcal{P}$ be the $(N+1)\times(N+1)$ lower triangular matrix

$$\left[(-1)^{n-i}\binom{n}{i}a(a+ib)^{n-1}\right]_{0\leq n,i\leq N}.$$

For example, when $N=3$, $\mathcal{P}$ is the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -a & a & 0 & 0 \\ a^2 & -2a(a+b) & a(a+2b) & 0 \\ -a^3 & 3a(a+b)^2 & -3a(a+2b)^2 & a(a+3b)^2 \end{bmatrix}$$

Using the vector notation introduced in Section 2, we can rewrite equation (9.2) as

$$\mathcal{P}\overrightarrow{E_1(i;a,b)}=\overrightarrow{\frac{i!ab^{i-1}}{2}\left[b\binom{i}{2}+ai+1\right]}. \tag{9.3}$$

Our next step is to find the inverse of $\mathcal{P}$. Let $\mathcal{Q}$ be the $(N+1)\times(N+1)$ lower triangular matrix

$$\left[\binom{i}{j}\frac{j!b^j}{(a+ib)^{j-1}}\right]_{0\leq i,j\leq N}.$$

For example, when $N=3$, $\mathcal{Q}$ is the matrix

$$\begin{bmatrix} a & 0 & 0 & 0 \\ a+b & b & 0 & 0 \\ a+2b & 2b & \dfrac{2b^2}{a+2b} & 0 \\ a+3b & 3b & \dfrac{6b^2}{a+3b} & \dfrac{6b^3}{(a+3b)^2} \end{bmatrix}$$

(9.2) Lemma.

$$\mathcal{P}\mathcal{Q}=\mathcal{N}\mathcal{L}$$

where $\mathcal{N}$ is the diagonal matrix whose iith entry is $ab^ii!$ and $\mathcal{L}$ is the lower triangular matrix with all ijth entries equal to 1 whenever $i\geq j$ and 0 otherwise.

Proof. The njth entry of the product $\mathcal{P}\mathcal{Q}$ equals

$$\sum_{i=j}^{n}(-1)^{n-i}\binom{n}{i}\binom{i}{j}j!b^j a(a+ib)^{n-j}.$$

Changing indices from i to $i-j$ and regrouping terms, this can be simplified to

$$\frac{n!b^j}{(n-j)!}\sum_{i=0}^{n-j}(-1)^{n-j-i}\binom{n-j}{i}a((a+jb)+ib)^{n-j}.$$

By the case $n-j$ of Lemma 9.1, the sum equals $(n-j)!ab^{n-j}$ and the lemma follows.

Lemma 9.2 can be rephrased as follows.

(9.3) Lemma.

$$\mathcal{P}^{-1}=\mathcal{Q}\mathcal{L}^{-1}\mathcal{N}^{-1}.$$

The inverse matrices $\mathcal{N}^{-1}$ and $\mathcal{L}^{-1}$ have simple interpretations when acting on a column vector $\overrightarrow{a_i}$. Multiplying on the left by $\mathcal{N}^{-1}$ divides the ith coordinate a_i by $ab^i i!$. The matrix $\mathcal{L}$ is the summation matrix and sends the vector $\overrightarrow{a_i}$ to the vector whose ith coordinate is $a_0+a_1+\ldots+a_i$. Hence, the inverse $\mathcal{L}^{-1}$ is the backward difference matrix, with all diagonal entries 1, all subdiagonal entries -1, and all other entries zero. Hence, multiplying the vector $\overrightarrow{a_i}$ on the left by $\mathcal{L}^{-1}$ results in the vector $\overrightarrow{a_i-a_{i-1}}$, obtained by taking the backward difference of the coordinates a_i, with the convention that $a_{-1}=0$.

Hence, when we apply Lemma 9.3 to the matrix equation (9.3), we obtain

$$\overrightarrow{E_1(i;a,b)}=\frac{1}{2b}Q(0,a+1,a+b,a+2b,\ldots,a+(i-1)b,\ldots,a+(N-1)b)^T.$$

Writing out the nth coordinate explicitly, we obtain the formula in Theorem 8.4.

To obtain Theorem 8.3, we need to "precondition" and consider the *adjusted* sum S_n^*, defined by

$$S_n^*=\frac{n(a+1)}{2}+b\binom{n}{2}-S_n.$$

For ordinary parking functions, the adjusted sum is the reversed sum (defined in Section 6), but this is not true in general.

Substituting S_n^* into equation (9.1) and simplifying using Lemma 9.1, we obtain

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}a(a+ib)^{n-1}E[S_i^*]=-\frac{n!ab^n(n-1)}{2}.$$

This converts to the matrix equation

$$\mathcal{P}\overrightarrow{E[S_i^*]}=-\frac{1}{2}(0,0,2ab^2,12ab^3,\ldots,i!ab^i(i-1),\ldots,N!ab^N(N-1))^T.$$

Inverting this, we obtain

$$\overrightarrow{E[S_i^*]}=-\frac{1}{2}Q(0,0,1,1,\ldots,1)^T,$$

from which the formula in Theorem 8.3 follows immediately.

10 Order properties of expected sums

In this section, we shall consider the following conjecture.

(10.1) Conjecture. The expected sum $E_1(n; u_1, u_2, \ldots, u_n)$ is an increasing function of n and the gaps $u_{i+1} - u_i$.

This conjecture may seem obvious. However, there are two factors to consider when n and the gaps are increased. On the positive side, the parking functions are allowed to take on higher values. On the negative side, there are more parking functions, and since parking functions cannot take on too many higher values, the sample might consist mostly of parking functions with smaller sums. Our intuition is that the positive factor always predominates.

We begin with a simple general result supporting this conjecture.

(10.2) Proposition. If γ is a rational number greater than 1, then

$$E_1(n; u_1, u_2, \ldots, u_n) < E_1(n; \gamma u_1, \gamma u_2, \ldots, \gamma u_n).$$

Proof. Writing $u_j(u_j + 1)$ as $u_j^2 + u_j$ in the formula in Theorem 7.5, we can write $E_1(n; \mathbf{u})P_n(\mathbf{u})$ as the sum $F(\mathbf{u}) + G(\mathbf{u})$ of two homogeneous functions in the variables $u_1, u_2, \ldots, u_n$, where $F(\mathbf{u})$ has total degree $n+1$ and $G(\mathbf{u})$ has total degree n. Using Corollary 5.6, we have

$$\begin{aligned} E_1(n; \gamma u_1, \gamma u_2, \ldots, \gamma u_n) &= \frac{\gamma^{n+1} F(\mathbf{u}) + \gamma^n G(\mathbf{u})}{\gamma^n P_n(\mathbf{u})} \\ &= \frac{(\gamma - 1)F(\mathbf{u})}{P_n(\mathbf{u})} + \frac{F(\mathbf{u}) + G(\mathbf{u})}{P_n(\mathbf{u})} \\ &= \frac{(\gamma - 1)F(\mathbf{u})}{P_n(\mathbf{u})} + E_1(n; u_1, u_2, \ldots, u_n). \end{aligned}$$

Since $\gamma > 1$, the proposition follows.

For the classical case, when $u_i = a + (i-1)b$, Conjecture 10.1 states that the expected sums $E_1(n; a, b)$ are increasing functions of n, a, and b. We shall verify this special case.

(10.3) Lemma. If $0 < a < c$,

$$E_1(n; a, b) < E_1(n; c, b).$$

Proof. Use the formula in Theorem 8.3 and observe that $-(c + nb)^{-1} > -(a + nb)^{-1}$.

Hence, $E_1(n; a, b)$ is an increasing function of a (for fixed n and b).

(10.4) Lemma.

$$E_1(n; a, b) < E_1(n+1; a, b).$$

Proof. Rewrite the formula in Theorem 8.3 in the form

$$E_1(n; a, b) = \frac{n(a+1)}{2} + b\binom{n}{2} - \frac{b}{2}\sum_{i=2}^{n} \frac{(n)_i}{(\gamma + n)^{i-1}},$$

where $(n)_i$ is a falling factorial and $\gamma = a/b$. Then, the forward difference

$$E_1(n+1; a, b) - E_1(n; a, b)$$

equals

$$\frac{a+1}{2} + bn - \frac{b}{2}\sum_{i=2}^{n}\left[\frac{(n+1)_i}{(n+1+\gamma)^{i-1}} - \frac{(n)_i}{(n+\gamma)^{i-1}}\right] - \frac{b}{2}\frac{(n+1)!}{(n+1+\gamma)^n}.$$

By an elementary induction argument, one can show that for γ a positive real number and $i = 2, 3, \ldots, n$,

$$\frac{(n+1)_i}{(n+1+\gamma)^{i-1}} - \frac{(n)_i}{(n+\gamma)^{i-1}} < 1. \tag{10.1}$$

The induction argument runs as follows. If $i = 2$,

$$\frac{(n+1)n}{n+1+\gamma} - \frac{n(n-1)}{n+\gamma} = \frac{n^2+2\gamma n+n}{n^2+2\gamma n+n+\gamma^2+\gamma} < 1.$$

Now assume that the inequality is true for $i \le k$. When $i = k+1$,

$$\begin{aligned}
& \frac{(n+1)_{k+1}}{(n+1+\gamma)^k} - \frac{(n)_{k+1}}{(n+\gamma)^k} \\
= \quad & \left[\frac{(n+1)_k}{(n+1+\gamma)^{k-1}} - \frac{(n)_k}{(n+\gamma)^{k-1}}\right]\frac{n-k+1}{n+\gamma+1} + \frac{(n)_k}{(n+\gamma)^{k-1}}\left[\frac{n-k+1}{n+\gamma+1} - \frac{n-k}{n+\gamma}\right] \\
< \quad & \frac{n-k+1}{n+\gamma+1} + \frac{(n)_k}{(n+\gamma)^{k-1}}\left[\frac{(\gamma+k)}{(n+\gamma)(n+\gamma+1)}\right] \\
= \quad & 1 - \frac{\gamma+k}{n+\gamma+1} + \frac{(n)_k}{(n+\gamma)^k}\cdot\frac{\gamma+k}{n+\gamma+1} \\
< \quad & 1.
\end{aligned}$$

From inequality (10.1), we conclude that

$$\begin{aligned}
E_1(n+1;a,b) - E_1(n;a,b) \quad & > \quad bn + \frac{a+1}{2} - \frac{b(n-1)}{2} - \frac{b}{2} \\
& = \quad \frac{bn+a+1}{2} \\
& > \quad 0.
\end{aligned}$$

(10.5) Lemma. If c is an integer strictly greater than b, then

$$E_1(n;a,b) < E_1(n;a,c).$$

Proof. Rewrite the formula for the expected sum $E_1(n; a, a+b, \ldots, a+(n-1)b)$ given in Theorem 8.4 in the form

$$\frac{n(a+1)}{2} + \frac{1}{2}\sum_{j=2}^{n}\frac{n!}{(n-j)!}\left[\frac{ab^{j-1}}{(a+nb)^{j-1}} + \frac{(j-1)b^j}{(a+nb)^{j-1}}\right].$$

From this, the lemma follows from the easy inequality: if $a > 0$ and $c > b$, then

$$\frac{b}{a+nb} < \frac{c}{a+nc}.$$

The three lemmas imply the following theorem.

(10.6) Theorem. The expected sum $E_1(n; a, a+b, \ldots, a+(n-1)b)$ is a strictly increasing function of n, a and b.

11 Factorial moments of parking functions

The decomposition in Section 5 can also be used to obtain linear recursions for higher factorial moments of sums of random parking functions. Let $\mathbf{u}$ be a sequence of non-decreasing positive integers. Let $\mathbf{a}$ be the sequence defined by $a_j = x - u_{j+1}$ and let

$$e_i^{(k)}(x; a_0, \ldots, a_{n-1}) = E[(S_i)_k],$$

the k-factorial moment of the sum of a random **u**-parking function as a function of $x, a_0, a_1, \ldots, a_{i-1}$. The *factorial moment generating function* $\mathcal{S}_i(t; \mathbf{a})$ for **u**-parking functions of length i is defined by the following formula:

$$\mathcal{S}_i(t; \mathbf{a}) = \sum_{k=0}^{\infty} e_i^{(k)}(x; \mathbf{a}) \frac{t^k}{k!}.$$

Given a discrete interval $[\alpha, \beta]$, let $U_i(\alpha, \beta)$ be the sum of a random (integer) sequence chosen with uniform distribution from the space $[\alpha, \beta]^i$ of all length-i sequences with terms in $[\alpha, \beta]$. Then $U_i(\alpha, \beta)$ can also be thought of as a length-i random sequence obtained by choosing each term independently with uniform distribution from $[\alpha, \beta]$. The factorial moments of $U_i(\alpha, \beta)$ are known and they can be expressed in a compact form by exponential generating functions (see, for example, [8]). Let

$$\mathcal{U}_i(t; \alpha, \beta) = \sum_{k=0}^{\infty} E[(U_i(\alpha, \beta))_k] \frac{t^k}{k!}.$$

Then

$$\mathcal{U}_i(t; \alpha, \beta) = \left(\frac{(1+t)^{\beta+1} - (1+t)^{\alpha}}{(\beta - \alpha + 1)t} \right)^i.$$

(11.1) Theorem. Let k be a positive integer. Then the factorial moments of the sum of a random **u**-parking function of length n satisfies the following linear recursion:

$$E[(U_n(1,x))_k] = \sum_{m=0}^{n} \binom{n}{m} \frac{a_m^{n-m} g_m(x; \mathbf{a})}{x^n} \left(\sum_{j=0}^{k} \binom{k}{j} e_m^{(j)}(x; \mathbf{a}) E[(U_{n-m}(u_{m+1}+1, x))_{k-j}] \right).$$

Proof. The proof is almost the same as the proof of Theorem 7.1. Consider the event that the maximum subsequence forming a **u**-parking function is indexed by $\{i_1, i_2, \ldots, i_m\}$. Because the length-$m$ **u**-parking function and the length-$(n-m)$ sequence from $[u_{m+1}+1, x]^{n-m}$ are chosen independently and an analogue of the binomial theorem holds for falling factorials, the expected value of $(U_n(1,x))_k$ conditioned on this event is

$$\sum_{j=0}^{k} \binom{k}{j} e_m^{(j)}(x; \mathbf{a}) E[(U_{n-m}(u_{m+1}+1, x))_{k-j}].$$

Summing over the conditional expectations, we obtain the linear recursion.

As with the expected sum, it follows from the linear recursion that $g_i(x; \mathbf{a}) e_i^{(k)}(x; \mathbf{a})$ is a sum of $k+1$ homogeneous polynomial in the variables $x, a_0, a_1, \ldots, a_{i-1}$ having total degree i, $i+1$, $\ldots$, and $i+k$.

When Theorem 11.1 is restated in terms of exponential generating functions, we obtain the following linear recursion for the factorial moment generating functions $\mathcal{S}_i(t; \mathbf{a})$:

$$x^n \mathcal{U}_n(t; 1, x) = \sum_{i=0}^{n} \binom{n}{i} a_i^{n-i} g_i(x; \mathbf{a}) \mathcal{S}_i(t; \mathbf{a}) \mathcal{U}_{n-i}(t; u_{i+1}+1, x), \tag{11.1}$$

We can use the matrix method in Section 7 to rewrite equation (11.1) in the following more compact form:

$$\mathcal{M} \overrightarrow{g_i(x; \mathbf{a}) \mathcal{S}_i(t; \mathbf{a})} = \overrightarrow{x^i \mathcal{U}_i(t; 1, x)},$$

where $\mathcal{M}$ is the lower triangular matrix with ijth entry equal to

$$\binom{i}{j} a_j^{i-j} \mathcal{U}_{i-j}(t; u_{j+1}+1, x)$$

if $i \geq j$ and zero if $i < j$. From this linear equation, one can obtain by Cramer's rule a rather complicated determinantal formula for $\mathcal{S}_i(t;\mathbf{a})$. This determinantal formula seems to have no simple form.

After this, it is a pleasant surprise that there is a simple Appell relation for $\mathcal{S}_i(t;\mathbf{a})$. First observe that

$$\begin{aligned}\sum_{n=0}^{\infty} x^n \mathcal{U}_n(t;1,x)\frac{q^n}{n!} &= \sum_{n=0}^{\infty}\left[\frac{(1+t)^{x+1}-(1+t)}{t}\right]^n \frac{q^n}{n!} \\ &= \exp(\frac{q}{t}((1+t)^{x+1}-(1+t)))\end{aligned}$$

and

$$\sum_{n=i}^{\infty} a_i^{n-i}\mathcal{U}_{n-i}(t;u_{i+1}+1,x)\frac{q^{n-i}}{(n-i)!} = \exp(\frac{q}{t}((1+t)^{x+1}-(1+t)^{1+x-a_i})).$$

Hence, multiplying equation (11.1) by $q^n/n!$, summing over all non-negative integers n, and dividing both sides by $\exp(q(i+t)^{x+1}/t)$, we obtain

$$\exp(-\frac{q}{t}(1+t)) = \sum_{i=0}^{\infty} g_i\mathcal{S}_i(t)\exp(-\frac{q}{t}(1+t)^{1+x-a_i})\frac{q^i}{i!}.$$

Changing variables from q to qt, we obtain the following Appell relation.

(11.3) Theorem.

$$\exp(-q(1+t)) = \sum_{i=0}^{\infty} g_i(x;\mathbf{a})\mathcal{S}_i(t;\mathbf{a})\exp(-q(1+t)^{1+x-a_i})\frac{t^iq^i}{i!}.$$

The left hand side of the Appell relation does not depend on x (which is not surprising, as the linear recursion from which it is derived holds for all sufficiently large integer x). Hence, simpler Appell relations can be obtained by setting x to be 0 or any convenient constant or variable.

12 Second factorial moments of sums of parking functions

The second power moment is of particular importance in estimating the spread of a distribution. In this section, we shall derive an explicit formula for the second factorial moment of the sum of a random $\mathbf{u}$-parking function with the matrix method used in Section 7.

We start with the linear recursion for the second factorial moment.

(12.1) Lemma. The second factorial moment $e_n^{(2)}(x;\mathbf{a})$ satisfies the linear recursion

$$\begin{aligned}&x^n E[(U_n(1,x))_2] \\ =& \sum_{m=0}^{n}\binom{n}{m}a_m^{n-m}g_m(x;\mathbf{a}) \\ &\cdot(e_m^{(2)}(x;\mathbf{a})+2e_m^{(1)}(x;\mathbf{a})E[U_{n-m}(x-a_m+1,x)])+E[(U_{n-m}(x-a_m+1,x))_2]\end{aligned}$$

with

$$E[U_{n-m}(x-a_m+1,x)] = \frac{(n-m)(2x-a_m+1)}{2},$$

$$E[(U_{n-m}(x-a_m+1,x))_2] = \frac{(2x-a_m+1)^2(n-m)^2}{4} + (n-m)\left[\frac{a_m^2}{12}-\frac{2x-a_m}{2}-\frac{7}{12}\right],$$

and

$$E[(U_n(1,x))_2] = \frac{n(n-1)(x+1)^2}{4} + \frac{n(x^2-1)}{3}.$$

Next, we rewrite the linear recursion as the following system of linear equations:

$$\begin{aligned}
& \frac{1}{4}(x+1)^2\overrightarrow{i(i-1)x^i} + \frac{1}{3}(x^2-1)\overrightarrow{ix^i} \\
= \quad & \mathcal{A}\overrightarrow{g_i(x;\mathbf{a})e_i^{(2)}(x;\mathbf{a})} \\
& +\mathcal{B}\overrightarrow{g_i(x;\mathbf{a})e_i^{(1)}(x;\mathbf{a})(2x-a_i+1) + g_i(x;\mathbf{a})(x^2-xa_i+\frac{1}{3}(a_i^2-1))} \\
& + \quad \mathcal{C}\overrightarrow{\left(\frac{2x-a_i+1}{2}\right)^2 g_i(x;\mathbf{a})},
\end{aligned}$$

where $\mathcal{A}$, $\mathcal{B}$ are the lower triangular matrices described in Section 7 and $\mathcal{C}$ is the lower triangular matrix with ijth entry equal to

$$i(i-1)\binom{i-2}{j}a_j^{i-j}$$

if $i > j+1$ and zero otherwise.

As in Section 7, we apply $\mathcal{A}^{-1}$ to both sides of the linear equation. Using Lemma 4.1, the left hand side simplifies to

$$\frac{x^2(x+1)^2}{4}\overrightarrow{D^2 g_i(x;\mathbf{a})} + \frac{x(x^2-1)}{3}\overrightarrow{Dg_i(x;\mathbf{a})}.$$

To simplify the right hand side, we need the entries of $\mathcal{A}^{-1}\mathcal{B}$ and $\mathcal{A}^{-1}\mathcal{C}$. The entries of $\mathcal{A}^{-1}\mathcal{B}$ were calculated in Section 7. The entries of $\mathcal{A}^{-1}\mathcal{C}$ can be calculated using a similar method. Indeed, the ijth entry of $\mathcal{A}^{-1}\mathcal{C}$ is

$$\binom{i}{j}a_j^2 D^2 g_{i-j}(a_j; a_j, a_{j+1}, \ldots, a_{i-1})$$

if $i \geq j$ and zero otherwise. Using these facts and the differential relation for Gončarov polynomials, we obtain the equation:

$$\begin{aligned}
& g_n(x;a_0,a_1,\ldots,a_{n-1})e_n^{(2)}(x;a_0,a_1,\ldots,a_{n-1}) \\
= \quad & \frac{x^2(x+1)^2}{4}n(n-1)g_{n-2}(x;a_2,a_3,\ldots,a_{n-1}) + \frac{x(x^2-1)}{3}ng_{n-1}(x;a_1,a_2,\ldots,a_{n-1}) \\
& -\frac{n(n-1)}{4}\sum_{i=0}^{n-2}\binom{n-2}{i}a_i^2(2x-a_i+1)^2 \\
& \cdot g_{n-i-2}(a_i;a_{i+2},a_{i+3},\ldots,a_{n-1})g_i(x;a_0,a_1,\ldots,a_{i-1}) \\
& -n\sum_{i=0}^{n-1}\binom{n-1}{i}a_i g_{n-i-1}(a_i;a_{i+1},a_{i+2},\ldots,a_{n-1}) \\
& \quad \cdot\Big[(2x-a_i+1)g_i(x;a_0,a_1,\ldots,a_{i-1})e_i^{(1)}(x;a_0,a_1,\ldots,a_{i-1}) \\
& \quad + \left(x^2-xa_i+\frac{a_i^2-1}{3}\right)g_i(x;a_0,a_1,\ldots,a_{i-1})\Big].
\end{aligned}$$

Setting $x = 0$ and $a_i = -u_{i+1}$, we obtain the following theorem.

(12.2) Theorem

$$
\begin{aligned}
& P(u_1, u_2, \ldots, u_n) E_2(n; u_1, u_2, \ldots, u_n) \\
= \quad & n \sum_{i=0}^{n-1} \binom{n-1}{i} u_{i+1} P_{n-i-1}(u_{i+2} - u_{i+1}, u_{i+3} - u_{i+1}, \ldots, u_n - u_{i+1}) \\
& \quad \cdot \left[(u_{i+1} + 1) P_i(u_1, \ldots, u_i) E_1(i; u_1, \ldots, u_i) + \left(\frac{u_{i+1}^2 - 1}{3} \right) P_i(u_1, \ldots, u_i) \right] \\
& - \frac{n(n-1)}{4} \sum_{i=0}^{n-2} \binom{n-2}{i} u_{i+1}^2 (u_{i+1} + 1)^2 \\
& \quad \cdot P_{n-i-2}(u_{i+3} - u_{i+1}, u_{i+4} - u_{i+1}, \ldots, u_n - u_{i+1}) P_i(u_1, \ldots, u_i).
\end{aligned}
$$

On comparing the formulas in Theorems 7.4 and 12.2, it is evident that one can obtain, in a mechanical way, formulas for any higher moments and that these formulas has a recognizable pattern.

For many applications, asymptotic formulas are much more useful than explicit formulas. The only known results are asymptotic formulas for the expected sum and second moment of random ordinary parking functions. These formulas can be extracted from [20, 24, 30] (see also [4]). Briefly,

$$
\begin{aligned}
\mu_n = E[S_n] \quad & \sim \quad \binom{n+1}{2} - \sqrt{\frac{\pi}{8}} n^{3/2}, \\
E[S_n^2] \quad & \sim \quad \binom{n+1}{2}^2 - n(n+1)\sqrt{\frac{\pi}{8}} n^{3/2} + \frac{5n^3}{12} + \sqrt{\frac{\pi}{8}} n^{3/2}.
\end{aligned}
$$

Hence, if σ_n is the variance of S_n, then

$$
\sigma_n^2 \sim (\frac{5}{12} - \frac{\pi}{8}) n^3 \approx 0.0239676 n^3.
$$

Using these formulas and Chebyshev's inequality,

$$
\Pr(|S_n - \mu_n| \geq \sqrt{\frac{\pi}{8}} n^{3/2}) \leq \frac{8\sigma^2}{\pi} \approx 0.061033,
$$

so that about 94% of ordinary parking functions have sums which are at least

$$
\binom{n+1}{2} - \sqrt{\frac{\pi}{2}} n^{3/2}.
$$

Moreover, as $K \to \infty$,

$$
\Pr(|S_n - \mu_n| > K n^{3/2}) \leq \frac{\sigma_n^2}{K^2} \to 0,
$$

in other words, when n is large, most ordinary parking functions have sums which are close to $\binom{n+1}{2}$, the largest possible value. Can one prove a similar result for $\mathbf{u}$-parking functions? A less speculative unsolved problem is to extend asymptotical results for ordinary parking functions to the classical case when $\mathbf{u}$ is an arithmetic progression.

13 Variants of parking functions

13.1. Reversed parking functions.

Let $\mathbf{u}$ be a sequence of non-decreasing positive integers. A *reversed* $\mathbf{u}$*-parking function* of length n on the discrete interval $[1, x]$ is a sequence $(x_1, x_2, \ldots, x_n)$ with terms in $[1, x]$ whose sequence of order statistics satisfies $x_{(i)} \geq u_i$. The *suites majeures* of Kreweras [12] are special cases of reversed parking functions.

As the astute reader might have noticed, Gončarov polynomials are better matched with reversed parking functions. For example, if $x \geq u_n$, the number of reversed $\mathbf{u}$-parking functions on $[1, x]$ is simply $g_n(x; u_1, u_2, \ldots, u_n)$. Almost all the results about Gončarov polynomials stated in this paper can be given combinatorial proofs using reversed parking functions. We note also that the slight incompatibility of Gončarov polynomials and parking functions is overcome in this paper by the substitution $u_{i+1} = x - a_i$. This allows us to shift and reflect the domain, so that we are essentially working with reversed parking functions.

13.2. Injective parking functions.

An ordinary parking function is injective or one-to-one if and only if it is a permutation. Thus, injective $\mathbf{u}$-parking functions may be considered generalizations of permutations.

Let $Q_n(u_1, u_2, \ldots, u_n)$ be the number of injective $\mathbf{u}$-parking functions of length n. Since it is almost immediate that the decomposition for integer sequences or discrete functions described in Section 5 works when restricted to injective functions, we have the following theorem.

(13.1) Theorem. Let x be an integer greater than or equal to u_n. Then

$$(x)_n = \sum_{m=0}^{n} \binom{n}{m} (x - u_{m+1})_{n-m} Q_m(u_1, u_2, \ldots, u_m).$$

Injective parking functions has a theory parallel to the one given in this paper. It is based on "difference" Gončarov polynomials (see [13] for the definition). For example, $Q_n(u_1, u_2, \ldots, u_n)$ equals $n! \det \mathcal{F}$, where $\mathcal{F}$ is the matrix with ijth entry equal to

$$\frac{(-u_i)_{j-i+1}}{(j-i+1)!}$$

if $j - i + 1 \geq 0$ and 0 otherwise.

13.3. Real-valued parking functions.

Let $\mathbf{u}$ be a non-decreasing sequence of non-negative real numbers. A *real-valued parking function of length* n is a sequence $(x_1, x_2, \ldots, x_n)$ of non-negative real numbers whose sequence of order statistics satisfies $x_{(i)} \leq u_i$. Using exactly the same proof, one can prove that sequences of length n with terms in the continuous interval $[0, x]$ satisfies the following decomposition, analogous to the one given in Corollary 5.2.

(13.2) Theorem. There is a bijection between the set $[0, x]^n$ of all length-n sequences with terms in the continuous interval $[0, x]$ and the disjoint union of cartesian products

$$\bigcup_{\{i_1, i_2, \ldots, i_m\}} \mathrm{Park}(i_1, i_2, \ldots, i_m) \times (u_{m+1}, x]^{n-m},$$

where $\mathrm{Park}(i_1, i_2, \ldots, i_m)$ is the set of real-valued length-m $\mathbf{u}$-parking functions indexed by $\{i_1, i_2, \ldots, i_m\}$ and $(u_{m+1}, x]^{n-m}$ is the set of length-$(m - n)$ sequences with terms in the continuous half-open interval $(u_{m+1}, x]$ indexed by the complement of $\{i_1, i_2, \ldots, i_m\}$.

Let $\bar{P}_n(\mathbf{u})$ be the probability that a random sequence $(X_1, X_2, \ldots, X_n)$ with the terms X_i chosen independently with uniform distribution from $[0, x]$ is a $\mathbf{u}$-parking function. Then, by Theorem 13.2, $\bar{P}_n(\mathbf{u})$ satisfies the following linear recursion:

$$1 = \sum_{m=0}^{n} \binom{n}{m} \frac{(x - u_{m+1})^{n-m}}{x^{n-m}} \bar{P}_m(u_1, u_2, \ldots, u_m).$$

Comparing this recursion with the recursion in Corollary 5.3, we obtain the following theorem.

(13.3) Theorem.

$$\bar{P}_n(u_1, u_2, \ldots, u_n) = \frac{P_n(u_1, u_2, \ldots, u_n)}{x^n}.$$

This theorem has appeared earlier in the paper [19]. Pitman and Stanley proved this theorem using their decomposition for $\mathbf{u}$-parking functions (which works for real numbers u_i also) described in Section 6.

Theorems 5.4 and 13.3 together imply that

$$\bar{P}_n(u_1, u_2, \ldots, u_n) = \frac{(-1)^n g_n(0; u_1, u_2, \ldots, u_n)}{x^n}.$$

The analogue for "reversed" real-valued $\mathbf{u}$-parking functions is usually stated in terms of an integral.

(13.4) Theorem. Let $0 \le u_n \le u_{n-1} \le \ldots \le u_1 \le x$. Then the probability that a length-n sequence $(X_1, X_2, \ldots, X_n)$ with terms X_i chosen independently with uniform distribution from $[0, x]$ satisfies the conditions $X_{(i)} \ge u_i, i = 1, 2, \ldots, n$, is

$$\frac{n!}{x^n} \int_{u_1}^{x} \int_{u_2}^{t_1} \cdots \int_{u_n}^{t_{n-1}} dt_n dt_{n-1} \ldots dt_1.$$

Proof. Condition on the size of the $(n-1)$st order statistics and use some well-known facts (see, for example, [3], Section 1.7) about independence and densities of the order statistics for a sequence of independent uniformly distributed random variables on $[0, x]$. See, for example, [28], [17], or [13].

This theorem seems to be first stated and proved by Steck [28] and rediscovered by many others.

Using the decomposition, we can also obtain the following recursion for the expected sums $\bar{E}_1(n; \mathbf{u})$ of random length-n real-valued $\mathbf{u}$-parking functions:

$$\frac{nx}{2} = \sum_{m=0}^{n} \binom{n}{m} \frac{(x - u_{m+1})^{n-m} P_m(\mathbf{u})}{x^n} \left(\bar{E}_1(m; \mathbf{u}) + \frac{(n-m)(x + u_{m+1})}{2} \right).$$

This recursion can be obtained from the recursion for integer-valued parking functions by deleting all terms not of total degree $n+1$. Hence, $P_n(\mathbf{u})\bar{E}_1(n; \mathbf{u})$ is a homogeneous polynomial in the variables $u_1, u_2, \ldots, u_n$ of total degree $n+1$ and equals

$$\frac{n}{2} \sum_{j=1}^{n} \binom{n-1}{j-1} u_j^2 \frac{P_{n-j}(u_{j+1} - u_j, u_{j+2} - u_j, \ldots, u_n - u_j) P_{j-1}(u_1, u_2, \ldots, u_{j-1})}{P_n(u_1, u_2, \ldots, u_n)}.$$

In the ordinary case, we have the formula

$$\bar{E}_1(n; a, a+b, \ldots, a+(n-1)b) = \frac{na}{2} + \frac{1}{2} \sum_{j=2}^{n} \frac{n!}{(n-j)!} \frac{b^{j-1}(a + (j-1)b)}{(a+nb)^{j-1}}.$$

14 Historical remarks

The idea behind parking functions has occurred in many different subjects and its history is replete with rediscoveries. No one paper contains a complete overview, but if one combines four papers, one can obtain a reasonably complete picture. The first paper is Niederhausen [17]. It offers a good survey of the use of real-valued parking functions in statistics up to around 1980. A comprehensive account of how parking functions occur in statistics and the study of certain polytopes and arrangements of hyperplanes can be found in Pitman and Stanley [19]. An excellent bibliography of work on bijections between ordinary parking functions and labelled trees (to around 2000) can be found in Gilbey and Kalikow [6]. Finally, a clear discussion of hashing and its relations to ordinary parking functions can be found in the paper of Flajolet, Poblete and Viola [4].

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