The Asymptotic Capacity of Multi-Dimensional Runlength-Limited Constraints and Independent Sets in Hypergraphs

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Let $\boldsymbol{C}(n,d)$ be the Shannon capacity of the n-dimensional $(d,$T$)$-runlength-limited (RLL) constraint. Denote by $\boldsymbol{I}(n,q)$ the number of independent sets in the Hamming graph with vertices consisting of all n-tuples over an alphabet of size q and edges connecting pairs of vertices with Hamming distance 1. We show that $\lim_{n}$∆T $\boldsymbol{C}(n,d)$ = $\lim_{n}$∆T $(d+1)^{-n} \log_2 I(n,d+1)=1/(d+1)$. Our method also leads to an improvement of a previous bound by Alon on the number of independent sets in regular graphs and to a generalization of this bound to a family of hypergraphs, of which the Hamming graphs can be thought of as a special case.

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Abstract

Let $\mathcal{C}(n,d)$ be the Shannon capacity of the n-dimensional (d,∞)-runlength-limited (RLL) constraint. Denote by $I(n,q)$ the number of independent sets in the Hamming graph with vertices consisting of all n-tuples over an alphabet of size q and edges connecting pairs of vertices with Hamming distance 1. We show that $\lim_{n\to\infty} \mathcal{C}(n,d) = \lim_{n\to\infty} (d+1)^{-n} \log_2 I(n,d+1) = 1/(d+1)$. Our method also leads to an improvement of a previous bound by Alon on the number of independent sets in regular graphs and to a generalization of this bound to a family of hypergraphs, of which the Hamming graphs can be thought of as a special case.

Keywords: Regular graphs; Hamming graphs; Linear hypergraphs; Multi-dimensional constraints; Runlength-limited constraints.

1 Introduction

For any n-tuple of positive integers $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ let Γ be an n-dimensional $m_1 \times m_2 \times \ldots \times m_n$ binary array whose entries are indexed by n-tuples of integers

$$\mathbf{j} \in \{0, 1, \ldots, m_1-1\} \times \{0, 1, \ldots, m_2-1\} \times \ldots \times \{0, 1, \ldots, m_n-1\}.$$

We say that Γ satisfies the (d,∞)-runlength-limited (RLL) constraint if and only if for any two indexes $\mathbf{j}$ and $\mathbf{j}'$ that differ in only one component and differ by less than $d+1$ in that

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component, either $\Gamma(\mathbf{j}) = 0$ or $\Gamma(\mathbf{j}') = 0$. That is, every one-dimensional sub-array of Γ satisfies the one-dimensional (d,∞)-RLL constraint. Let $\mathcal{A}(n,d,\mathbf{m})$ be the set of all such arrays. The Shannon capacity of the n-dimensional (d,∞)-RLL constraint is defined by

$$\begin{aligned} \mathcal{C}(n,d) &= \lim_{i\to\infty} \frac{\log_2 |\mathcal{A}(n,d,\mathbf{m}^{(i)})|}{\prod_{\ell=1}^{n} m_\ell^{(i)}} & (1) \\ &= \inf_{\mathbf{m}} \frac{\log_2 |\mathcal{A}(n,d,\mathbf{m})|}{\prod_{\ell=1}^{n} m_\ell}, & (2) \end{aligned}$$

where $\mathbf{m}^{(i)} = (m_1^{(i)}, m_2^{(i)}, \ldots, m_n^{(i)})$ is any sequence of n-tuples of integers satisfying $\min_\ell m_\ell^{(i)} \to \infty$. That the right-hand side of (1) is independent of how the limit is taken and coincides with (2) follows from sub-additivity arguments; see [4], [6].

The following facts about $\mathcal{C}(n,d)$ are known:

1. $\mathcal{C}(1,d) = \log_2 \alpha_d$, where α_d is the positive real root of the polynomial $x^{d+1} - x^d - 1$ [11, p. 65].
2. $\mathcal{C}(2,d) \sim (\log_2 d)/d$ (namely, $\lim_{d\to\infty} \mathcal{C}(2,d) \cdot (d/\log_2 d) = 1$) [6].
3. $0.5878911617 \le \mathcal{C}(2,1) \le 0.5878911619$ [3], [9], [13].
4. $0.5225 \le \mathcal{C}(3,1) \le 0.5269$ [9].
5. $\mathcal{C}(n,d) \ge 1/(d+1)$ for all n [4], [6]. This follows by further constraining the 1's in Γ to have indexes $j_1, j_2, \ldots, j_n$ satisfying $j_1 + j_2 + \ldots + j_n \equiv 0 \pmod{(d+1)}$.

The last fact, together with the simple observation that $\mathcal{C}(n,d)$ is decreasing in n for fixed d (implied by the infimum-based specification of $\mathcal{C}(n,d)$ in (2)), raises the possibility that $\mathcal{C}(n,d)$ decreases with n all the way down to $1/(d+1)$. Our contribution to the emerging picture of $\mathcal{C}(n,d)$ is to prove that this is the case.

Our approach to proving the aforementioned convergence is based on showing that the crude sequence of upper bounds,

$$\mathcal{C}(n,d) \le \frac{\log_2 |\mathcal{A}(n,d,(d+1)\mathbf{1})|}{(d+1)^n},$$

where $\mathbf{1}$ denotes the n-tuple consisting of all 1's, also converges to $1/(d+1)$, thereby closing the gap.

2 Statement of results

We first recall some basic terms from graph theory. For an (undirected) graph G, let V_G (respectively, E_G) denote the set of vertices (respectively, edges) of G, where $E_G \subseteq \{e \subseteq V_G :$

$|e| = 2\}$. For a vertex v in V_G, denote by $\delta_G(v)$ the degree of v in G, i.e., $\delta_G(v)$ is the number edges that are incident on v in G. A graph G is called s-regular if $\delta_G(v) = s$ for all $v \in V_G$. A graph G is bipartite if V_G can be partitioned into a 'left' set X and a 'right' set Y such that $|e \cap X| = |e \cap Y| = 1$ for all $e \in E_G$. An independent set in G is a subset $T \subseteq V_G$ such that $|e \cap T| \leq 1$ for all $e \in E_G$. The number of independent sets in G will be denoted by $I(G)$.

Consider the family of graphs $\mathcal{H}(n, q)$ whose vertices are all indexes $\mathbf{j} \in \{0, 1, \ldots, q-1\}^n$ and two vertices are connected by an edge if and only if they are at Hamming distance 1 apart. The graphs $\mathcal{H}(n, q)$ are known in the literature as the Hamming graphs. It is not hard to see that the set of locations of 1's in any array in $\mathcal{A}(n, d, (d+1)\mathbf{1})$ corresponds to an independent set in the graph $\mathcal{H}(n, d+1)$. The reverse is also true. Thus, $|\mathcal{A}(n, d, (d+1)\mathbf{1})|$ is equal to the number of independent sets in $\mathcal{H}(n, q)$ for $q = d+1$. We shall henceforth use the shorthand notation $I(n, q)$ for the number of independent sets, $I(\mathcal{H}(n, q))$, in $\mathcal{H}(n, q)$. Note that $I(n, q)$ is also the number of codes of length n and minimum Hamming distance ≥ 2 over an alphabet of size q.

The quantity $I(n, q)$ has received some attention in the literature, particularly for the case $q = 2$ ($\mathcal{H}(n, 2)$ is more commonly known as the binary Hamming hypercube). The strongest result for $q = 2$ is due to Korshunov and Sapozhenko [7] (see also [10]), who show that

$$I(n, 2) \sim 2\sqrt{\mathrm{e}}\, 2^{2^{n-1}},$$

where e is the base of natural logarithms. It readily follows that $2^{-n} \log_2 I(n, 2) \to 1/2$, thus proving our result for $d = 1$.

The case $d = 1$ also follows from more general bounds on the number of independent sets in regular graphs with large degree, due to Alon [1], and on the number of independent sets in regular bipartite graphs due to Kahn [5]. For example, [1] shows that the number of independent sets in an s-regular graph G with r vertices satisfies

$$\frac{\log_2 I(G)}{r} \leq \frac{1}{2} + O(s^{-0.1}). \tag{3}$$

For $\mathcal{H}(n, 2)$ it is easy to see that $r = 2^n$ and $s = n$, thereby implying the desired convergence result for $d = 1$. The bound of [5] for bipartite graphs can also be applied since $\mathcal{H}(n, 2)$ is a bipartite graph with a left vertex set $\{\mathbf{j} = (j_1, j_2, \ldots, j_n) : j_1 + j_2 + \ldots + j_n \equiv 0 \pmod 2\}$ and a right vertex set $\{\mathbf{j} : j_1 + j_2 + \ldots + j_n \equiv 1 \pmod 2\}$. This results in a bound similar to (3) except with a smaller error term of $O(1/s)$.

Little seems to be known about $I(n, q)$ for $q > 2$. Numerical computations of $I(n, q)$ for $q = 2, 3, 4$ and small n have been carried out [12]. We are not aware of any asymptotic analysis of $I(n, q)$ for $q > 2$ beyond what we derive here. Specifically, we prove the following.

Theorem 2.1 *The number of independent sets $I(n, q)$ in the Hamming graph $\mathcal{H}(n, q)$ satisfies*

$$\frac{\log_2 I(n, q)}{q^n} \leq \frac{1}{q} + O\left(\frac{\log^2(qn)}{qn}\right),$$

for all q.

Setting $q = d + 1$ in the last theorem shows that the right-hand side of (1) does indeed converge to $1/(d+1)$ as n tends to infinity, thereby proving the following corollary.

Corollary 2.2

$$\lim_{n\to\infty} \mathcal{C}(n,d) = \frac{1}{d+1}.$$

Our method also yields the following improvement over (3) for regular graphs.

Theorem 2.3 *For an s-regular graph G with r vertices,*

$$\frac{\log_2 I(G)}{r} \le \frac{1}{2} + O\left(\frac{\log^2 s}{s}\right). \tag{4}$$

Unfortunately, (4) is not tight for the widely conjectured worst-case graph consisting of a disjoint union of complete bipartite graphs with degree s [1], [5]. Thus, there is still room for improvement.

Both Theorems 2.1 and 2.3 will follow from Theorem 2.4 below, which provides an upper bound on the number of independent sets in a certain family of hypergraphs (see [2]). For a hypergraph G, let V_G and E_G, respectively, denote the set of vertices and set of hyperedges of G, where $E_G \subseteq \{e \subseteq V_G : |e| \ge 2\}$. For a vertex v in V_G let $N_G(v)$ denote the set of vertices that are adjacent to v in G, namely,

$$N_G(v) = \Big\{v' \in V_G \setminus \{v\} : \{v, v'\} \subseteq e \text{ for some } e \in E_G\Big\},$$

and let $\delta_G(v) = |N_G(v)|$ be the degree of v in G. Similarly to (ordinary) graphs, we say that $T \subseteq V_G$ is an independent set in G if $|e \cap T| \le 1$ for all $e \in E_G$, and we denote by $I(G)$ the number of independent sets in G.

A hypergraph G is *t-uniform* if each hyperedge contains t vertices, and is called s-regular if each vertex is contained in s hyperedges. If the intersection of any two hyperedges of G contains at most one vertex then G is said to be *linear*.

Theorem 2.4 *Let G be a t-uniform s-regular linear hypergraph with r vertices. The number of independent sets $I(G)$ in G satisfies*

$$\log_2 I(G) \le \frac{r}{t}\left(1 + O\left(\frac{\log^2(ts)}{s}\right)\right).$$

The proof of Theorem 2.4 is given in Section 3. We note that a subset of the Hamming graph $\mathcal{H}(n,q)$ is an independent set if and only if it is also an independent set in the q-uniform, n-regular, linear hypergraph with the same vertex set as $\mathcal{H}(n,q)$ and with hyperedges being the subsets of vertices of $\mathcal{H}(n,q)$ that agree in all but one component. Hence, Theorem 2.1 follows from Theorem 2.4 by setting $r = q^n$, $s = n$, and $t = q$. Theorem 2.3 follows by setting $t = 2$. The implied constants in the $O(\cdot)$ terms can also be derived for each case by substituting the appropriate values for r, s, and t in our analysis in Section 3.

In Section 4 we present a generalization of Theorem 2.4 to uniform linear hypergraphs that are not necessarily regular.

3 Independent sets in uniform, regular, linear hypergraphs

Given a hypergraph G and a subset $Y \subseteq V_G$, let G_Y be the induced (i.e., maximally connected) sub-hypergraph of G on the vertices Y, that is,

$$V_{G_Y} = Y \quad \text{and} \quad E_{G_Y} = \Big\{ e \cap Y : e \in E_G, |e \cap Y| \geq 2 \Big\}.$$

Let $\mathcal{S}_i(G)$ be the set of all induced sub-hypergraphs of G on i vertices, namely,

$$\mathcal{S}_i(G) = \{G_Y : Y \subseteq V_G, |Y| = i\}.$$

Define $f_i(G)$ as

$$f_i(G) = \max_{H \in \mathcal{S}_i(G)} I(H). \tag{5}$$

Note that $f_1(G) = 2$, $f_{|V_G|}(G) = I(G)$, $f_i(G) \geq f_{i-1}(G)$ for $1 < i \leq |V_G|$, and

$$f_i(G) \leq 2^i. \tag{6}$$

We also define $f_0(G) = 1$ as standing for the empty independent set in an 'empty' sub-hypergraph. Let $\mathcal{S}_i^*(G)$ denote the subset of sub-hypergraphs in $\mathcal{S}_i(G)$ that achieve the maximum in (5). We then have the following simple lemma.

Lemma 3.1 *Given a hypergraph G and an integer i in the range $1 \leq i \leq |V_G|$, let Δ be a nonnegative integer that satisfies $\Delta \leq \delta_H(v)$ for some vertex v of some sub-hypergraph $H \in \mathcal{S}_i^*(G)$. Then*

$$f_i(G) \leq f_{i-1}(G) + f_{i-\Delta-1}(G). \tag{7}$$

Proof. For any sub-hypergraph $H \in \mathcal{S}_i^*(G)$ and any vertex $v \in V_H$, the number of independent sets $I(H) = f_i(G)$ is equal to the sum of the number of independent sets that contain v and the number of independent sets that do not contain v. The latter is

$$I(H_{V_H \setminus \{v\}}) \leq f_{i-1}(G)$$

and the former is

$$I(H_{V_H \setminus (\{v\} \cup N_H(v))}) \leq f_{i-\delta_H(v)-1}(G).$$

The lemma follows from the fact that $f_i(G)$ is non-decreasing in i. □

The idea behind the proof of Theorem 2.4 is to start the recursion (7) with the bound $f_{i_0}(G) \leq 2^{i_0}$ for some i_0 and then proceed by bounding the result of iterating the recursion (7) up to $i = |V_G|$. The key to obtaining a good final bound is, for each i, to choose H and v to make Δ in (7) as large as possible. The extent to which this can be done depends on the structure of G.

Specializing to uniform, regular, linear hypergraphs, the following lemma provides a lower bound on the largest possible choice for Δ, for each i.

Lemma 3.2 *Let G be a t-uniform, s-regular, linear hypergraph with r vertices. Then for every $H \in \mathcal{S}_i(G)$*

$$\max_{v \in H} \delta_H(v) \geq \left\lceil s\left(\frac{ti}{r} - 1\right)\right\rceil. \tag{8}$$

Proof. Fix a sub-hypergraph $H \in \mathcal{S}_i(G)$. We prove the lemma by counting ordered pairs of adjacent vertices in V_H in two different ways. Let

$$P = \Big\{(v, v') \in V_H \times V_H : v \neq v' \text{ and } \{v, v'\} \subseteq e \text{ for some } e \in E_G\Big\},$$

and for every $e \in E_G$ let $\beta_e = |e \cap V_H|$. Then $|P| = \sum_{e \in E_G} \beta_e(\beta_e - 1)$; that is, for each hyperedge in G we count the number of ordered pairs of elements of V_H in that hyperedge and sum this over all hyperedges. By the linearity of G each ordered pair is counted only once. Further, $\sum_{e \in E_G} \beta_e = si$ since each vertex $v \in V_H$ contributes to the sum for precisely the s hyperedges that contain it.

Since the function $(\beta_e)_{e \in E_G} \mapsto \sum_{e \in E_G} \beta_e(\beta_e - 1)$ is Schur convex [8] in the variables β_e, its minimum value subject to the constraint $\sum_{e \in E_G} \beta_e = si$ is achieved when β_e is constant-valued.[1] And, since $|E_G| = rs/t$, the minimizing β_e is $si/(rs/t) = ti/r$. Thus,

$$\begin{aligned}
\min_{\beta_e} \sum_{e \in E_G} \beta_e(\beta_e - 1) &= \frac{rs}{t}\frac{ti}{r}\left(\frac{ti}{r} - 1\right) \\
&= si\left(\frac{ti}{r} - 1\right) \\
&\leq |P|.
\end{aligned}$$

[1] We can obtain a tighter bound on $\max_{v \in H} \delta_H(v)$ by not ignoring the fact that β_e is integer-valued. In this case, the minimizing β_e takes on at most two values that differ by 1. The resulting bound, however, is more complicated and only slightly improves our bounds on the asymptotic number of independent sets.

On the other hand, letting $\Delta = \max_{v \in H} \delta_H(v)$, we clearly have $|P| \le \Delta |V_H| = \Delta i$. Combining the two bounds on $|P|$ and dividing by i gives (8). $\square$

We also need the following two elementary propositions.

Proposition 3.3 *The equation $x^{m+1} = x^m + 1$ has only one positive real solution α_m, which is decreasing in m. Further, $\alpha_m \le m^{1/m}$ for $m \ge 3$.*

Proof. Write the equation as $x^m(x-1) = 1$. The left-hand side is non-positive for x in the range $0 \le x \le 1$ and monotonically increasing for $x \ge 1$, implying that there is only one solution $\alpha_m > 1$. By definition $\alpha_m^m(\alpha_m - 1) = 1$ so that $\alpha_m^{m+1}(\alpha_m - 1) > 1$, implying, in turn, that $\alpha_{m+1} < \alpha_m$. Finally, for every $m \ge 3$ we have

$$x^m(x-1)|_{x=m^{1/m}} = m\left(m^{1/m} - 1\right) = m\left(\mathrm{e}^{(\log_{\mathrm{e}} m)/m} - 1\right) \ge m \cdot \frac{\log_{\mathrm{e}} m}{m} = \log_{\mathrm{e}} m > 1,$$

thus implying that $\alpha_m \le m^{1/m}$. $\square$

Proposition 3.4 *Let $0 = m_0 < m_1 < \ldots < m_\ell$ and $0 = i_{-1} < i_0 < i_1 < \ldots < i_\ell$ be integers such that $i_{j-1} \ge m_j$ for $j = 1, 2, \ldots, \ell$, and suppose that the integer sequence $(f_i)_{i=0}^{i_\ell}$ satisfies*

$$f_i \le f_{i-1} + f_{i-m_j-1}, \qquad 1 \le i \le i_\ell,$$

where $j = j(i)$ is the unique index such that $(m_j \le)\ i_{j-1} < i \le i_j$. Let the real sequence $(g_i)_{i=0}^{i_\ell}$ be defined recursively by $g_0 = f_0$ and

$$g_i = \alpha_{m_j} g_{i-1}, \qquad 1 \le i \le i_\ell,$$

where j is such that $i_{j-1} < i \le i_j$ and α_{m_j} is the positive real solution of $x^{m_j+1} = x^{m_j} + 1$. Then $f_i \le g_i$ for all $0 \le i \le i_\ell$.

Proof. We prove by induction on i, where the induction base $i = 0$ is obvious. Turning to the induction step, suppose that $f_{i'} \le g_{i'}$ holds for all $0 \le i' < i$ and let j be such that $i_{j-1} < i \le i_j$. Then

$$\begin{aligned} f_i &\le f_{i-1} + f_{i-m_j-1} \\ &\le g_{i-1} + g_{i-m_j-1} && (9) \\ &\le (1 + \alpha_{m_j}^{-m_j}) g_{i-1} && (10) \\ &= \alpha_{m_j} g_{i-1} && (11) \\ &= g_i, \end{aligned}$$

where (9) follows from the induction hypothesis, (10) follows from the definition of g_i and the fact that α_{m_j} is decreasing in j (Proposition 3.3), and (11) follows from the definition of α_m. $\square$

Proof of Theorem 2.4. Let $\Delta(i)$ equal the right-hand side of (8). For $j = 0, 1, \ldots, \ell$, let $m_0 < m_1 < \ldots < m_\ell$ be the nonnegative values taken on by $\Delta(i)$ as i increases from 0 to r; clearly, $m_0 = 0$ and $m_\ell = s(t-1)$. Denote by i_j the largest i for which $\Delta(i) = m_j$. Thus,

$$i_j = \left\lfloor \left(\frac{m_j}{s} + 1 \right) \frac{r}{t} \right\rfloor \tag{12}$$

and, in particular, $i_0 = \lfloor r/t \rfloor$ and $i_\ell = r$. Since $|N_G(v)| \leq i-1$ for every vertex v in every $H \in \mathcal{S}_i(G)$ and since $|N_G(v)| \geq m_j$ for some v when $i = i_{j-1}+1$, we have $i_{j-1} \geq m_j$. Therefore, by Lemma 3.1, the sequence $(f_i(G))_{i=0}^{i_\ell}$ with the integers m_j and i_j satisfy the assumptions of Proposition 3.4. Hence,

$$\begin{aligned} \log_2 f_r(G) &= \log_2 f_{i_\ell}(G) \\ &\leq \log_2 f_{i_0}(G) + \sum_{j=1}^{\ell} (i_j - i_{j-1}) \log_2 \alpha_{m_j} \\ &\leq i_0 + \sum_{j=1}^{\ell} (i_j - i_{j-1}) \log_2 \alpha_{m_j}, \end{aligned} \tag{13}$$

where α_{m_j} is the positive real solution of $x^{m_j+1} = x^{m_j} + 1$ and (13) follows from (6). Incorporating $i_j - i_{j-1} \leq (m_j - m_{j-1})r/(ts) + 1$ (from (12)) and $i_0 \leq r/t$ into (13) yields

$$\begin{aligned} \log_2 f_r(G) &\leq \frac{r}{t} + \sum_{j=1}^{\ell} \left((m_j - m_{j-1}) \frac{r}{ts} + 1 \right) \log_2 \alpha_{m_j} \\ &\leq \frac{r}{t} + \sum_{m=1}^{m_\ell} \left(\frac{r}{ts} + 1 \right) \log_2 \alpha_m & (14) \\ &\leq \frac{r}{t} + \left(\frac{r}{ts} + 1 \right) \left(2 + \sum_{m=3}^{m_\ell} \frac{\log_2 m}{m} \right) & (15) \\ &\leq \frac{r}{t} + \left(\frac{r}{ts} + 1 \right) \left(2 + \frac{\log_2^2 (s(t-1))}{\log_2 \mathrm{e}} \right) & (16) \\ &\leq \frac{r}{t} \left(1 + O\left(\frac{\log^2(ts)}{s} \right) \right), & (17) \end{aligned}$$

where (14) follows since α_m is decreasing in m, (15) follows since $\alpha_2 < \alpha_1 < 2$ and $\log_2 \alpha_m \leq (1/m) \log_2 m$ for $m \geq 3$ (Proposition 3.3), and (16) follows from the fact that $\sum_{m=3}^{m_\ell} 1/m \leq \log_{\mathrm{e}} m_\ell = \log_{\mathrm{e}} s(t-1)$. The bound $r \geq m_\ell = s(t-1)$ justifies (17). The proof is completed by noting that $I(G) = f_r(G)$. □

4 Irregular hypergraphs

In this section, we generalize Theorem 2.4 to uniform linear hypergraphs that are not necessarily regular.

Given a hypergraph G, let $v_1, v_2, \ldots, v_{|V_G|}$ be a labeling of the vertices of G satisfying $\delta_G(v_1) \leq \delta_G(v_2) \leq \ldots \leq \delta_G(v_{|V_G|})$. For $i = 1, 2, \ldots, |V_G|$ define

$$\sigma_G(i) = \frac{1}{i} \sum_{j=1}^{i} \delta_G(v_j).$$

That is, $\sigma_G(i)$ is the average degree among the i vertices with smallest degrees in G.

Following is a version of Lemma 3.2 for irregular hypergraphs.

Lemma 4.1 *Let G be a t-uniform linear hypergraph with r vertices. Then for all $H \in \mathcal{S}_i(G)$*

$$\max_{v \in V_H} \delta_H(v) \geq \left\lceil \sigma(i) \left(\frac{ti}{r} \frac{\sigma(i)}{\sigma(r)} - 1 \right) \right\rceil, \tag{18}$$

where $\sigma(i) = \sigma_G(i)$.

Proof. Replace $\sum_{e \in E_G} \beta_e = si$ with $\sum_{e \in E_G} \beta_e \geq i\sigma(i)$ and $|E_G| = rs/t$ with $|E_G| = r\sigma(r)/t$ in the proof of Lemma 3.2. □

For the case of s-regular hypergraphs $\sigma_G(i) = s$, so Lemma 3.2 is a special case of Lemma 4.1.

Next we combine Lemma 4.1 with Lemma 3.1, to obtain the following irregular counterpart of Theorem 2.4.

Theorem 4.2 *Let G be a t-uniform linear hypergraph with r vertices. The number of independent sets $I(G)$ in G satisfies*

$$\begin{aligned} \log_2 I(G) &\leq i_0 + \frac{r}{t} \cdot O\left(\frac{\log^2(ts)}{s_1^2/s} \right) &(19)\\ &\leq \frac{r}{t} \cdot \frac{s}{s_0} \cdot \left(1 + O\left(\frac{\log^2(ts)}{s_1} \right) \right) &(20)\\ &\leq \frac{r}{t} \cdot \frac{s}{s_0} \cdot \left(1 + O\left(\frac{t \log^2(ts)}{s} \right) \right), &(21) \end{aligned}$$

where $s = \sigma_G(r)$ is the average degree in G, i_0 is the largest i for which $i\sigma_G(i) \leq rs/t$, $s_0 = \sigma_G(i_0)$, and $s_1 = \sigma_G(i_0 + 1)$.

Proof. We proceed as in the proof of Theorem 2.4, but this time we let $\Delta(i)$ equal the right-hand side of (18). Also, let $0 = m_0 < m_1 < \ldots < m_\ell = s(t-1)$ be the nonnegative values taken on by $\Delta(i)$ as i ranges from 0 to r.

Denote by i_j the largest i for which $\Delta(i) = m_j$; in particular, for $j = 0$ we get that i_0 is indeed the largest i for which $i\sigma_G(i) \leq rs/t$, and for $j = \ell$ we get $i_\ell = r$. We note that $\sigma(i) = \sigma_G(i)$ is non-decreasing in i and hence so is $\sigma(i)(ti\sigma(i)/(rs) - 1)$. Therefore, i_j is the largest integer i satisfying

$$\sigma(i)\left(\frac{ti}{r}\frac{\sigma(i)}{s} - 1\right) \leq m_j$$

or, equivalently, the largest integer i satisfying

$$i \leq \left(\frac{m_j}{\sigma(i)} + 1\right)\frac{rs}{t\sigma(i)} \; . \tag{22}$$

This characterization of i_j implies that

$$i_j > \left(\frac{m_j}{\sigma(i_j+1)} + 1\right)\frac{rs}{t\sigma(i_j+1)} - 1. \tag{23}$$

By (22) and (23) we have, for $j \geq 1$,

$$\begin{aligned} i_j - i_{j-1} &\leq \frac{rs}{t}\left(\frac{m_j}{(\sigma(i_j))^2} - \frac{m_{j-1}}{(\sigma(i_{j-1}+1))^2} + \frac{1}{\sigma(i_j)} - \frac{1}{\sigma(i_{j-1}+1)}\right) + 1 \\ &\leq \frac{rs}{t(\sigma(i_j))^2}\,(m_j - m_{j-1}) + 1 \qquad (24) \\ &\leq \frac{rs}{t(\sigma(i_0+1))^2}\,(m_j - m_{j-1}) + 1 \qquad (25) \\ &= \frac{rs}{ts_1^2}\,(m_j - m_{j-1}) + 1, \qquad (26) \end{aligned}$$

where (24) and (25) follow from the fact that $\sigma(i)$ is non-decreasing in i and that $i_0 + 1 \leq i_{j-1} + 1 \leq i_j$.

Inequality (13) from the proof of Theorem 2.4 applies verbatim here, and incorporating the bound (26) on $i_j - i_{j-1}$ yields

$$\begin{aligned} \log_2 f_r(G) &\leq i_0 + \sum_{j=1}^{\ell}\left((m_j - m_{j-1})\frac{rs}{ts_1^2} + 1\right)\log_2 \alpha_{m_j} \\ &\leq i_0 + \frac{r}{t}\cdot O\biggl(\frac{\log^2(ts)}{s_1^2/s}\biggr), \qquad (27) \end{aligned}$$

where (27) follows from the same reasoning used to obtain (17): the only difference is that here $r \geq m_\ell = (t-1)s \geq (t-1)s_1^2/s$, which we need to assert that $rs/(ts_1^2)$ is bounded away from 0.

Turning to (20), by the definition of i_0 we get that $i_0 s_0 = i_0\sigma(i_0) \leq rs/t$, i.e., $i_0 \leq (r/t)(s/s_0)$. In addition, since $\sigma(i)$ is non-decreasing in i we have $s_0 s_1 \leq s_1^2$. Combining these two observations with (19) yields (20). Finally, the definition of i_0 also implies that $rs_1 \geq (i_0+1)s_1 > rs/t$; so, $s_1 > s/t$, which readily leads to (21). $\square$

In general, if more is known about the behavior of $\sigma_G(i)$ for $i > i_0$, the $O(\cdot)$ term in (19) can be improved. We obtained (19) by using the pessimistic bound of $\sigma_G(i) \geq \sigma_G(i_0+1)$ for $i > i_0$. We do note, however, that (19) is tight to first order (the i_0 term) for a bipartite graph G in which the degree of any left vertex is smaller than the degree of any right vertex. In such a graph, there are necessarily more left vertices than right vertices and i_0 is easily seen to be the number of left vertices, which in turn is smaller than $\log_2 I(G)$.

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