On the Enumeration of Linear Threshold Functions [1]

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Abstract: The problem of enumerating linear threshold functions of n Boolean variables is reduced to that of enumerating cells in a central, hyper-octahedrally symmetric arrangement of hyperplanes in $n+1$-dimensional Euclidean space. Because of the symmetry, enumeration of equivalence classes of cells is sufficient. To this end we extend Zaslavsky's theorem on counting cells in a central arrangement of hyperplanes in two ways. First, the theorem is extended to symmetric arrangements so that the number of cells is related to a generalised Möbius function defined on a symmetry-adapted poset of hyperplane intersections (SAPHI). Second, we show how a SAPHI can be unfolded into a symmetry-adapted face poset whose maximal elements correspond to and fully characterise equivalence classes of linear threshold functions.

Key Words: Arrangement of hyperplanes, geometric lattice, hyper-octahedral group, linear threshold function, Möbius function, symmetry-adapted poset of hyperplane intersections, zeta function.
Category: G2.1

1 Introduction

Consider a threshold gate or a McCulloch-Pitts model neuron shown in figure 1. For a given set of weights, $\{w_i, i = 0, 1, 2, \ldots, n\}$, the device computes a Boolean function of its inputs. Not all Boolean functions may be computed by varying the weights; the ones which may are called linear threshold functions. For a given n, the problem is to enumerate all linear threshold functions of n Boolean variables.

This problem received a great deal of attention during a twelve-year period from 1960 to 1972, mostly from electrical engineers. This research was motivated by the expectation that the use of threshold gates (rather than conventional logic gates, AND, OR and NOT) in digital circuits would lead to less complex circuits with fewer elements. However, by 1972 it was clear that the engineering tolerance required for the construction of reliable circuits was beyond reach and researchers in this area turned their attention to other problems. Nevertheless, a substantial body of mathematical results had been accumulated and a number of authoritative monographs were published during this period [2, 3, 6].

As part of this work, much effort was expended in tabulating linear threshold functions of up to 8 variables. Some of these results are summarised in table 1. Practical tabulation exploits the hyper-octahedral symmetry inherent in the

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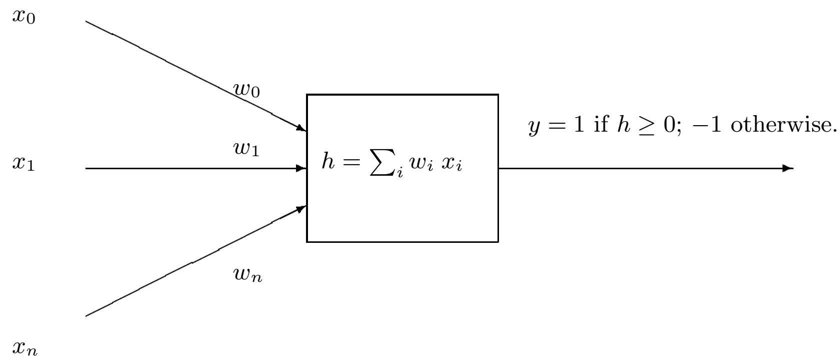


Figure 1: A threshold gate or a McCulloch-Pitts model neuron. The input x_0 is clamped to 1 and other inputs and the output are bipolar ($x_i = \pm 1$ for $1 \leq i \leq n$, and $y = \pm 1$). The device is characterised by real-valued weights $\{w_i, i = 0, 1, 2, \ldots, n\}$ and computes a Boolean function of the inputs.

n	$LTF(n)$	$CLTF(n)$	$CBF(n)$
1	4	1	1
2	14	2	3
3	104	3	7
4	1,882	7	83
5	94,572	21	109,950
6	15,028,134	135	28,613,442,061,634
7	8,378,070,864	2470	*
8	17,561,539,552,946	175,428	*

Table 1: Some results on the enumeration of linear threshold functions [5]. $LTF(n)$ is the number of linear threshold functions of n Boolean variables and $CLTF(n)$ and $CBF(n)$ are the number of equivalence classes of linear threshold and Boolean functions respectively. $CBF(n)$ for n up to 5 had been calculated previously; the sequence has been extended up to $n = 8$ recently by Strectch [11]. The last two numbers in this sequence, $CBF(7)$ and $CBF(8)$ are too large to be included here but can be found in [10]. For comparison, the total number of Boolean functions of n Boolean variables is 2^{2^n}.

problem. [2] One therefore tabulates equivalence classes of linear threshold functions. If an exemplar of a class is given, other functions in the class can be generated by symmetry operations.

Column 3 in table 1 lists the number of equivalence classes of LTFs and for comparison, column 4 lists the number of equivalence classes of Boolean

[2] It so happens that the symmetry group associated with LTFs of n variables is the hyper-octahedral group in $n+1$ dimensions, denoted here by $HO(n+1)$. A typical element of this group permutes the $n+1$ coordinate axes and changes the sign of some of these. The order of the group is $2^{n+1}(n+1)!$. It is the invariance group of the $n+1$-dimensional coordinate frame.

functions with respect to the same group, $HO(n+1)$. The following upper and lower bounds on $LTF(n)$ were also established:

$$LTF(n) \leq 2\sum_{j=0}^{n} \binom{2^n - 1}{j} \leq 2^{n^2},$$

$$LTF(n) > 2^{\frac{1}{2}n(n-1)+16} \text{ for } n \geq 8.$$

The upper bound follows immediately from Cover's theorem [1] on the number of linearly separable partitions of p points in a general position in $n+1$ dimensional space. [3] The lower bound is due to Muroga [4]. More recently, Zuev [13] has improved the lower bound for sufficiently large n:

$$\frac{1}{n^2}\, log_2\, LTF(n) > 1 - 10/log\ n.$$

Upper and lower bounds are thus asymptotically tight and we have

$$\frac{1}{n^2}\, log_2\, LTF(n) \rightarrow 1.$$

Cover's upper bound is based on input vectors being disposed in a general position whereas bipolar input vectors (for $n \geq 3$) are not in a general position. It may therefore be possible to improve the upper bound (for finite n). We will return to this possibility later. Previous work has also not tried to place any bounds on $CLTF(n)$ or analyse its asymptotic behaviour. At the end of this paper, we will return to this question as well.

2 Enumeration in weight space

In this paper, we analyse the enumeration problem in weight space. Given an $n+1$ dimensional input vector $\mathbf{X}$, what choice of weights would cause the threshold gate to respond with $y = 1$? Note that the equation $\mathbf{X} \cdot \mathbf{W} = 0$ defines a hyperplane through the origin in weight space (fig 2). The discriminant $h = \mathbf{X} \cdot \mathbf{W}$ is +ve on one side of the hyperplane and −ve on the other. Thus any $\mathbf{W}$ in the positive half or on the hyperplane causes $y = 1$ whereas any $\mathbf{W}$ in the negative half causes $y = -1$. The discriminant hyperplane partitions the weight space into two $n+1$-dimensional convex regions (corresponding to $h > 0$ and $h < 0$) and an n-dimensional region (where $h = 0$). All weights in the +ve half or on the hyperplane compute the same function of the input; likewise for all weights in the −ve half. When there are several inputs, they collectively define a central arrangement of hyperplanes, one hyperplane for each input. [4] The weight space partition contains a number of $n+1$-dimensional open convex cells; threshold gates with weights in the same cell respond to all inputs in exactly the same way and therefore compute the same function of the given inputs. The problem of enumerating functions of given inputs computable by a threshold gate is thus reduced to the problem of enumerating $n+1$-dimensional cells in a central arrangement of hyperplanes. [5]

[3] The points are in a general position if none of the m-dimensional subspaces ($m < n+1$) contains more than $m+1$ points.

[4] In a central arrangement all hyperplanes pass through a common point.

[5] Any function that can be computed by a weight on one or more hyperplanes can also be computed by weights in one of the open cells.

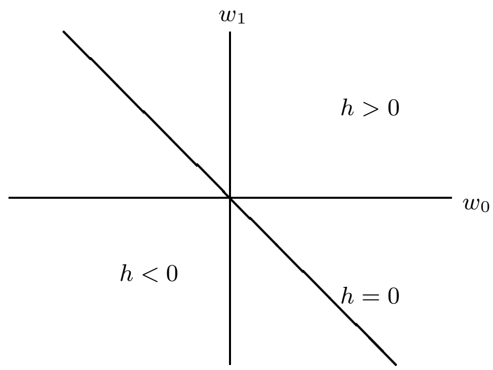


Figure 2: Weight-space interpretation of computation by a threshold unit. An input vector defines a discriminant hyperplane ($h = \sum_i x_i \ w_i = 0$) which partitions the weight space into +ve and −ve halves. Threshold gates with weights in the positive half or on the hyperplane respond with $y = 1$ whereas those with weights in the negative half respond with $y = -1$.

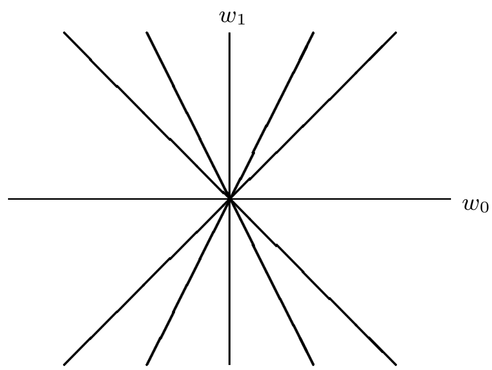


Figure 3: A central arrangement of hyperplanes.

There are two aspects to the enumeration of cells in an arrangement of hyperplanes. First, one must count the total number of cells, and second, one must characterise the cells fully so that a weight vector inside the cell may be calculated. In 1975, Zaslavsky [12] derived a combinatorial cell-counting formula which is discussed in section 2.1. Complete characterisation of cells requires the construction of a face poset. This is discussed in section 2.2.

2.1 Zaslavsky's cell-counting formula

We briefly summarise the exposition of Zaslavsky's original work by Siu, Roychowdhury and Kailath [9]. Interested readers should consult the latter reference for further details and proofs.

Let $\mathcal{A}$ be a central arrangement of p hyperplanes in $n + 1$-dimensional Euclidean space of weight vectors $\mathbf{W}$. It is assumed that all hyperplanes pass through the origin and the direction vectors are $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_p\}$.

Next consider $\mathcal{L}$, the set comprising the entire weight space W and all intersections of the hyperplanes in $\mathcal{A}$. This set is partially ordered with respect to set inclusion, i.e. for $s, t \in \mathcal{L}, s \leq t$ if $s \subseteq t$.

$\mathcal{L}$ is in fact a lattice because every pair of elements $s, t \in \mathcal{L}$ has a unique greatest lower bound (called meet and denoted $s \wedge t$) and a unique least upper bound (called join and denoted $s \vee t$). These are defined as follows:

$$\begin{aligned} s \wedge t &= s \cap t; \\ s \vee t &= \cap\{u \in \mathcal{L} : s \cup t \subseteq u\}. \end{aligned}$$

$\mathcal{L}$, being a lattice, has unique minimal and maximal elements (labelled O and W in the rest of this paper). All chains between any two fixed elements have the same length and $\mathcal{L}$ is therefore a geometric lattice. The rank $r(s)$ of $s \in \mathcal{L}$ is the dimension of s.

The zeta function for a pair of variables $s, t \in \mathcal{L}$ is defined so that

$$\begin{aligned} \zeta_{s,t} &= 1 \text{ if } s \leq t \\ &= 0 \text{ otherwise.} \end{aligned}$$

We can write the ζ-function as a matrix and arrange its rows and columns so that it has upper triangular form. The Möbius function $\mu_{s,t}$ is defined recursively as follows:

$$\begin{aligned} \mu_{t,t} &= 1; \\ \mu_{s,t} &= -\sum_{s<u\leq t} \mu_{u,t}. \end{aligned}$$

Notice that the Möbius function is integer-valued and it is defined only for pairs of comparable elements. Its associated matrix also has upper triangular form and is the inverse of the ζ-matrix.

The following theorem is the key combinatorial result for counting cells in the partition of weight space by the arrangement $\mathcal{A}$.

Theorem 1. *Zaslavsky's Theorem [12]:* The number of cells in a central arrangement of hyperplanes in $\mathcal{R}^{n+1}$ is given by $\sum_{s\in\mathcal{L}} |\mu_{s,W}|$.

A simple example comprising a two-dimensional arrangement is shown in figure 4. More elaborate examples are given in [7].

2.2 Face poset of an arrangement

In order to completely characterise the partition of weight space by an arrangement of hyperplanes, we need to recognise that in addition to the $n+1$-dimensional cells considered in the previous section, there are other open, convex regions whose dimensions range from 1 to n. We will use the term 'face' for all such regions and for the origin as well which we will regard as a 0-dimensional face. A partial order of faces is naturally defined and the weight-space partition is fully characterised by the face poset, hereafter denoted by $\mathcal{F}$..

Note that each face is characterised by a signature which is a string of $n+1$ symbols where each symbol is one of $\{+, -, 0\}$; the i-th symbol is determined

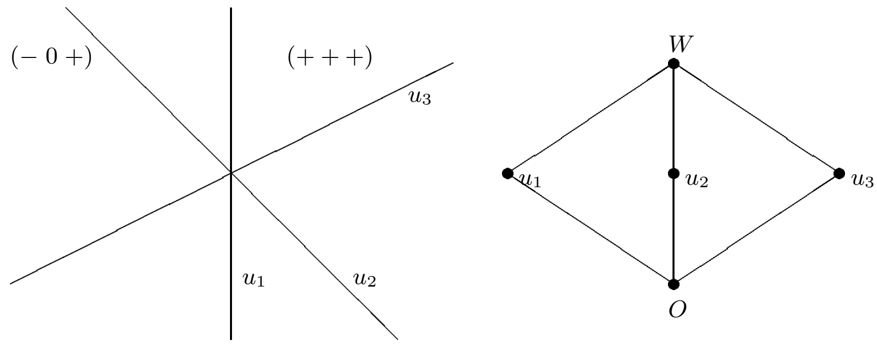


Figure 4: A central arrangement of three lines in two-dimensional Euclidean space and its associated lattice of line intersections. The labels u_1, u_2 and u_3 refer to the three lines. The signatures of one cell and one line segment are shown for illustration. The signature of the origin is (000). The number of cells is the sum of the magnitudes of the follwing values of the Möbius function: $\mu_{W,W} = 1$, $\mu_{u_1,W} = \mu_{u_2,W} = \mu_{u_3,W} = -1$ and $\mu_{O,W} = 2$.

by the value of the i-th discriminant. We will define a partial order of faces by asserting that for any pair f_1 and f_2, $f_1 \leq f_2$ if the signature of f_1 can be obtained by replacing some $\pm$s in the signature of f_2 by 0s. The resulting face poset encapsulates the essential topological characteristics of the partititon. In particular, the cells in an arrangement are the maximal elements of the face poset.

The face poset for the arrangement in figure 4 is shown in figure 5.

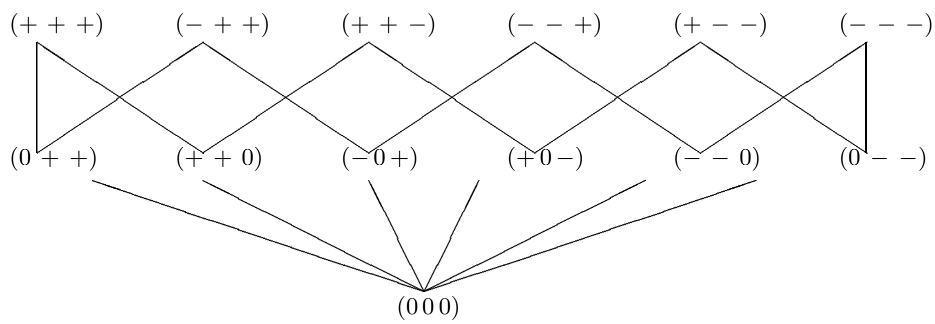


Figure 5: The face poset of the arrangement in figure 4.

3 Enumeration of cells in symmetric arrangements

The scheme outlined in the previous section for enumerating linear threshold functions runs into an immediate difficulty because for n-variable inputs, there are 2^n hyperplanes (in $n+1$ dimensions) and the number of elements in the lattice of hyperplane intersections is of the order $O(2^{n^2})$. This combinatorial explosion is ameliorated (but not entirely tamed) by utilising the symmetry of the arrangement. Instead of considering the set $\mathcal{L}$ of hyperplane intersections, we consider the set Λ of equivalence classes of hyperplane intersections. A partial order of the elements of Λ is defined with reference to the partial order of the elements of $\mathcal{L}$; Λ is thus a symmetry-adapted poset of hyperplane intersections. Likewise, instead of considering the set $\mathcal{F}$ of faces, we consider the set Π of equivalence classes of faces and define a partial order on elements of Π by referring to the partial order of the underlying face poset. Π is thus a symmetry-adapted face poset. Its maximal elements are the equivalence classes of linear threshold functions.

The enumeration problem thus reduces to the construction of the symmetry-adapted poset of hyperplane intersections. Once this is known, it can be unfolded to yield the symmetry-adapted face poset, and thereby equivalence classes of linear threshold functions. The schematiccs of this computation are shown in figure 6.

$$\begin{tikzcd}
\mathcal{L} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
\Lambda \arrow[r] & \Pi
\end{tikzcd}$$

Figure 6: Use of symmetry in the enumeration of linear threshold functions. Instead of constructing $\mathcal{L}$ and unfolding it into $\mathcal{F}$, it is computationally easier to construct Λ and unfold it into Π.

3.1 Extension of Zaslavsky's formula to symmetric arrangements

We first define a partial order on the elements of Λ as follows: for $\alpha, \beta \in \Lambda$, if there is a corresponding pair of elements $s, t \in \mathcal{L}$ such that $s \in \alpha$, $t \in \beta$ and $s \leq t$ then $\alpha \leq \beta$; otherwise α and β are not comparable.

We next define a generalised zeta function $\underline{\zeta}_{\alpha,\beta}$ on Λ. For each pair of comparable classes $\alpha \leq \beta$, it is the number of elements in α with which a typical element $t \in \beta$ can be compared; thus

$$\underline{\zeta}_{\alpha,\beta} = \sum_{s \in \alpha} \zeta_{s,t}$$

where t is any fixed element in β. Note that $\underline{\zeta}_{\alpha,\alpha} = 1$ for all α and $\underline{\zeta}_{\alpha,\beta} = 0$ when α and β can not be compared. [6]

[6] Another complementary generalised zeta function may be defined by

$$\overline{\zeta}_{\alpha,\beta} = \sum_{t \in \beta} \zeta_{s,t}$$

A generalised Möbius functions is defined recursively as follows:

$$\begin{aligned}\underline{\mu}_{\alpha,\alpha} &= 1,\\ \underline{\mu}_{\alpha,\beta} &= -\sum_{\alpha<\gamma\leq\beta} \underline{\zeta}_{\alpha,\gamma}\ \underline{\mu}_{\gamma,\beta},\end{aligned} \tag{1}$$

whereby all elements of $\underline{\mu}$ are easily computed.

From this and the previous recursion for the Möbius function on $\mathcal{L}$, it follows that

$$\underline{\mu}_{\alpha,\beta} = \sum_{s\in\alpha} \mu_{s,t}$$

(t is any fixed element of β). Thus the computation of the Möbius function on $\mathcal{L}$ is simplified considerably by computing the closely related generalised Möbius function on the symmetry-adapted poset of hyperplane intersections because the latter has far fewer elements than the former. Zaslavsky's theorem is now restated as follows:

Theorem 2. *Zaslavsky's Theorem Restated:* The number of cells in a central, symmetric arrangement of hyperplanes in $\mathcal{R}^{n+1}$ is given by $\sum_{\alpha\in\Lambda} |\underline{\mu}_{\alpha,W}|$.

We have calculated the SAPHI Λ_3, Λ_4 and Λ_5 corresponding to 3, 4, and 5 variable LTFs and computed the corresponding generalised zeta and Möbius functions. The results for 3 variable LTFs are given in figure 7 and table 2. Other results may be found in [7].

3.2 Unfolding Λ into Π

Having constructed the symmety-adapted poset of hyperplane intersections, the next task is to unfold it into a symmetry-adapted face poset. To this end, we need to devise a signature for equivalence classes of faces. Dertouzos [2] has shown that the vector sum of direction vectors weighted by their class label ($y = \pm 1$),

$$\mathbf{C} = \sum_i y_i\ \mathbf{X}_i,$$

is distinct for distinct linear threshold functions and may therefore be used to characterise them. [7] Thus equivalence classes of linear threshold functions may be labelled by the characteristic vector of an exemplar. Dertouzos has used the canonical form where all elements of $\mathbf{C}$ are non-negative and non-increasing ($c_j \leq c_i$ if $j > i$) and we will keep to his convention. The number of linear threshold functions in class $\mathbf{C}$ is the number of distinct vectors that are generated from $\mathbf{C}$ by permuting its elements and changing the sign of some of them.

An extended characteristic vector defined below can be used to characterise equivalence classes of all faces – not just cells – in the arrangement. Consider a

where s is any fixed element in α. It is useful for checking the calculations reported in this paper.

[7] The characteristic vector contains enough information to permit calculation of weights which are sufficiently deep in the interior of the cell so as to allow robust computation of the associated LTF.

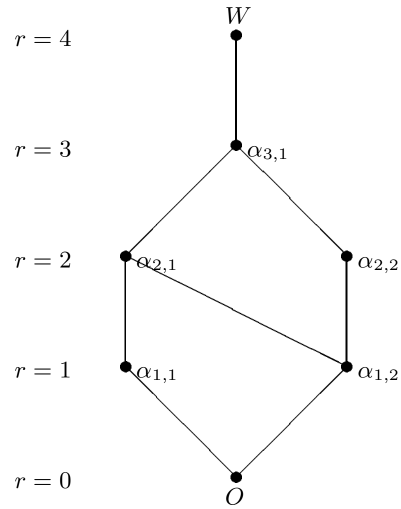


Figure 7: The symmetry-adapted poset of hyperplane intersections, Λ_3, for LTFs of 3 variables. W is the 4-dimensional weight space; each of the other elements is an equivalence class of hyperplane intersections. The generic label for each element is $\alpha_{r,i}$ where r is the rank and i distinguishes elements of same rank. Note that although this poset happens to be a geometric lattice, it is not so for LTFs of more than 3 variables.

face f whose affine extension is some r-dimensional intersection of hyperplanes, say $s \in \mathcal{L}$ and let $\mathbf{W} \in f$ be some weight vector. Ostensibly, $\mathbf{W} \cdot \mathbf{X} = 0$ if $s \subseteq \mathbf{X}$ and non-zero otherwise. The chracteristic vector for f is defined as

$$\mathbf{C} = \sum_{\{\mathbf{X}_i | \mathbf{W} \cdot \mathbf{X}_i \neq 0\}} sign(\mathbf{W} \cdot \mathbf{X}_i)\ \mathbf{X}_i.$$

It is easy to show that this vector is distinct for distinct faces in s and may be used to characterise them. Moreover, if two faces in s are related by a symmetry transformation, then so are their characteristic vectors. Thus, equivalence classes of faces may be labelled by canonical-form characteristic vectors.

The unfolding of Λ_n into Π_n is carried out step-by-step proceeding from low-rank elements to high-rank elements using what one might call the “climbing-frame algorithm”. Construction of Π_n, particularly computation of the characteristic vectors of its maximal elements completes the task of enumerating n-variable LTFs.

Figure 8 shows Π_3, the symmetry-adapted face poset for 3-variable linear threshold functions. Further examples and details of the climbing-frame algorithm are given in [7].

4 Conclusions

We have extended Zaslavsky’s work [12] on counting the number of cells in a partition of Euclidean space by a central arrangement of hyperplanes to sym-

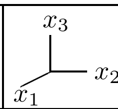
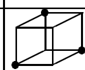
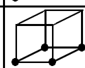
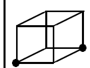
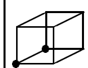
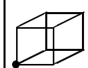
			O	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{2,1}$	$\alpha_{2,2}$	$\alpha_{3,1}$	W	$\underline{\mu}_{\alpha,W}$
										
		O	1	1	1	1	1	1	1	23
$\mathbf{W}_1^T = (1111)$		$\alpha_{1,1}$	0	1	0	2	0	3	8	-8
$\mathbf{W}_1^T = (1001)$		$\alpha_{1,2}$	0	0	1	2	3	6	12	-36
$\mathbf{W}_1^T = (1001)$ $\mathbf{W}_2^T = (0110)$		$\alpha_{2,1}$	0	0	0	1	0	3	12	12
$\mathbf{W}_1^T = (1001)$ $\mathbf{W}_2^T = (1010)$		$\alpha_{2,2}$	0	0	0	0	1	4	16	16
$\mathbf{W}_1^T = (1001)$ $\mathbf{W}_2^T = (1010)$ $\mathbf{W}_3^T = (1111)$		$\alpha_{3,1}$	0	0	0	0	0	1	8	-8
		W	0	0	0	0	0	0	1	1
										104

Table 2: Generalised zeta and Möbius functions for the the symmetry-adapted poset of hyperplane intersections of figure 7. The direction vectors of hyperplanes whose intersection gives an exemplar of equivalence class $\alpha_{r,i}$ are shown in column 2. (It is assumed that $x_0 = 1$ in all cases.) A set of r weight vectors which span the intersection is given in column 1. The main body of the table contains the $\underline{\zeta}$ function. The Möbius function $\underline{\mu}_{\alpha,W}$ is listed in the last column; Its last entry is $\sum_\alpha |\underline{\mu}_{\alpha,W}|$, the number of linear threshold functions of three variables.

metric arrangements. We have also shown how such arrangements may be used to enumerate equivalence classes of n-variable linear threshold functions. Two of the most promising directions for future work are outlined below.

First, Zaslavsky's formula extends Cover's theorem in the sense that it allows calculation of the number of linearly separable partitions of p points in n dimensions even when the points are not in a general position. We are hopeful that it can be reorganised as a sequence of decreasingly significant corrections to Cover's result, where each correction accounts for the fact that points in some class of subsets of the p points are not in a general position. [8] When the complete latice of hyperplane intersections is known, one can calculate all corrections and thereby compute the precise number of linearly separable partitions. When the lattice is partially constructed, one can calculate only some of the corrections

[8] Something very similar was done way back in 1888 by Roberts [8] for an arrangement of lines in the plane.

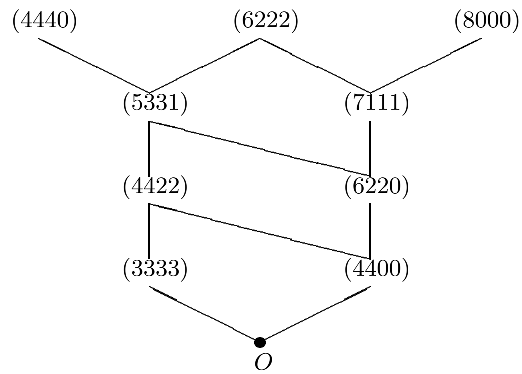


Figure 8: Symmetry-adapted face poset, Π_3, for LTFs of 3 variables. Each element is identified by a characteristic vector. There are three equivalence classes of 3-variable LTFs corresponding to the three maximal elements of Π_3.

and obtain an upper bound on the number of partitions which is an improvement on the upper bound from Cover's formula. The final observation in this regard is that a given $\mathcal{L}_m$ is embedded in all lattices $\mathcal{L}_n$ where $n > m$; to have constructed a particular $\mathcal{L}_m$ is to have partially constructed all $\mathcal{L}_n$ with $n > m$. Thus, by embedding $\mathcal{L}_5$ in 'higher-order' lattices, we may be able to improve the current upper bound on $LTF(n)$ for $n > 5$.

Second, it follows from the 'climbing-frame' algorithm that $CLTF(n) \leq 2^n \times Ch_n$, where Ch_n is the number of chains from O to W in Λ_n. Next, we observe that $\mathcal{L}_n$ may be re-interpreted as a sub-lattice of the lattice of subgroups of $HO(n+1)$ and likewise, Λ_n may be re-interpreted as a subset of the poset of equivalence classes of subgroups of $HO(n+1)$. This connection may allow us to place an upper bound on Ch_n. However, it is probably unlikely that the resulting upper bound on $CLTF(n)$ will be am improvement on $CBF(n)$ which is a simple natural upper bound.

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