

THE NUMBER OF M -SEQUENCES AND f -VECTORS

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ABSTRACT. We give a recursive formula for the number of M -sequences (a.k.a. f -vectors for multicomplexes or O -sequences) given the number of variables and a maximum degree. In particular, it is shown that the number of M -sequences for at most 2 variables are powers of two and for at most 3 variables are Bell numbers. We give an asymptotic estimate of the number of M -sequences when the number of variables is fixed. This leads to a new lower bound for the number of polytopes with few vertices. We also prove a similar recursive formula for the number of f -vectors for simplicial complexes. Keeping the maximum degree fixed we get the number of M -sequences and the number of f -vectors for simplicial complexes as polynomials in the number of variables and it is shown that these numbers are asymptotically equal.

1. INTRODUCTION

A *multicomplex* is a collection \mathcal{M} of finite multisets satisfying $A \subseteq B \in \mathcal{M} \implies A \in \mathcal{M}$. It is often convenient to think of the underlying ground set as variables and of the sets in \mathcal{M} as monomials. Then a multicomplex is a collection of monomials closed under division. Given a multicomplex \mathcal{M} , let $m_i := |\{A \in \mathcal{M} : \deg A = i\}|$. The sequence $m = (m_0, m_1, m_2, \dots)$ is called the M -sequence of \mathcal{M} . The purpose of this paper is to study the number of M -sequences given upper bounds on the number of variables and on the degree for the monomials.

M -sequences play an important role in different mathematical theories. Theorem 1.1 below is a summary of how the enumeration of M -sequences can be interpreted. It is a consequence of deep theorems (such as Macaulay's theorem, the g -theorem etc.) by Billera, Lee, Macaulay, McMullen and Stanley. We refer to Ziegler [Z, Chapter 8] and Stanley [S1] for an account of the underlying definitions and theorems. In this paper we will use (ii) to do the counting.

Theorem 1.1. *Fix $n, p \geq 0$. Then the following are equal.*

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- (i) *The number of M -sequences with $m_1 \leq p$ and $m_j = 0$ for all $j > n$, not counting $(0, 0, 0, \dots)$.*
- (ii) *The number of non-empty compressed multicomplexes on at most p variables and with no monomial of degree higher than n .*
- (iii) *The number of f -vectors of $n-1$ dimensional shellable (or Cohen-Macaulay) simplicial complexes on at most $n + p$ vertices.*
- (iv) *The number of f -vectors of simplicial $2n$ -polytopes ($2n+1$ -polytopes) with at most $p + 2n + 1$ ($p + 2n + 2$) vertices.*
- (v) *The number of Hilbert functions for standard graded \mathbf{k} -algebras $R = R_0 + R_1 + \dots + R_d$, with $d \leq n$ and $\dim R_1 \leq p$. \square*

Let $M^p(n) - 1$ denote the common number in Theorem 1.1.

Our basic result on the number of M -sequences, from which the other results will follow, is the recursion in Theorem 2.1. Corollary 2.3 shows that when fixing p and expressing $M^p(n)$ in terms of n , we get the sequence of functions:

linear, powers of 2, Bell numbers, \dots ,

for $p = 1, 2$ and 3 respectively. Using Theorem 1.1 we can deduce that the number of f -vectors for a simplicial d -polytope with at most $d + 3$ vertices is $2^{\lfloor d/2 \rfloor + 1} - 1$ and with at most $d + 4$ vertices is $B(\lfloor d/2 \rfloor + 2) - 1$, where $B(n)$ is the Bell-number. It would be very interesting if someone could give a direct proof of this, avoiding the g -theorem.

We estimate the asymptotic growth of $M^p(n)$ for fixed $p \geq 4$, which gives a lower bound $d^{(p-2-o(1))d/2}$ for the number of simplicial d -polytopes with $d+p+1$ vertices, Corollary 2.6.

In Section 3 we give a recursion for the number of f -vectors for simplicial complexes. In Corollary 3.4 we prove the perhaps somewhat surprising result that the number of f -vectors for simplicial complexes and the number of M -sequences for multicomplexes have asymptotically equal growth for each fixed n (maximal cardinality). From this we deduce, Corollary 3.5, that for a fixed dimension $n - 1$ and a large number of vertices p , almost every f -vector for simplicial complexes is also an f -vector for a shellable simplicial complex.

2. THE NUMBER OF M -SEQUENCES

2.1. Basic recursion. After Theorem 1.1 we defined $M^p(n)$ to be one more than the number of M -sequences for non-empty multicomplexes. We think of this extra one as coming from the sequence $(0, 0, 0, 0, \dots)$ for the empty multicomplex. This sequence does not have a proper non-empty counterpart when counting f -vectors of simplicial polytopes, shellable simplicial complexes etc. in Theorem 1.1, but is included to obtain the nicest looking recursions. Hence, we will have $M^p(0) = 2$ for all $p \geq 0$. We also define $M^p(-1) := 1$ for

all $p \geq 0$. On the other boundary we have $M^0(n) = 2$, for all $n \geq 0$, again counting both $(0, 0, 0, \dots)$ and $(1, 0, 0, \dots)$.

Given two monomials $x_1^{a_1} \dots x_p^{a_p}$ and $x_1^{b_1} \dots x_p^{b_p}$ we say that $x_1^{a_1} \dots x_p^{a_p}$ comes before $x_1^{b_1} \dots x_p^{b_p}$ in *reverse lexicographic order* if $a_p = b_p, a_{p-1} = b_{p-1}, \dots, a_{i+1} = b_{i+1}$, but $a_i < b_i$. A multicomplex \mathcal{M} is said to be *compressed* if $B \in \mathcal{M}, \deg(A) = \deg(B)$ and A comes before B in reverse lexicographic order implies that $A \in \mathcal{M}$.

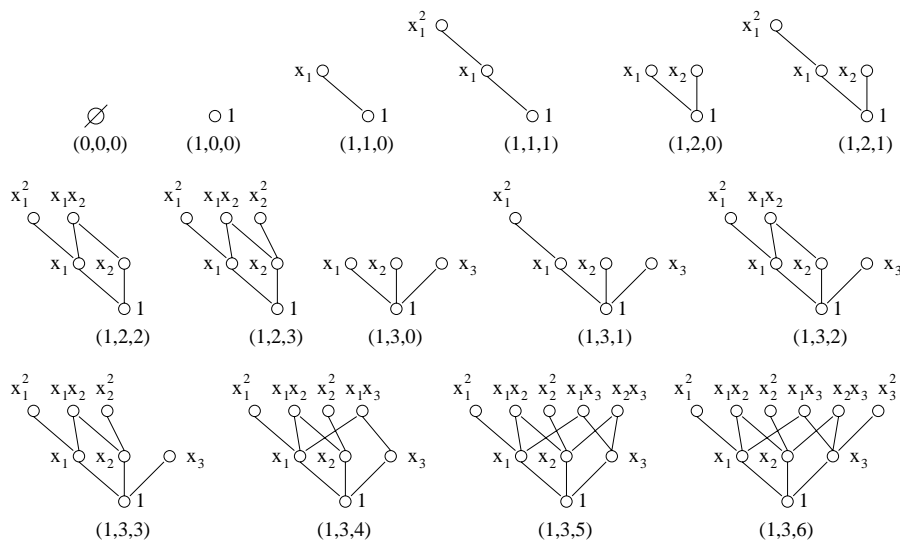


FIGURE 1. All the M -sequences and the corresponding compressed complexes when $p = 3$ and $n = 2$.

We also need to have a notation for the number of M -sequences corresponding to multicomplexes that for a fixed number of variables have all the monomials up to a fixed degree k . For $n \geq k \geq -1, p \geq 1$ define

$L^p(n, k) :=$ the number of M -sequences with at most p variables and degree at most n that has maximal value for m_i when $i \leq k$ but not for m_{k+1} , i.e., $m_i = \binom{p+i-1}{i}$ for $i \leq k$ and $m_{k+1} < \binom{k+p}{k+1}$.

The boundary conditions are $L^p(n, n) = L^p(n, -1) = 1$ for all $p \geq 1, n \geq -1$. For consistency we define $L^0(n, n) := L^0(n, -1) := 1$ and $L^0(n, k) := 0$ for $k \neq -1, n$. It follows from these definitions that

$$M^p(n) = \sum_{k=-1}^n L^p(n, k), \tag{1}$$

for all $n, p \geq 0$.

The numbers $L^p(n, k)$ also have interesting interpretations along the lines of Theorem 1.1. In polytope theory for example we get from the bijection

between (i) and (iv) of Theorem 1.1 that $L^p(n, k)$ is the number of f -vectors for simplicial $2n$ - (or $2n + 1$)-polytopes with $p + 2n + 1$ ($p + 2n + 2$) vertices that are k -neighborly, i.e., they have all possible r -sets as faces for $r \leq k$, but not $k + 1$ -neighborly.

The basic theorem from which the other results will follow is the following.

Theorem 2.1. *The number of M -sequences satisfies the following recursions for all $p, n \geq 1, k \geq 0$:*

$$M^p(n) = 1 + \sum_{i=0}^n L^{p-1}(n, i) M^p(i-1) \quad (2)$$

and

$$L^p(n, k) = \sum_{i=k}^n L^{p-1}(n, i) L^p(i-1, k-1). \quad (3)$$

PROOF The theorem is, once discovered, easy to prove. From Theorem 1.1 we have that when counting M -sequences we can count compressed multicomplexes instead. See Figure 1. Let \mathcal{M} be a compressed multicomplex on the p variables x_1, \dots, x_p of degree at most n that is totally filled exactly to level k . Partition the multisets in \mathcal{M} into two disjoint parts depending on whether the multiset contains x_p or not, i.e., $\mathcal{M}_1 := \{A \in \mathcal{M} : x_p \notin A\}$ and $\mathcal{M}_2 := \{A \in \mathcal{M} : x_p \in A\}$. See Figure 2.

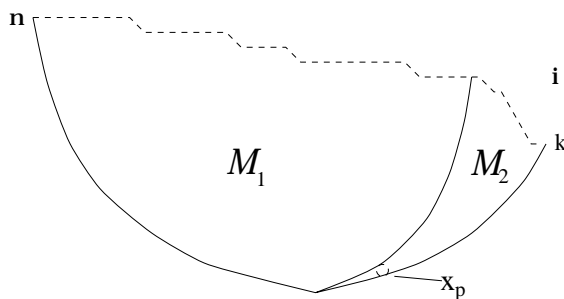


FIGURE 2. The partition of a compressed multicomplex \mathcal{M} as in the proof of Theorem 2.1.

Note that \mathcal{M}_1 is a compressed multicomplex on at most $p - 1$ variables and that dividing every monomial in \mathcal{M}_2 by x_p we get a compressed multicomplex on at most p variables. Let $i, i \geq k$, be the largest level in \mathcal{M}_1 that is totally filled. Then there are $L^{p-1}(n, i)$ possibilities for \mathcal{M}_1 and $L^p(i - 1, k - 1)$ possibilities for \mathcal{M}_2 and all these possibilities occur for some \mathcal{M} . Summing over i we get recursion (3). Recursion (2) follows from (1) and (3). \square

| $p \setminus n$ | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------------|----|---|---|----|-----|-------|--------|----------|------------|
| 0 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 |
| 3 | 1 | 2 | 5 | 15 | 52 | 203 | 877 | 4140 | 21147 |
| 4 | 1 | 2 | 6 | 26 | 152 | 1144 | 10742 | 122772 | 1673856 |
| 5 | 1 | 2 | 7 | 42 | 392 | 5345 | 102050 | 2632429 | 89026966 |
| 6 | 1 | 2 | 8 | 64 | 904 | 20926 | 753994 | 40412530 | 3099627142 |

TABLE 1. Table of $M^p(n)$, the number of M -sequences with $m_1 \leq p$ and $m_i = 0$ for $i > n$.

Corollary 2.2. *As special cases we get,*

$$L^p(n, 0) = M^{p-1}(n) - 1$$

and
$$L^p(n, n-1) = \binom{p+n-1}{n}.$$

PROOF Follows directly from Theorem 2.1. Both formulas are also easily understandable directly from the definition of $L^p(n, k)$. \square

2.2. Keeping p fixed. Recall that the Stirling number of the second kind $S(n, k)$ is the number of ways to partition $\{1, 2, \dots, n\}$ into k blocks, and that the Bell number $B(n) = \sum_{k=1}^n S(n, k)$ is the number of all possible partitions.

Corollary 2.3. *For $n \geq k \geq -1$, the number of M -sequences with at most 1, 2 and 3 variables are*

$$\begin{aligned} M^1(n) &= n + 2, & L^1(n, k) &= 1 \\ M^2(n) &= 2^{n+1}, & L^2(n, k) &= \binom{n+1}{k+1} \\ M^3(n) &= B(n+2), & L^3(n, k) &= S(n+2, k+2), \end{aligned}$$

where $B(n)$ are the Bell numbers and $S(n, k)$ are the Stirling numbers of the second kind.

PROOF Easy consequence of Theorem 2.1. \square

REMARK The result $M^2(n) = 2^{n+1}$ has been previously calculated by Björner [B1]. \square

We will now proceed to study the asymptotics of $M^p(n)$ when p is fixed.

Lemma 2.4. *For any $r, p, n > 0$, we have*

$$L^p(n+r, n) \geq \frac{n^{r(p-1)}}{(p-1)!r!}.$$

PROOF If there are i monomials of degree k in a compressed multicomplex \mathcal{M} , then we can always have at least anywhere between 0 and i monomials of degree $k + 1$ in \mathcal{M} . Hence $L^p(n + r, n)$ is bounded below by the number of weakly decreasing sequences of non-negative integers of length r starting with an integer smaller than $\binom{n+p}{p-1}$, which is

$$\binom{\binom{n+p}{p-1} - 1 + r}{r}.$$

The lemma follows. \square

Theorem 2.5. *For any $p \geq 3$ and $\epsilon > 0$ we have that for all sufficiently large n*

$$\frac{\log M^p(n)}{n \log n} > p - 2 - \epsilon.$$

PROOF Fix an integer $r > 2p/\epsilon$. Select one term of recursion (2) and apply Lemma 2.4 to get

$$\frac{M^p(n)}{M^p(n-r-1)} > L^{p-1}(n, n-r) \geq \frac{(n-r)^{r(p-2)}}{(p-2)!r!},$$

for any n . Applying this recursively we obtain,

$$\begin{aligned} M^p(n) &> \prod_{i=1}^{\lfloor \frac{n}{r+1} \rfloor} \frac{(n - (r+1)i + 1)^{r(p-2)}}{(p-2)!r!} \geq \\ &\geq \prod_{i=1}^{\lfloor \frac{n}{r+1} \rfloor} \frac{((n - (r+1)i + 1) \cdots (n - (r+1)i - r + 1))^{(p-2)r/(r+1)}}{(p-2)!r!} \geq \\ &\geq \frac{(n-r)!^{(p-2)(1-1/(r+1))}}{(p-2)!n r!^{(n/r)}}. \end{aligned}$$

The statement is now true for any $n > (pr)^r$ and the theorem follows. \square

Theorem 2.5 is close to best possible. Mireille Bousquet-Mélou [BM] has proved that for any $p \geq 2$ we have $M^p(n) < 2^{n+1}(n+1)!^{p-2}$.

2.3. Polytopes with few vertices. Let $c_s(d + p + 1, d)$ be the number of different combinatorial types of simplicial d -polytopes on $d + p + 1$ labeled vertices. Over the years, a lot of attention has been given to the problem of estimating $c_s(m, d)$, see [A][G, pp. 288–290]. Even the asymptotic behavior was a big open question, until Goodman and Pollack [GP] obtained the upper bound $c_s(m, d) \leq m^{d(d+1)m}$. The lower bound $\left(\frac{m-d}{d}\right)^{dm/4} \leq c_s(m, d)$ is due to Alon [A], who also improved the upper bound and generalized to arbitrary polytopes.

The problem of estimating $c_s(d + p + 1, d)$ for fixed p was posed at the end of [A], where the inequality

$$c_s(d + p + 1, d) \leq (d + p + 1)^{p(p-1)d(1+o(1))},$$

was given for any fixed p . From Theorem 1.1 we know that $c_s(2n + p + 1) \geq M^p(n) - M^{p-1}(n)$, so Theorem 2.5 above implies the following corollary.

Corollary 2.6. *For a fixed $p \geq 3$, we have*

$$d^{(p-2-o(1))d/2} < c_s(d + p + 1, d), \quad d \rightarrow \infty.$$

□

For $p = 3$ a better bound was obtained by Shemer [Sh] who shows that the number of simplicial polytopes in dimension $d = 2m$ on $d + 4$ vertices that have maximal f -vector, is at least $(d/e)^{d/2}$. I find it remarkable that the two bounds that undercount seriously in two different ways get estimates that are so close to each other. Even though the number of f -vectors for simplicial polytopes intuitively should be a poor estimate for $c_s(d + p + 1, d)$, I have not found any better bounds in the literature than Corollary 2.6 for $p \geq 4$.

2.4. Keeping n fixed. Next we study $M^p(n)$ and $L^p(n, k)$ in terms of p while keeping n and k fixed. Surprisingly enough they turn out to be polynomials.

Theorem 2.7. *$L^p(n, k)$ is a polynomial in p of degree $\binom{n+1}{2} - \binom{k+1}{2}$ and $M^p(n)$ is a polynomial in p of degree $\binom{n+1}{2}$, for each fixed pair $n, k \geq 0$.*

PROOF We will prove the theorem by double induction over n and $n - k$ using recursion (3). The statement is trivially true for $n = 0$ and for $n = k$ since $L^p(n, n) = 1$. Now, write (3) as

$$L^p(n, k) - L^{p-1}(n, k) = \sum_{i=k+1}^n L^{p-1}(n, i) L^p(i-1, k-1).$$

By induction we see that $L^p(n, k) - L^{p-1}(n, k)$ is a polynomial of degree

$$\begin{aligned} \max \left\{ \left[\binom{n+1}{2} - \binom{i+1}{2} \right] + \left[\binom{i}{2} - \binom{k}{2} \right] \right\}_{i=k+1}^n &= \\ = \binom{n+1}{2} - \binom{k+1}{2} - 1. \end{aligned}$$

Hence we get that $L^p(n, k)$ is a polynomial of degree $\binom{n+1}{2} - \binom{k+1}{2}$. From Corollary 2.2 it follows that $M^p(n)$ is a polynomial of degree $\binom{n+1}{2}$. □

REMARK A weaker formulation of the polynomial growth of $M^p(n)$ appears without proof in [B3]. □

Next, we calculate the leading coefficients, which we will need in Section 3.

Proposition 2.8. *For $n \geq 1$, the leading coefficient in $M^p(n)$ is*

$$\frac{\prod_{i=0}^{n-2} \left(\binom{n+1}{2} - \binom{i+1}{2} - 1 \right)}{\binom{n+1}{2}!}. \quad (4)$$

PROOF Let $c(n, k)$ be the leading coefficient of $L^p(n, k)$. We will show that $c(n, k) = \prod_{i=k}^{n-2} \left(\binom{n+1}{2} - \binom{i+1}{2} - 1 \right) / \left(\binom{n+1}{2} - \binom{k+1}{2} \right)!$, copying the double induction of the proof above. We get that $\left(\binom{n+1}{2} - \binom{k+1}{2} \right) c(n, k) = c(n, k+1)c(k, k-1)$. From the induction assumptions we have

$$c(n, k+1)c(k, k-1) = \frac{\prod_{i=k+1}^{n-2} \left(\binom{n+1}{2} - \binom{i+1}{2} - 1 \right)}{\left(\binom{n+1}{2} - \binom{k+2}{2} \right)!} \frac{1}{k!} = \frac{\prod_{i=k}^{n-2} \left(\binom{n+1}{2} - \binom{i+1}{2} - 1 \right)}{\left(\binom{n+1}{2} - \binom{k+1}{2} - 1 \right)!}.$$

The formula for $c(n, k)$ follows. The result for $M^p(n)$ is easily extracted from Corollary 2.2. \square

REMARK Similarly for fixed $p \geq 2, r \geq 0$, $L^p(n, n-r)$ is a polynomial in n of degree $r(p-1)$, with leading coefficient $1/(p-1)!r!$. \square

3. THE NUMBER OF f -VECTORS FOR SIMPLICIAL COMPLEXES

A *simplicial complex* is a collection \mathcal{F} of finite sets satisfying $B \in \mathcal{F}, A \subseteq B \implies A \in \mathcal{F}$. A set in \mathcal{F} is called a *simplex* or a *face*. A set in \mathcal{F} of cardinality one is called a *vertex*. A face that is not contained in any other face is called a *facet*. Given a simplicial complex \mathcal{F} , we define $f_i := |\{A \in \mathcal{F} : |A| = i\}|$. Note that f_i 's are indexed by cardinality and not by dimension.

In this section we will study the number of f -vectors for simplicial complexes given upper bounds $p \geq 0$ on the number of vertices and $n \geq 0$ on the cardinality for the simplices. We may assume that $n \leq p$ without loss of generality. Let

$F^p(n) :=$ the number of f -vectors of simplicial complexes with $f_1 \leq p$ and $f_j = 0$ for all $j > n$.

We include both $(0, 0, \dots)$, the f -vector of the empty simplicial complex, and $(1, 0, 0, \dots)$, the f -vector for the simplicial complex containing only the empty set, when computing $F^p(n)$. Hence we get $F^p(-1) = 1, F^p(0) = 2$ for all $p \geq 0$ and $F^0(n) = 2$ for all $n \geq 0$.

Part (ii) in the theorem below is what we need to carry out the same counting argument as for M -sequences.

Theorem 3.1. (Clements-Lindström [CL][CL]) *Fix $0 \leq n \leq p$. Then the following are equal to $F^p(n)$.*

- (i) *The number of f -vectors for simplicial complexes on at most p vertices and with no set of cardinality higher than n .*
- (ii) *The number of compressed simplicial complexes on at most p vertices and with no set of cardinality higher than n .*
- (iii) *The number of facet-vectors for simplicial complexes on at most p vertices and with no set of cardinality higher than n .* \square

For the definition of “compressed” see Section 2. The *facet-vector* of \mathcal{F} is the sequence $v = (v_0, v_1, v_2, \dots)$, where $v_i := |\{A \in \mathcal{F} : A \text{ a facet, } |A| = i\}|$.

As in Section 2 we need a refinement to state the recursion. Let

$E^p(n, k)$:= the number of f -vectors for simplicial complexes with $f_1 \leq p$ and $f_j = 0$ for all $j > n$ and with maximal value for f_j when $j \leq k$ but not for f_{k+1} , i.e., $f_i = \binom{p}{i}$ for $i \leq k$ but $f_{k+1} < \binom{p}{k+1}$.

From the definition we have for every pair $p \geq 0, n \geq -1$, that $E^p(n, -1) = 1$, $E^p(n, n) = 1$ and $E^p(n, k) = 0$ for $k > n$, and the simplicial equivalent of (1)

$$F^p(n) = \sum_{k=-1}^n E^p(n, k), \quad \text{for all } p \geq n \geq 0. \quad (5)$$

Theorem 3.2. *For $0 \leq k, 1 \leq n \leq p$ we have the recursions*

$$F^p(n) = \begin{cases} 1 + \sum_{i=0}^n E^{p-1}(n, i)F^{p-1}(i-1) & , \text{ if } n < p \\ 1 + F^p(p-1) & , \text{ if } n = p \end{cases} \quad (6)$$

and

$$E^p(n, k) = \begin{cases} \sum_{i=k}^n E^{p-1}(n, i)E^{p-1}(i-1, k-1) & , \text{ if } k \leq n < p \\ E^p(p-1, k) & , \text{ if } k < n = p \\ 1 & , \text{ if } k = n = p. \end{cases} \quad (7)$$

PROOF Similar to proof of Theorem 2.1. \square

As the careful reader has noticed Theorems 2.1 and 3.2 are very similar. The difference does not effect the proofs of Theorem 2.7 and Proposition 2.8.

Theorem 3.3. *$E^p(n, k)$ is a polynomial in p of degree $\binom{n+1}{2} - \binom{k+1}{2}$ and $F^p(n)$ is a polynomial in p of degree $\binom{n+1}{2}$, for each fixed $n, k \geq 0$. Moreover, $E^p(n, k)$ and $F^p(n)$ has the same leading coefficients as $L^p(n, k)$ and $M^p(n)$, respectively.*

PROOF Identical to the proofs for $L^p(n, k)$ and $M^p(n)$. \square

Corollary 3.4. *Fix $n \geq 0$. When p is large enough, almost every M -sequence is also an f -vectors for a simplicial complex. More precisely*

$$\lim_{p \rightarrow \infty} \frac{M^p(n)}{F^p(n)} = 1, \quad \text{for each } n \geq 0.$$

PROOF Immediate from Theorem 3.3. \square

Finally we will use that by Theorem 1.1 (iii), the number of f -vectors of $n - 1$ -dimensional shellable simplicial complexes on at most p vertices is equal to $M^{p-n}(n) - 1$.

Corollary 3.5. *Fix $n \geq 0$. When the number of vertices increases, almost every f -vector for an $n - 1$ -dimensional simplicial complex is also an f -vector for an $n - 1$ -dimensional shellable simplicial complex.*

The same is true when replacing shellable by Cohen-Macaulay, partitionable, pure or other weaker conditions on simplicial complexes.

PROOF We have the following chain of inequalities:

$$M^{p-n}(n) \leq F^p(n) - F^p(n-1) \leq F^p(n) \leq M^p(n).$$

The result follows, since for a fixed n , all four are polynomials in p of degree $\binom{n+1}{2}$ by Theorem 2.7 and Theorem 3.3. \square

Note that in Corollary 3.5 “shellable” complexes are pure. If we use “shellable” in the generalized nonpure sense of [B2], then every f -vector of a simplicial complex is the f -vector of some shellable complex.

4. REMARKS AND OPEN PROBLEMS

Remark 1: The total number of possible f -vectors of a simplicial complex with at most p vertices, i.e. $F^p(p)$, gives rise to an interesting sequence. It starts 2, 3, 5, 10, 26, 96, 553, 5461, 100709, \dots . This sequence has been calculated as the number of facet-vectors for simplicial complexes on p vertices by Knuth using a more complicated recursion that will appear in [K]. We found the references to [C] and [K] in [SP].

Remark 2: Using the recursions it is not difficult to define bijections from the M -sequences on two or three variables to subsets and partitions, respectively. It is a fun exercise to define “iterated partitions” as suggested by the recursion to obtain another combinatorial object enumerated by $M^p(n)$, see [L].

Remark 3: It would be nice to understand the sequence $M^4(n)$ better. Let $H_p(x) := \sum_{n=-1}^{\infty} M^p(n) \frac{x^{n+1}}{(n+1)!}$. I have not found a closed formula for this generating function, but it satisfies the following functional equation

$$H_4(x) = e^x + e^x \int e^x H_4(e^x - 1) dx.$$

Remark 4: Assume we are given an integer $p \geq 0$ and $n_1, n_2, \dots, n_p \in \{1, 2, \dots\} \cup \{\infty\}$. A set of monomials, $x_1^{a_1} \dots x_p^{a_p}$ satisfying $0 \leq a_i \leq n_i$ for $i = 1, \dots, p$, which is closed under division will be called a *Clements-Lindström complex of type n_1, \dots, n_p* . When $n_i = \infty$ for all i we have multicomplexes on p variables and when $n_i = 1$ for all i we have simplicial complexes on p vertices. One can ask the generalized question of the number of f -vectors of

a Clements-Lindström complex of type n_1, \dots, n_p and formulate a common generalization of Theorem 2.1 and Theorem 6. For details, see [L].

Problem: Give a direct proof of the fact that the number of possible f -vectors of simplicial d -polytopes with at most $d + 4$ vertices is the Bell number $B(\lfloor d/2 \rfloor + 2) - 1$.

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