ANALYTIC CONTINUATION OF MULTIPLE ZETA-FUNCTIONS AND THEIR VALUES AT NON-POSITIVE INTEGERS

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Abstract. Analytic continuation of the multiple zeta-function is established by a simple application of the Euler-Maclaurin summation formula. Multiple zeta values at non-positive integers are defined and their properties are investigated.

1. Introduction

The multiple zeta values due to D. Zagier are defined by

$$\zeta_k(s_1, s_2, \dots, s_k) = \sum_{0<n_1<n_2<\dots<n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \dots n_k^{s_k}}$$

with positive integers s_i $(i = 1, 2, \dots, k)$ and $s_k \geq 2$. These values have a certain connection with topology and physics, and algebraic relations among them are extensively studied (see [18], [19], [6], [7] and [14]). Recently, Y. Ohno developed a unified algebraic relation in [16]. It is also interesting to consider it for complex variables s_i.

In this paper, we treat analytic continuation of $\zeta_k(s_1, s_2, \dots, s_k)$. Analytic continuation of $\zeta_2(s_1, s_2)$ was proved by F.V. Atkinson [5] with applications to the study of the asymptotic behavior of the 'mean values' of zeta-functions. See also Y. Motohashi [15] and M. Katsurada & K. Matsumoto [13]. In [4], T. Arakawa & M. Kaneko used analytic continuation of $\zeta_k(s_1, s_2, \dots, s_k)$ as a function of one variable s_k when $s_1, s_2, \dots, s_{k-1}$ are positive integers, and discussed the relation among generalized Bernoulli numbers. On the other hand, S. Egami discussed the relationship among various multiple zeta-functions introduced by E.W. Barnes, T. Shintani and D. Zagier. (See [9] and [10].)

However, for a general k, we cannot find the proof of analytic continuation of $\zeta_k(s_1, s_2, \dots, s_k)$ as a function of k variables in literature (but see the comment of Zagier [18, p. 509, lines 14–19]). We shall show that the multiple zeta-function can be continued analytically to $\mathbb{C}^k$ and discuss interesting properties of multiple zeta values at non-positive integers.

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Remark 1. After submitting the first version of our paper, we found a recent work of J. Zhao [20] treating analytic continuation of multiple zeta-function. This fact was also pointed out by the referee. With the help of the theory of generalized function in the sense of I.M. Gel'fand and G.E. Shilov, he gave their possible singularities as well as the residues. However our method is apparently simple and reveals the *exact* location of singularities, which seems to be an advantage.

2. Analytic continuation

Let l and m be positive integers. Define an entire function:

$$\phi_l(m,s)=\sum_{n=1}^{m}\frac{1}{n^s}-\left\{\frac{m^{1-s}-1}{1-s}+\frac{1}{2m^s}-\sum_{q=1}^{l}\frac{(s)_q a_q}{m^{s+q}}+\zeta(s)-\frac{1}{s-1}\right\} \tag{1}$$

with $(s)_n=s(s+1)\cdots(s+n-1)$ and $a_q=B_{q+1}/(q+1)!$. Here B_q are Bernoulli numbers defined by $z/(e^z-1)=\sum_{q=0}^{\infty}B_q z^q/q!$ and $\zeta(s)$ is the Riemann zeta-function. By using the Euler-Maclaurin summation formula, we have $\phi_l(m,s)=O(|(s)_{l+1}|m^{-\Re(s)-l-1})$ when s is a complex number. Considering s as a complex variable and $m\to\infty$, we get an analytic continuation of $\zeta(s)$ in $\Re(s+l+1)>0$. Note that (1) is also valid when $s\to 1$, if we replace $(m^{1-s}-1)/(1-s)$ by its limit $\log m$.

This is one of the oldest way of the analytic continuation of the Riemann zeta-function, which provides us with a method of numerical calculations in the critical strip $0<\Re s<1$. (c.f. [8], [12]). It does not give us the celebrated functional equation of $\zeta(s)$ directly, but it is possible to derive it by more precise observations (see Chapter 2 of [17]). Hereafter we will use (1) in the form:

$$\sum_{n=m+1}^{\infty}\frac{1}{n^s}=-\phi_l(m,s)+\frac{m^{1-s}}{s-1}-\frac{1}{2m^s}+\sum_{q=1}^{l}\frac{(s)_q a_q}{m^{s+q}}, \tag{2}$$

for $\Re(s)>1$. Consider the multiple zeta-function in two variables:

$$\zeta_2(s_1,s_2)=\sum_{0<n_1<n_2}\frac{1}{n_1^{s_1}n_2^{s_2}}$$

with $\Re s_i > 1$ $(i = 1, 2)$. By (2),

$$\begin{aligned}
\zeta_2(s_1, s_2) &= \sum_{n_1=1}^{\infty} \frac{1}{n_1^{s_1}} \sum_{n_2=n_1+1}^{\infty} \frac{1}{n_2^{s_2}} \\
&= \sum_{n_1=1}^{\infty} \frac{1}{n_1^{s_1}} \left\{ -\phi_l(n_1, s_2) + \frac{n_1^{1-s_2}}{s_2 - 1} - \frac{1}{2n_1^{s_2}} + \sum_{q=1}^{l} \frac{(s_2)_q a_q}{n_1^{s_2+q}} \right\} \\
&= \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} - \frac{\zeta(s_1 + s_2)}{2} \\
&\quad + \sum_{q=1}^{l} (s_2)_q a_q \zeta(s_1 + s_2 + q) - \sum_{n_1=1}^{\infty} \frac{\phi_l(n_1, s_2)}{n_1^{s_1}} \qquad (3)
\end{aligned}$$

holds for $\Re(s_i) > 1$ $(i = 1, 2)$. The terms on the right hand side have meromorphic continuations except the last one. The last sum is absolutely convergent, and hence holomorphic, in $\Re(s_1 + s_2 + l) > 0$. Thus we now have a meromorphic continuation of $\zeta_2(s_1, s_2)$ to $\Re(s_1 + s_2 + l) > 0$. Since we can choose arbitrary large l, we get a meromorphic continuation of $\zeta_2(s_1, s_2)$ to $\mathbb{C}^2$, which is holomorphic in

$$\left\{ (s_1, s_2) \in \mathbb{C}^2 \mid s_2 \neq 1, s_1 + s_2 \notin \{2, 1, 0, -2, -4, -6, \dots\} \right\}.$$

One can see easily that this trick can be applied to a multiple zeta-function with k variables. In fact,

$$\begin{aligned}
\zeta_k(s_1, s_2, \dots, s_k) &= \sum_{n_1=1}^{\infty} \frac{1}{n_1^{s_1}} \sum_{n_2=n_1+1}^{\infty} \frac{1}{n_2^{s_2}} \cdots \sum_{n_{k-1}=n_{k-2}+1}^{\infty} \frac{1}{n_{k-1}^{s_{k-1}}} \sum_{n_k=n_{k-1}+1}^{\infty} \frac{1}{n_k^{s_k}} \\
&= \sum_{n_1=1}^{\infty} \frac{1}{n_1^{s_1}} \sum_{n_2=n_1+1}^{\infty} \frac{1}{n_2^{s_2}} \cdots \sum_{n_{k-1}=n_{k-2}+1}^{\infty} \frac{1}{n_{k-1}^{s_{k-1}}} \times \\
&\qquad \times \left\{ -\phi_l(n_{k-1}, s_k) + \frac{n_{k-1}^{1-s_k}}{s_k - 1} - \frac{1}{2n_{k-1}^{s_k}} + \sum_{q=1}^{l} \frac{(s_k)_q a_q}{n_{k-1}^{s_k+q}} \right\} \\
&= \frac{\zeta_{k-1}(s_1, s_2, \dots, s_{k-2}, s_{k-1} + s_k - 1)}{s_k - 1} - \frac{\zeta_{k-1}(s_1, s_2, \dots, s_{k-2}, s_{k-1} + s_k)}{2} \\
&\quad + \sum_{q=1}^{l} (s_k)_q a_q \zeta_{k-1}(s_1, s_2, \dots, s_{k-2}, s_{k-1} + s_k + q) \\
&\quad - \sum_{0<n_1<n_2<\dots<n_{k-1}} \frac{\phi_l(n_{k-1}, s_k)}{n_1^{s_1} n_2^{s_2} \dots n_{k-1}^{s_{k-1}}} \qquad (4)
\end{aligned}$$

for $\Re(s_i) > 1$ $(i = 1, 2, \ldots, k)$. Since

$$\sum_{0<n_1<n_2<\ldots<n_{k-1}} \frac{\phi_l(n_{k-1}, s_k)}{n_1^{s_1} n_2^{s_2} \ldots n_{k-1}^{s_{k-1}}} \ll \sum_{n_{k-1}} \frac{n_{k-1}^{-l-\Re(s_k)+k-3}}{n_{k-1}^{L}}$$

with $L = \Re(s_{k-1}) + \sum_{\substack{1\le j\le k-2,\\ \Re(s_i)\le 0}} \Re(s_i)$, the last summation is convergent absolutely in

$$l - k + 2 + \Re(s_k) + \Re(s_{k-1}) + \sum_{\substack{1\le i\le k-2,\\ \Re(s_i)\le 0}} \Re(s_i) > 0. \tag{5}$$

Since l can be taken arbitrarily large, we get an analytic continuation of $\zeta_k(s_1, s_2, \ldots, s_k)$ to $\mathbb{C}^k$. Now we consider the set of singularities. For simplicity, we put $(s)_0 = 1$. Then the 'singular part' of $\zeta_2(s_1, s_2)$ is given by

$$\frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} + \sum_{q_1 \ge 0} \frac{a_{q_1}(s_2)_{q_1}}{s_1 + s_2 + q_1 - 1}.$$

Note that this sum is formal and only indicates local singularities. From this expression, we see

$$s_2 = 1, \quad s_1 + s_2 \in \{2, 1, 0, -2, -4, -6, \ldots\}$$

forms the set of whole singularities. For the case $\zeta_3(s_1, s_2, s_3)$, by using the singular part of ζ_2, we see that singularities lie on

$$s_3 = 1, \quad s_2 + s_3 \in \{2, 1, 0, -2, -4, -6, \ldots\}$$

and

$$s_1 + s_2 + s_3 \in \{3, 2, 1, 0, -1, -2, -3, \ldots\}.$$

We want to show that these are the whole singularities. It suffices to show that no singularities defined by one of above equations will identically vanish. This can be shown by replacing variables:

$$u_1 = s_1, \quad u_2 = s_2 + s_3, \quad u_3 = s_3.$$

In fact, we see that the singular part of $\zeta_3(u_1, u_2 - u_3, u_3)$ is given by

$$\frac{1}{u_3 - 1}\zeta_2(u_1, u_2 - 1) + \sum_{q_2 \ge 0} (u_3)_{q_2} a_{q_2} \zeta_2(u_1, u_2 + q_2).$$

By this expression we see that the singularities of $\zeta_2(u_1, u_2 + q)$ are summed with functions of u_3 of different degree. Thus these singularities, as a weighted sum by another variable u_3, will not vanish identically. Similarly, we see

Theorem 1. *The multiple zeta-function $\zeta_k(s_1, s_2, \ldots, s_k)$ is meromorphically continued to $\mathbb{C}^k$ and has singularities on*

$$s_k = 1, \quad s_{k-1} + s_k = 2, 1, 0, -2, -4, \ldots,$$

and

$$\sum_{i=1}^{j} s_{k-i+1} \in \mathbb{Z}_{\leq j} \, (j = 3, 4, \ldots, k),$$

where $\mathbb{Z}_{\leq j}$ is the set of integers less than or equal to j.

3. Zeta values at non-positive integers

In this section, we use the notation $(s)_0 = 1$ and $(s)_{-1} = 1/(s-1)$ for the sake of simplicity. We also put $a_q = B_{q+1}/(q+1)!$ for $q = 0$ and -1 as in §2. A point of $\mathbb{C}^n$ $(n \geq 2)$ is said to be *a point of indeterminacy* of a meromorphic function if both the local denominator and the local numerator vanish there. See p.164 of [11] for the precise definition. For instance, let $f(s_1, s_2) = s_1/(s_1 + s_2)$. Then $s_1 = s_2 = 0$ is a point of indeterminacy of f. So the value of f at $(0, 0)$ depends on a limiting process, for example $\lim_{s_2 \to 0} \lim_{s_1 \to 0} f(s_1, s_2) = 0$ while $\lim_{s_1 \to 0} \lim_{s_2 \to 0} f(s_1, s_2) = 1$.

Let r_i $(i = 1, 2, \ldots, k)$ be non-negative integers. Recall from Theorem 1 that each point $(-r_1, -r_2, \ldots, -r_k)$ except when $k = 2$ and $r_1 + r_2$ is odd, lies on the set of singularities. Moreover, such a point is a point of indeterminacy. To prove this, it suffices to show that ζ_k has a finite value at $(-r_1, -r_2, \ldots, -r_k)$ by a specific limiting process. Now we give the definition which we will employ in this paper.

Definition . We define the multiple zeta values at non-positive integers by

$$\zeta_k(-r_1, -r_2, \ldots, -r_k) = \lim_{s_1 \to -r_1} \lim_{s_2 \to -r_2} \cdots \lim_{s_k \to -r_k} \zeta_k(s_1, s_1, \ldots, s_k).$$

From (4) and the above definition, we have

$$\zeta_k(-r_1, -r_2, \ldots, -r_k) = \sum_{q=-1}^{r_k} (-r_k)_q a_q \zeta_{k-1}(-r_1, -r_2, \ldots, -r_{k-2}, -r_{k-1} - r_k + q). \tag{6}$$

Here we used the fact that $\phi_r(m, l) = 0$ for $l \geq r$. This formula shows that the value $\zeta_k(-r_1, -r_2, \ldots, -r_k)$ is determined recursively as a finite number, hence each point $(-r_1, -r_2, \ldots, -r_k)$ is a point of indeterminacy. The formula (6) also gives us a simple way of calculation of multiple zeta values $\zeta_k(-r_1, -r_2, \ldots, -r_k)$. For example, we have $\zeta_2(0, 0) = 1/3$, $\zeta_3(0, 0, 0) = -1/4$, $\zeta_4(0, 0, 0, 0) = 1/5$, $\zeta_2(-1, -1) = 1/360$, $\zeta_3(-1, -1, -1) = 83/30240$. One may expect that $\zeta_k(0, 0, \ldots, 0) = (-1)^k/(1+k)$. This assertion will be proved in the forthcoming paper [2]. Here we shall show some other interesting properties.

Theorem 2. *Let r_i $(i = 1, 2, \ldots, k)$ be non-negative integers. Then the value $\zeta_k(-r_1, -r_2, \ldots, -r_k)$ is a rational number whose denominator has prime factors less than or equal to $1 + k + \sum_{i=1}^{k} r_i$.*

Proof. It is well known that $\zeta(0) = -1/2, \zeta(-2r) = 0$ and $\zeta(1-2r) = -B_{2r}/2r$ for positive integers r. By using the theorem of von Staudt & Clausen, the assertion for $k = 1$ is obvious. From (6), the proof is completed by the induction on k. □

Theorem 3. *Let r_i $(i = 1, 2, \ldots, k)$ be positive integers and n_i $(i = 1, 2, \ldots, k)$ be non-negative integers. If $\sum_{i=1}^{k}(r_i + n_i + 1)$ is odd then*

$$\sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma)\zeta_k(-r_{\sigma(1)} - n_1, -r_{\sigma(2)} - n_2, \ldots, -r_{\sigma(k)} - n_k) = 0, \tag{7}$$

where $\mathfrak{S}_k$ is the symmetric group of degree k and $\operatorname{sgn}(\sigma)$ *is the signature of $\sigma \in \mathfrak{S}_k$*

The statement is trivial when r_i are not distinct. We will show some examples when $n_1 = n_2 = n_3 = 0$ before proving the theorem:

$$\begin{aligned}
&\zeta_2(-1,-2) - \zeta_2(-2,-1) = -\frac{1}{240} + \frac{1}{240} = 0,\\
&\zeta_3(-1,-2,-3) + \zeta_3(-2,-3,-1) + \zeta_3(-3,-1,-2)\\
&\qquad - \zeta_3(-1,-3,-2) - \zeta_3(-2,-1,-3) - \zeta_3(-3,-2,-1)\\
&\quad = -\frac{101}{100800} + \frac{149}{302400} + \frac{107}{302400} + \frac{19}{30240} + \frac{17}{43200} - \frac{131}{151200} = 0.
\end{aligned}$$

Proof. In the following, we shall only prove the case $n_1 = n_2 = \ldots = n_k = 0$. The generalization is quite easy and is left to the reader. Let

$$I_k = \left\{(-r_1, \ldots, -r_k) \middle| r_i \text{ are positive integers and } \textstyle\sum_{i=1}^{k}(r_i + 1) \text{ is odd.}\right\}.$$

For $1 \le a < b \le k$, we define a vector space $\mathfrak{F}_k(a,b)$ consisting of $\mathbb{C}$-valued functions $f(\xi_1, \xi_2, \ldots, \xi_k)$ such that

$$\sum_{\sigma \in \mathfrak{S}_{a:b}} \operatorname{sgn}(\sigma) f(-r_{\sigma(1)}, -r_{\sigma(2)}, \ldots, -r_{\sigma(k)}) = 0$$

for any $(-r_1, \ldots, -r_k) \in I_k$, where $\mathfrak{S}_{a:b}$ is a subgroup of $\mathfrak{S}_k$ whose elements stabilize $\{1, 2, \ldots, k\} \setminus \{a, a+1, \ldots, b\}$. Our task is to show that the function $\zeta_k(\xi_1, \xi_2, \ldots, \xi_k)$ is contained in $\mathfrak{F}_k(1,k)$. Considering the coset decomposition $\mathfrak{S}_k/\mathfrak{S}_{a:b}$, we see $\mathfrak{F}_k(a,b)$ is a subspace of $\mathfrak{F}_k(1,k)$. Thus it is enough to show that the multiple zeta-function is contained in a sum of subspaces $\mathfrak{F}_k(a,b)$.

First we prove the case $k \le 3$. The assertion (7) is valid when $k=1$, since $\zeta(-2r)=0$ for any positive integers r. When $k=2$ and $(-r_1,-r_2)\in I_2$, we have

$$\begin{aligned}\zeta_2(-r_1,-r_2) &= \sum_{q=-1}^{r_2}(-r_2)_q a_q \zeta(-r_1-r_2+q)\\ &= -\frac{1}{2}\zeta(-r_1-r_2), \qquad (8)\end{aligned}$$

which shows the assertion for $k=2$. When $k=3$ and $(-r_1,-r_2,-r_3)\in I_3$, we have from (8) that

$$\begin{aligned}\zeta_3(-r_1,-r_2,-r_3) &= \sum_{q=-1}^{r_3}(-r_3)_q a_q \zeta_2(-r_1,-r_2-r_3+q)\\ &= -\frac{1}{2}\zeta_2(-r_1,-r_2-r_3) - \frac{1}{2}\sum_{\substack{q=-1\\ q\,:\,\mathrm{odd}}}^{r_3}(-r_3)_q a_q \zeta(-r_1-r_2-r_3+q). \qquad (9)\end{aligned}$$

Hence $\zeta_3(\xi_1,\xi_2,\xi_3)\in\mathfrak{F}_3(2,3)+\mathfrak{F}_3(1,2)$.

Let $k\ge 3$ and $(-r_1,\dots,-r_k)\in I_k$. Then by induction on k, we can easily see that

$$\begin{aligned}&\zeta_k(\xi_1,\xi_2,\dots,\xi_k)+\frac{1}{2}\zeta_{k-1}(\xi_1,\xi_2,\dots,\xi_{k-2},\xi_{k-1}+\xi_k)\\ &\in \mathfrak{F}_k(1,2)+\mathfrak{F}_k(2,3)+\ldots+\mathfrak{F}_k(k-2,k-1). \qquad (10)\end{aligned}$$

The assertion of the theorem follows immediately from (10). □

Suppose that $k=3$, r_i are non-negative integers, $r_1>0$ and $r_1+r_2+r_3$ is even. Then from (8) and (9), we have

$$\zeta_3(-r_1,-r_2,-r_3) = -\frac{1}{2}\Big\{\zeta_2(-r_1-r_2,-r_3)+\zeta_2(-r_1,-r_2-r_3)\Big\}. \qquad (11)$$

One may expect that symmetric expressions like (8) and (11) would give us a deeper understanding of Theorem 3. Further calculation suggests us the following conjecture. To state it, we shall prepare some notation. Let S be the ordered index set $\{1,2,\cdots,k\}$ of k elements and let $\mathcal{D}_l^k$ be the set of all ways of dividing S into l parts. Clearly the set $\mathcal{D}_l^k$ consists of $\binom{k-1}{l-1}$ elements. The element J in $\mathcal{D}_l^k$ can be expressed as

$$J=(1,\cdots,i_1\big|\, i_1+1,\cdots,i_2\big|\, i_2+1,\cdots,i_{l-1}\big|\, i_{l-1}+1,\cdots,k).$$

Let $A = (-r_1, -r_2, \cdots, -r_k)$ be a sequence of k non-positive integers. For $J \in \mathcal{D}_l^k$ as above, we set

$$A^J = (-r_1 - r_2 - \cdots - r_{i_1},\ -r_{i_1+1} - \cdots - r_{i_2}, \cdots, -r_{i_{l-1}+1} - \cdots - r_k)$$

and

$$\zeta_l(A^J) = \zeta_l\big(-r_1 - r_2 - \cdots - r_{i_1},\ -r_{i_1+1} - \cdots - r_{i_2}, \cdots, -r_{i_{l-1}+1} - \cdots - r_k\big).$$

Now we can state our

Conjecture . *Let r_i be non-negative integers, $r_1 > 0$ and $\sum_{i=1}^k (r_i + 1)$ is odd. Let $A = (-r_1, -r_2, \cdots, -r_k)$. Then we have*

$$\zeta_k(A) = -2 \sum_{j=1}^{k-1} (2^{j+1} - 1) \frac{B_{j+1}}{j+1} \Big(\sum_{J \in \mathcal{D}_{k-j}^k} \zeta_{k-j}(A^J) \Big). \tag{12}$$

Further discussion [1] will be found in [2]. We would like to note that we could find the conjectural form of (12) by the home page 'Sloane's On-Line Encyclopedia of Integer Sequences'.

Theorem 4. *For a positive integer r, we have*

$$\frac{\zeta(-4r-1)}{\zeta_2(-2r, -2r)} = (2r+1) \binom{4r+2}{2r+1}.$$

Proof. From (6) and the definition of a_q, we have

$$\zeta_2(-2r, -2r) = \frac{B_{4r+2}}{2(2r+1)^2} + \frac{1}{2r+1} \sum_{j=1}^{r} \binom{2r+1}{2j} \frac{B_{2j} B_{4r+2-2j}}{4r+2-2j}.$$

We note the following identity of Bernoulli numbers:

$$2(2r+1) \sum_{j=0}^{r} \binom{2r+1}{2j} \frac{B_{2j} B_{4r+2-2j}}{4r+2-2j} + \frac{((2r+1)!)^2}{(4r+2)!} B_{4r+2} = 0,$$

obtained by putting $m = n = 2r+1$ and $x = 0$ in Apostol [3, p.276, 19 (b)]. Hence,we have

$$\zeta_2(-2r, -2r) = -\frac{1}{2(2r+1)^2} \frac{((2r+1)!)^2}{(4r+2)!} B_{4r+2}.$$

On the other hand, $\zeta(-4r-1) = -B_{4r+2}/(4r+2)$, and this gives the assertion of Theorem 4. □

Finally we want to add several remarks.

[1] Addendum for the revised version: Finally we have succeeded in proving the validity of this Conjecture. See [2] for details.

Remark 2. There are many other possibilities for the definition of multiple zeta values at non-positive integers. For instance, define the value ζ_k^* by

$$\zeta_k^*(-r_1, -r_2, \dots, -r_k) = \lim_{\varepsilon \to 0} \zeta_k(-r_1 + \varepsilon, -r_2 + \varepsilon, \dots, -r_k + \varepsilon).$$

When $k = 2$, this is equivalent to define by

$$\zeta_2^*(-r_1, -r_2) = \sum_{q=-1}^{r_1} (-r_2)_q a_q \zeta(-r_1 - r_2 + q) + \frac{(-1)^{r_1} r_1!\, r_2!\, a_{r_1+r_2+1}}{2} \tag{13}$$

for non-negative integers r_i $(i = 1, 2)$. This definition seems to be better than our former definition at least when $k = 2, 3$. In fact, when $r_1 + r_2$ is odd we have $\zeta_2(-r_1, -r_2) = \zeta_2^*(-r_1, -r_2)$ and

$$\zeta_2^*(-2u_1, -2u_2) + \zeta_2^*(-2u_2, -2u_1) = 0$$

for positive integers u_i $(i = 1, 2)$. Especially we have $\zeta_2^*(-2u, -2u) = 0$ with a positive integer u. We can also find a recursive formula for $k = 3$ and show that

$$\zeta_3^*(-2u, -2v, -2w) + \zeta_3^*(-2w, -2v, -2u) = 0$$

for positive integers u, v, w. However in the general case, it seems difficult to construct a recursive formula like (13), since there exist a lot of singularities to take into account. One may hope that

$$\zeta_k^*(-2u, -2u, \dots, -2u) = 0$$

for a positive integer u.

Remark 3. The set of points of indeterminacy forms a $k-2$ dimensional holomorphic subvariety of $\mathbb{C}^k$, by p.166 of [11]. We have shown that each non-positive points $(-r_1, \dots, -r_k)$ are actually on this subvariety, but there are another type of integer points on this set. For instance, it will be shown in [2] that $(-r_1, \dots, -r_{k-1}, 1)$ is a point of indeterminacy whose multiple zeta value in our sense is rational, when $r_i \in \mathbb{Z}_{\geq 0}$ and not all r_i is zero. Also we have

$$\zeta_3(4, -3, -2) = -\frac{461}{2520} - \frac{\pi^2}{144} + \frac{\pi^4}{45360} + \frac{\zeta(3)}{420}.$$

We could not determined yet the whole set of such integer points.

Remark 4. We can easily apply the Euler-Maclaurin summation formula to more general zeta-functions. For instance, let $\alpha_i > -1$ $(i = 1, 2, \dots, k)$ be real numbers and χ_i $(i = 1, 2, \dots, k)$ the Dirichlet characters. Define a function $\xi(s_1, s_2, \dots, s_k)$ for $\Re(s_i) > 1$ $(i = 1, 2, \dots, k)$, by a convergent sum:

$$\sum_{0 < n_1 < n_2 < \dots < n_k} \frac{\chi_1(n_1)\chi_2(n_2)\cdots\chi_k(n_k)}{(n_1 + \alpha_1)^{s_1}(n_2 + \alpha_2)^{s_2} \dots (n_k + \alpha_k)^{s_k}}.$$

Then ξ is meromorphically continued to $\mathbb{C}^k$. In fact, using binomial series expansion of $(n+\beta)^{-s} = n^{-s}(1+\beta/n)^{-s}$ for each variable, we see that ξ can be expressed in terms of absolutely convergent sums of multiple zeta functions. See [1] for further study of this kind of function.

References

[1] S. Akiyama and H. Ishikawa, An analytic continuation of multiple L-functions and related zeta-functions, to appear in 'Analytic Number Theory' ed. by C.Jia and K.Matsumoto.

[2] S. Akiyama and Y. Tanigawa, Multiple zeta values at non-positive integers, submitted.

[3] T.M. Apostol, *Introduction to Analytic Number Theory*, Springer 1976.

[4] T. Arakawa and M. Kaneko, Multiple zeta values, poly-Bernoulli numbers, and related zeta-functions, *Nagoya Math. J.* **153** (1999), 189–209.

[5] F.V. Atkinson, The mean value of the Riemann zeta-function, *Acta Math.*, **81** (1949), 353–376.

[6] J.M. Borwein, D.M. Bradley, D.J. Broadhurst and P. Lisoněk, Combinatorial aspects of multiple zeta values, *Electron. J. Combin.*, **5** (1998), no. 1, Research Paper 38, 12 pp.

[7] D.J. Broadhurst and D. Kreimer, Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops, *Phys. Lett.*, B 393 (1997), no. 3-4, 403–412.

[8] H.M. Edwards,*Riemann's Zeta Function*, Academic Press, New York and London 1974.

[9] S. Egami, Reciprocity laws of multiple zeta functions and generalized Dedekind sums, *Analytic number theory and related topics* (Tokyo, 1991), 17–27, World Sci. Publishing, River Edge, NJ, 1993.

[10] S. Egami, *Introduction to multiple zeta function*, Lecture Note at Niigata University (in Japanese), DVI and TeX files are available at http://mathalg.ge.niigata-u.ac.jp/Seminar/Intensive/Egami.html

[11] R.C. Gunning, *Introduction to holomorphic functions of several variables II*, Wadsworth & Brooks/Cole Mathematics Series.

[12] A. Ivić, *The Riemann Zeta-Function*, A Wiley-Interscience Publ. John Wiley & Sons 1985.

[13] M. Katsurada and K. Matsumoto, Asymptotic expansions of the mean values of Dirichlet L-functions. *Math. Z.*, **208** (1991), 23–39.

[14] T.Q.T. Le and J. Murakami, Kontsevich's integral for the Homfly polynomial and relations between values of multiple zeta functions, *Topology Appl.*, **62** (1995), 193–206.

[15] Y. Motohashi, A note on the mean value of the zeta and L-functions. I, *Proc. Japan Acad.*, Ser. A Math. Sci. **61** (1985), 222–224.

[16] Y. Ohno, A generalization of the duality and sum formulas on the multiple zeta values, *J. Number Theory*, **74** (1999), 39–43.

[17] E.C. Titchmarsh (revised by D.R. Heath-Brown), *The theory of the Riemann Zeta-function*, Clarendon Press, Oxford 1986.

[18] D. Zagier, Values of zeta-functions and their applications, *First European Congress of Mathematics*, Vol. II, Birkhäuser, (1994) 210–220

[19] D. Zagier, Periods of modular forms, traces of Hecke operators, and multiple zeta values, *Research into automorphic forms and L functions (in Japanese) (Kyoto, 1992), Sūrikaisekikenkyūsho Kōkyūroku*, **843** (1993), 162–170.

[20] J. Zhao, Analytic continuation of multiple zeta function, Proc. Amer. Math. Soc. **128** (2000), no. 5, 1275–1283.

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