Labelled and unlabelled enumeration of k-gonal 2-trees

Gilbert Labelle, Cédric Lamathe and Pierre Leroux

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Abstract

In this paper[1], we generalize 2-trees by replacing triangles by quadrilaterals, pentagons or k-sided polygons (k-gons), where $k \geq 3$ is given. This generalization, to k-gonal 2-trees, is natural and is closely related, in the planar case, to some specializations of the cell-growth problem. Our goal is the labelled and unlabelled enumeration, of k-gonal 2-trees according to the number n of k-gons. We give explicit formulas in the labelled case, and, in the unlabelled case, recursive and asymptotic formulas. We also enumerate these structures according to their perimeter.

1 Introduction

The class of *bidimensional trees*, or in brief *2-trees*, is extensively studied in the literature. For instance, see [7] and [5, 6] and their references; see also [10, 11]. Essentially, a 2-tree is a connected simple graph composed by triangles glued along their edges in a tree-like fashion, that is, without cycles (of triangles). In [8], Harary et al. enumerated a variant of the cell-growth problem, namely plane and planar (in the sense that all faces, except possibly the external face, are also k-sided polygons, also called outerplanar) 2-trees, in which triangles have been replaced by quadrilaterals, pentagons or k-sided polygons (k-gons), where $k \geq 3$ is fixed. Such 2-trees, built on k-gons, are called *k-gonal 2-trees*. This generalization is natural and the purpose of this work is the enumeration of free k-gonal 2-trees, *i.e.*, seen as simple graphs, without any condition of planarity. Figure 1, a) and b), and Figure 2 a) show examples of k-gonal 2-trees, for $k = 3, 5$ and 4, respectively.

Our goal is the labelled and unlabelled enumeraton of k-gonal 2-trees, according to the number of k-gons. We give explicit formulas in the labelled case and recursive and asymptotic formulas in the unlabelled case. This is the full version of a paper presented at the "Mathematics and computer science" conference in Versailles, France, in September 2002 (see [15]). More complete proofs are given, in particular for the asymptotic formulas, and a section has been added on the enumeration of k-gonal 2-trees according to their perimeter.

[1] With the support of FCAR (Québec) and NSERC (Canada).

It was recently brought to our attention that Ton Kloks [10, 11] had enumerated unlabelled *biconnected partial 2-trees* according to the number of vertices, in his 1993 thesis. These structures are more general than k-gonal 2-trees since various size of polygons can occur in the same graph and some polygons may have missing edges.

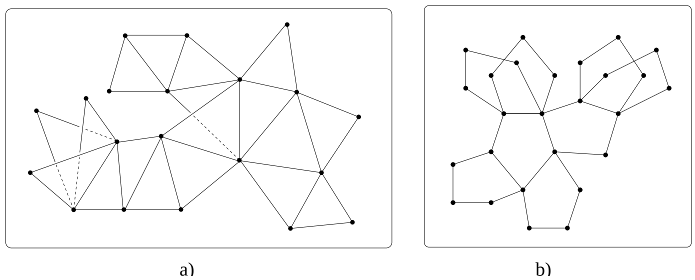


Figure 1: k-gonal 2-trees with $k = 3$ and $k = 5$

We say that a k-gonal 2-tree is *oriented* if its edges are oriented in such a way that each k-gon forms an oriented cycle; see Figure 2 b). In fact, for any k-gonal 2-tree s, the orientation of any one of its edges can be extended uniquely to all of s by first orienting all the polygons to which the edge belongs and then continuing recursively on all adjacent polygons. The coherence of the extension is ensured by the arborescent (acyclic) nature of 2-trees.

We denote by a and a_o the species of k-gonal 2-trees and of oriented k-gonal 2-trees. For these species, we use the symbols $-$, $\diamond$ and $\underline{\diamond}$ as upper indices to indicate that the structures are pointed at an edge, at a k-gon, and at a k-gon having itself a distinguished edge, respectively.

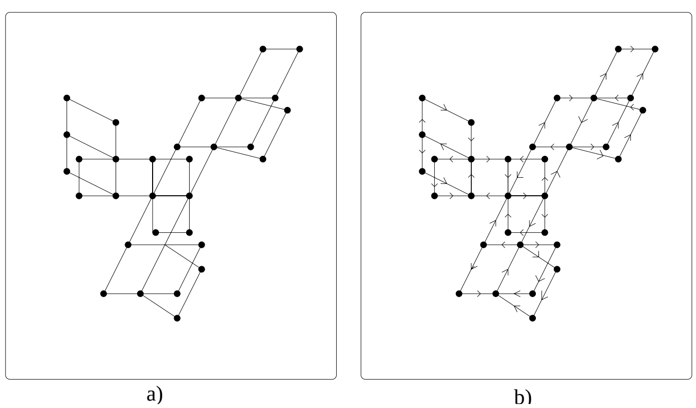


Figure 2: A unoriented and oriented 4-gonal 2-tree

Following the approach of Fowler et al. in [5, 6], which corresponds to the case $k = 3$, we label the 2-trees at their k-gons and give functional equations which relate these various pointed species together and eventually lead to their enumeration. The main difficulty of this extension from triangles to k-gons comes, as we will see later, from the case where k is an even integer.

The first step is an extension of the dissymmetry theorem for 2-trees to the k-gonal case. The proof is similar to the case $k = 3$ and is omitted (see [5, 6]).

Theorem 1. DISSYMMETRY THEOREM FOR k-GONAL 2-TREES. The species $\mathcal{a}_o$ and $\mathcal{a}$ of oriented and unoriented k-gonal 2-trees, respectively, satisfy the following isomorphisms of species:

$$\mathcal{a}_o^- + \mathcal{a}_o^\diamond = \mathcal{a}_o + \mathcal{a}_o^{\overset{\diamond}{-}}, \qquad (1)$$

$$\mathcal{a}^- + \mathcal{a}^\diamond = \mathcal{a} + \mathcal{a}^{\overset{\diamond}{-}}. \qquad (2)$$

There is yet another species to introduce, which plays an essential role in the process. It is the species $B = \mathcal{a}^{\rightarrow}$ of oriented-edge rooted (k-gonal) 2-trees, that is of 2-trees where an edge is selected and oriented. As mentionned above, the orientation of the rooted edge can be extended uniquely to an orientation of the 2-tree so that there is a canonical isomorphism $B = \mathcal{a}_o^-$ which can be used for all enumerative purposes. However, it is often useful not to perform this extension and to consider that only the rooted edge is oriented, as we will see.

In the next section, we characterize the species $B = \mathcal{a}^{\rightarrow}$ by a combinatorial functional equation and state some of its properties. The goal is to express the various pointed species occuring in the dissymmetry theorem in terms of B and to deduce enumerative results for the species $\mathcal{a}_o$ and $\mathcal{a}$. The oriented case is simpler, and carried out first, in Section 3. The unoriented case is analyzed in Section 4, distinguishing two cases according to the parity of the integer k. Enumeration of k-gonal 2-trees according to the perimeter is carried out in Section 5. Finally, asymptotic results are presented in Section 6.

This paper uses the framework of species theory. See Chapter 1 of [3] for an introduction. The main tool for our purposes is the composition theorem which can be stated as follows: let the species F be the (partitionnal) composition of two species, $F = G \circ H$. Then, the exponential generating function

$$F(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!},$$

where $f_n = |F[n]|$ is the number of labelled F-structures of order n, and the tilde generating function

$$\widetilde{F}(x) = \sum_{n \geq 0} \widetilde{f}_n x^n,$$

where $\widetilde{f}_n = |F[n]/\mathbb{S}_n|$ is the number of unlabelled F-structures of order n, satisfy the following equations:

$$F(x) = G(H((x)), \qquad (3)$$

$$\widetilde{F}(x) = Z_G(\widetilde{H}(x), \widetilde{H}(x^2), \ldots), \qquad (4)$$

where $Z_G(x_1, x_2, \ldots)$ is the cycle index series of G.

2 The species B of oriented-edge rooted 2-trees

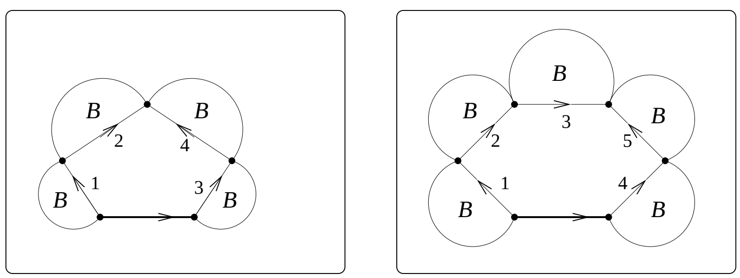
The species $B = \mathfrak{a}^{\rightarrow}$ plays a central role in the study of k-gonal 2-trees. The following theorem is an extension to a general k of the case $k = 3$. Note that formula (5) also makes sense for $k = 2$ and corresponds to edge-labelled (ordinary) rooted trees.

Theorem 2. The species $B = \mathfrak{a}^{\rightarrow}$ of oriented-edge rooted k-gonal 2-trees satisfies the following functional equation (isomorphism):

$$B = E(XB^{k-1}), \tag{5}$$

where E represents the species of sets and X is the species of singleton k-gons.

Proof. We decompose an $\mathfrak{a}^{\rightarrow}$-structure as a set of *pages*, that is, of maximal subgraphs sharing only one k-gon with the rooted edge. For each page, the orientation of the rooted edge permits to define a linear order and an orientation on the $k - 1$ remaining edges of the polygon having this edge, in some conventional way, for example in the fashion illustrated in Figure 3 a), for the odd case, and b), for the even case. These edges being oriented, we can glue on them some B-structures. We then deduce relation (5). ■



a) b)

Figure 3: An oriented page for a) $k = 5$, b) $k = 6$

We can easily relate the species $B = \mathfrak{a}^{\rightarrow}$ to the species of rooted trees denoted by A, characterized by the functional equation $A = XE(A)$, where X is now the species of singleton vertices. Indeed from (5), we deduce successively

$$(k-1)XB^{k-1} = (k-1)XE((k-1)XB^{k-1}), \tag{6}$$

knowing that $E^m(X) = E(mX)$, and, by unicity,

$$(k-1)XB^{k-1} = A((k-1)X). \tag{7}$$

Finally, we obtain the following expression for the species B in terms of the species of rooted trees.

Proposition 1. The species $B = a^{\rightarrow}$ of oriented-edge-rooted k-gonal 2-trees satisfies

$$B = \sqrt[k-1]{\frac{A((k-1)X)}{(k-1)X}}. \tag{8}$$

Corollary 1. The numbers $a_n^{\rightarrow}$, $a_{n_1,n_2,\ldots}^{\rightarrow}$, and $b_n = \widetilde{a}_n^{\rightarrow}$ of k-gonal 2-trees pointed at an oriented edge and having n k-gons, respectively labelled, fixed by a permutation of cycle type $1^{n_1}2^{n_2}\ldots$ and unlabelled, satisfy the following formulas and recurrence:

$$a_n^{\rightarrow} = ((k-1)n+1)^{n-1} = m^{n-1}, \tag{9}$$

where $m = (k-1)n+1$ is the number of edges,

$$a_{n_1,n_2,\ldots}^{\rightarrow} = \prod_{i=1}^{\infty}(1+(k-1)\sum_{d|i} dn_d)^{n_i-1}(1+(k-1)\sum_{d|i,d<i} dn_d), \tag{10}$$

and

$$b_n = \frac{1}{n}\sum_{1\leq j\leq n}\sum_{\alpha}(|\alpha|+1)b_{\alpha_1}b_{\alpha_2}\ldots b_{\alpha_{k-1}}b_{n-j}, \qquad b_0 = 1, \tag{11}$$

the last sum is running over $(k-1)$-tuples of integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{k-1})$ such that $|\alpha|+1$ divides the integer j, where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}$.

Proof. Formulas (9) and (10) are obtained by specializing with $\mu = (k-1)^{-1}$ the following formulas, given by Fowler et al. in [5, 6],

$$\left(\frac{A(x)}{x}\right)^{\mu} = \sum_{n\geq 0}\mu(\mu+n)^{n-1}\frac{x^n}{n!}, \tag{12}$$

$Z_{\left(\frac{A(X/\mu)}{X/\mu}\right)^{\mu}} =$

$$\sum_{n_1,n_2,\ldots}\frac{x_1^{n_1}x_2^{n_2}\ldots}{1^{n_1}n_1!2^{n_2}n_2!\ldots}\prod_{i=1}^{\infty}(1+\frac{1}{\mu}\sum_{d|i} dn_d)^{n_i-1}(1+\frac{1}{\mu}\sum_{d|i,d<i} dn_d). \tag{13}$$

Formula (9) can also be established by a Prüfer-like bijection. To obtain the recurrence (11), it suffices to take the logarithmic derivative of the equation

$$\widetilde{B}(x) = \exp\left(\sum_{i\geq 1}\frac{x^i\widetilde{B}^{k-1}(x^i)}{i}\right), \tag{14}$$

where $\widetilde{B}(x) = \sum_{n\geq 0} b_n x^n$, which follows from relation (5), using (4). ∎

It is interesting to note that the sequences $\{b_n\}_{n\in\mathbb{N}}$, for $k = 2,3,4,5$, are listed in the encyclopedia of integer sequences [18] and the equation (5), in the encyclopedia of combinatorial structures [9]. Also remark that, for each $n \geq 1$, b_n is a polynomial in k of degree $n-1$. This follows from (10) and the following explicit formula for b_n,

$$b_n = \sum_{n_1+2n_2+\cdots=n} \frac{a^{\rightarrow}_{n_1,n_2,\ldots}}{1^{n_1}n_1!2^{n_2}n_2!\ldots}, \tag{15}$$

which is a consequence of Burnside's lemma. The asymptotic behavior of the numbers b_n as $n \to \infty$, is studied, in particular as a function of k, in Section 7.

Remark 1. Equation (8) can also be used to compute the molecular expansion of the species B from the molecular expansion of A, using the binomial theorem. See [1] for more details.

3 Oriented case

We begin by determining relations for the pointed species appearing in the dissymmetry theorem. These relations are quite direct and the proof is left to the reader.

Proposition 2. The species $\mathfrak{a}_o^{-}$, $\mathfrak{a}_o^{\diamond}$, and $\mathfrak{a}_o^{\overset{\diamond}{-}}$ are characterized by the following isomorphisms:

$$\mathfrak{a}_o^{-} = B, \quad \mathfrak{a}_o^{\diamond} = XC_k(B), \quad \mathfrak{a}_o^{\overset{\diamond}{-}} = XB^k, \tag{16}$$

where $B = \mathfrak{a}^{\rightarrow}$ and C_k represents the species of oriented cycles of length k.

The dissymmetry theorem permits us to express the ordinary generating series $\widetilde{\mathfrak{a}}_o(x)$ of unlabelled oriented k-gonal 2-trees in terms of the corresponding series for the rooted species:

$$\widetilde{\mathfrak{a}}_o(x) = \widetilde{\mathfrak{a}}_o^{-}(x) + \widetilde{\mathfrak{a}}_o^{\diamond}(x) - \widetilde{\mathfrak{a}}_o^{\overset{\diamond}{-}}(x). \tag{17}$$

By Proposition 2, we can then express $\widetilde{\mathfrak{a}}_o(x)$ as function of $\widetilde{B}(x) = \widetilde{\mathfrak{a}}^{\rightarrow}(x)$.

Proposition 3. The ordinary generating series $\widetilde{\mathfrak{a}}_o(x)$ of unlabelled oriented k-gonal 2-trees is given by

$$\widetilde{\mathfrak{a}}_o(x) = \widetilde{B}(x) + \frac{x}{k}\sum_{\substack{d\mid k\\ d>1}} \phi(d)\widetilde{B}^{\frac{k}{d}}(x^d) - \frac{k-1}{k}x\widetilde{B}^k(x). \tag{18}$$

Corollary 2. The numbers $a_{o,n}$ and $\widetilde{a}_{o,n}$ of oriented k-gonal 2-trees labelled and unlabelled, over n k-gons, respectively, are given by

$$\begin{aligned} a_{o,n} &= ((k-1)n+1)^{n-2} = m^{n-2}, \quad n \geq 2, & (19)\\ \widetilde{a}_{o,n} &= b_n - \frac{k-1}{k}b^{(k)}_{n-1} + \frac{1}{k}\sum_{\substack{d\mid k\\ d>1}} \phi(d)b^{(\frac{k}{d})}_{\frac{n-1}{d}}, & (20)\end{aligned}$$

where

$$b_i^{(j)} = [x^i]\widetilde{B}^j(x) = \sum_{i_1+\cdots+i_j=i} b_{i_1}b_{i_2}\dots b_{i_j},$$

denotes the coefficient of x^i in the series $\widetilde{B}^j(x)$, with $b_r^{(j)} = 0$ if r is non-integral or negative.

Proof. For the labelled case, it suffices to remark that $a_n^{\rightarrow} = m a_{o,n}$. In the unlabelled case, equation (20) is directly obtained from (18). ■

4 Unoriented case

In the unoriented case, the number a_n of k-gonal 2-trees labelled over n polygons satisfies $2a_n = a_{o,n}+1$, since the only k-gonal 2-tree left fixed by a reversal of the orientation, for a given number of polygons, is the one in which every polygon share one common edge. We get

Proposition 4. The number a_n of labelled k-gonal 2-trees on n k-gons is given by

$$a_n = \frac{1}{2}\left(m^{n-2}+1\right), \qquad n \geq 2, \tag{21}$$

where $m = (k-1)n+1$.

For the unlabelled enumeration of k-gonal 2-trees (unoriented), we have to consider quotient species of the form $F/\mathbb{Z}_2$, where F is any species of "oriented" structures and $\mathbb{Z}_2 = \{1, \tau\}$, is the group where the action of τ is to reverse the orientation of the structure. A structure of such a species then consists in an orbit $\{s, \tau \cdot s\}$ of F-structures under the action of $\mathbb{Z}_2$.

For instance, the different pointed species of unoriented k-gonal 2-trees a^-, $a^\diamond$ and $a^{\underline{\diamond}}$, can be expressed as quotient species of the corresponding species of oriented k-gonal 2-trees:

$$a^- = \frac{a^{\rightarrow}}{\mathbb{Z}_2}, \quad a^\diamond = \frac{a_o^\diamond}{\mathbb{Z}_2} = \frac{XC_k(B)}{\mathbb{Z}_2}, \quad a^{\underline{\diamond}} = \frac{a_o^{\underline{\diamond}}}{\mathbb{Z}_2} = \frac{XB^k}{\mathbb{Z}_2}. \tag{22}$$

For the ordinary generating series (unlabelled structures) associated to such quotient species, we use the following formula, which is quite obvious,

$$(F/\mathbb{Z}_2)^{\sim}(x) = \frac{1}{2}(\widetilde{F}(x) + \widetilde{F}_\tau(x)), \tag{23}$$

where $\widetilde{F}_\tau(x) = \sum_{n\geq 0} |\mathrm{Fix}_{\widetilde{F}_n}(\tau)| x^n$ is the ordinary generating series of unlabelled F-structures left fixed by the action of τ, that is, by orientation reversal. However, the computation of the series $\widetilde{F}_\tau(x)$ is quite complicated and it is better to treat separately two cases according to the parity of k.

4.1 Case k odd

We can notice, observing Figures 3 a) and b), that in every k-gon containing the pointed (but not oriented) edge of an $\mathcal{a}^-$-structure, it is possible to orient the $k-1$ other edges in a "going away (from the root edge) direction" as in Figure 3 a), when k is odd, but there remains an ambiguous edge if k is even. This phenomenon permits us to introduce *skeleton* species, when k is odd, in analogy with the approach of Fowler et al. in [5, 6], where $k = 3$. They are the two-sort quotient species $Q(X,Y)$, $S(X,Y)$ and $U(X,Y)$, where X represents the species of k-gons and Y the species of oriented edges, defined by Figures 4 a), b) and c), where $k = 5$. In analogy with the case $k = 3$, we get the following

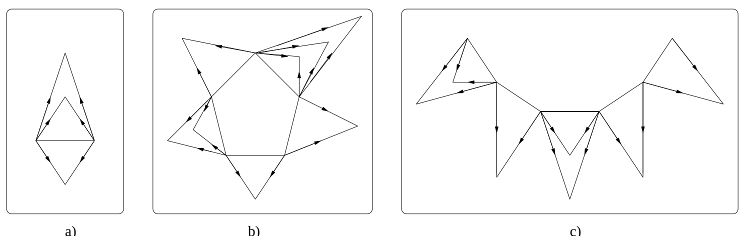


Figure 4: Skeleton species a) $Q(X,Y)$, b) $S(X,Y)$ and c) $U(X,Y)$

propositions.

Proposition 5. The skeleton species Q, S and U admit the following expressions in terms of quotients species

$$Q(X,Y) = E(XY^2)/\mathbb{Z}_2, \tag{24}$$
$$S(X,Y) = C_k(E(XY^2))/\mathbb{Z}_2, \tag{25}$$
$$U(X,Y) = (E(XY^2))^k/\mathbb{Z}_2. \tag{26}$$

Proposition 6. For k odd, $k \geq 3$, we have the following expressions for the pointed species of k-gonal 2-trees, where $B = \mathcal{a}^{\rightarrow}$:

$$\mathcal{a}^- = Q(X, B^{\frac{k-1}{2}}), \quad \mathcal{a}^{\diamond} = X \cdot S(X, B^{\frac{k-1}{2}}), \quad \mathcal{a}^{\hat{\diamond}} = X \cdot U(X, B^{\frac{k-1}{2}}). \tag{27}$$

In order to obtain enumeration formulas, we have first to compute the cycle index series of the species Q, S and U.

Proposition 7. The cycle index series of the species $Q(X,Y)$, $S(X,Y)$ and $U(X,Y)$ are given by

$$Z_Q = \frac{1}{2}\Big(Z_{E(XY^2)} + q\Big), \tag{28}$$
$$Z_S = \frac{1}{2}\Big(Z_{C_k(E(XY^2))} + q \cdot (p_2 \circ Z_{E(XY^2)})^{\frac{k-1}{2}}\Big), \tag{29}$$
$$Z_U = \frac{1}{2}\Big(Z_{(E(XY^2))^k} + q \cdot (p_2 \circ Z_{E(XY^2)})^{\frac{k-1}{2}}\Big), \tag{30}$$

where $q = h \circ (x_1 y_2 + p_2 \circ (x_1 \frac{y_1^2 - y_2}{2}))$, p_2 represents the power sum function of degree two, h the homogeneous symmetric function and $\circ$, the plethystic substitution.

Proof. Formula (28) and the method used can be found in [5, 6]. It is a matter of counting colored unlabelled $F(X,Y)$-structures left fixed by τ. In the case of S, we have to leave fixed a colored $C_k(E(XY^2))$-structure. For this, the basis cycle of length k must possess (at least) one symmetry axis passing through the middle of one of its sides. We can see that when such a structure has several axis of symmetry, the choice of the axis is arbitrary. On both sides of the axis, each colored $E(XY^2)$-structure must have its mirror image; this contributes for a term of $(p_2 \circ Z_{E(XY^2)})^{\frac{k-1}{2}}$. Next, the attached structure on the distinguished edge must be globally left fixed; this gives the factor q. The reasoning is very similar for the species U ■

Combining the dissymmetry theorem, equations (28), (29), (30) and the substitution rules of unlabelled enumeration, we obtain the ordinary generating series of the species of k-gonal 2-trees.

Proposition 8. Let $k \geq 3$ be an odd integer. The ordinary generating series $\widetilde{a}(x)$ of unlabelled k-gonal 2-trees is given by

$$\widetilde{a}(x) = \frac{1}{2}\Big(\widetilde{a}_o(x) + \exp\big(\sum_{i\geq 1} \frac{1}{2i}(2x^i \widetilde{B}^{\frac{k-1}{2}}(x^{2i}) + x^{2i}\widetilde{B}^{k-1}(x^{2i}) - x^{2i}\widetilde{B}^{\frac{k-1}{2}}(x^{4i}))\Big). \tag{31}$$

Corollary 3. For $k \geq 3$, odd, the number $\widetilde{a}_n$ of unlabelled k-gonal 2-trees over n k-gons, satisfy the following recurrence

$$\widetilde{a}_n = \frac{1}{2n}\sum_{j=1}^{n}\Big(\sum_{l|j} l\omega_l\Big)\Big(\widetilde{a}_{n-j} - \frac{1}{2}\widetilde{a}_{o,n-j}\Big) + \frac{1}{2}\widetilde{a}_{o,n}, \quad \widetilde{a}_0 = 1, \tag{32}$$

where, for all $n \geq 1$,

$$\omega_n = 2b_{\frac{n-1}{2}}^{(\frac{k-1}{2})} + b_{\frac{n-2}{2}}^{(k-1)} - b_{\frac{n-2}{4}}^{(\frac{k-1}{2})}, \tag{33}$$

and $b_i^{(j)}$ is defined in Corollary 2.

4.2 Case k even

The case k even is much more delicate. In order to express the ordinary generating functions of the three species a^-, $a^\diamond$ and $a^{\underline{\diamond}}$, we apply relation (23) to formulas (22). For the species a^-, we have

$$\widetilde{a}^-(x) = \frac{1}{2}(\widetilde{a}^{\rightarrow}(x) + \widetilde{a}_\tau^{\rightarrow}(x)), \tag{34}$$

where $\widetilde{\mathfrak{a}}_\tau^{\to}(x) = \sum_{n\geq 0} |\mathrm{Fix}_{\widetilde{\mathfrak{a}}_n^{\to}}(\tau)| x^n$ is the ordinary generating series of unlabelled oriented-edge-rooted 2-trees which are left fixed by reversing the orientation. Let $\mathfrak{a}_{\mathrm{S}}$ denotes the subspecies of $\mathfrak{a}^{\to}$ consisting of $\mathfrak{a}^{\to}$-structures s which are isomorphic to their image $\tau \cdot s$ under the orientation reversing map. We have to compute $\widetilde{\mathfrak{a}}_{\mathrm{S}}(x) = \widetilde{\mathfrak{a}}_\tau^{\to}(x)$. For this, let us introduce some auxiliary species. The first one, denoted $\mathfrak{a}_{\mathrm{TS}}$, is the class of $\mathfrak{a}_{\mathrm{S}}$-structures for which every page attached to the rooted edge is vertically symmetric without crossed symmetries (see below); we say *totally symmetric*. We can characterize this species by the

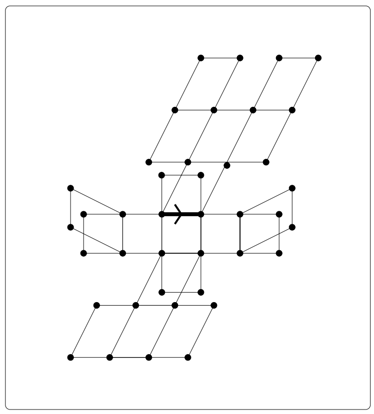


Figure 5: A structure of the species $\mathfrak{a}_{\mathrm{TS}}$

following functional equation (see Figure 5)

$$\mathfrak{a}_{\mathrm{TS}} = E(X \cdot X_{=}^{2} < B^{\frac{k-2}{2}} > \cdot \mathfrak{a}_{\mathrm{TS}}) = E(P_{\mathrm{TS}}), \tag{35}$$

where $X_{=}^{2} < F >$ represents the species of ordered pairs of isomorphic F-structures and P_{TS} is the species of *totally symmetric pages*. Translating this equation in terms of generating series, we get

$$\widetilde{\mathfrak{a}}_{\mathrm{TS}}(x) = \exp\left(\sum_{i\geq 1} \frac{1}{i} x^i \widetilde{B}^{\frac{k-2}{2}}(x^{2i}) \widetilde{\mathfrak{a}}_{\mathrm{TS}}(x^i)\right). \tag{36}$$

Proposition 9. The numbers $\beta_n = |\widetilde{\mathfrak{a}}_{\mathrm{TS}}[n]|$ of unlabelled $\mathfrak{a}_{\mathrm{TS}}$-structures on n polygons satisfy the recurrence

$$\beta_n = \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{d|i} d\pi_d\right)\beta_{n-i}, \;\; n\geq 1 \quad \beta_0 = 1, \tag{37}$$

where

$$\pi_n = \widetilde{P}_{\mathrm{TS},n} = \sum_{\substack{i+j=n-1 \\ i\ even}} b_{\frac{i}{2}}^{(\frac{k-2}{2})} \beta_j. \tag{38}$$

Proof. It suffices to take the logarithmic derivative of (36), that is

$$x\frac{(\widetilde{a}_{\mathrm{TS}})'(x)}{\widetilde{a}_{\mathrm{TS}}(x)} = x \cdot \sum_{i\geq 1} \Omega'(x^i)x^{i-1}, \tag{39}$$

where $\Omega(x) = \sum_{n\geq 1} \omega_n x^n = x\widetilde{B}^{\frac{k-2}{2}}(x^2)\widetilde{a}_{\mathrm{TS}}(x)$. Next, extracting the coefficient of x^n in both sides of

$$x(\widetilde{a}_{\mathrm{TS}})'(x) = \big(\sum_{i\geq 1} \Omega'(x^i)x^i\big)\,\widetilde{a}_{\mathrm{TS}}(x) \tag{40}$$

leads to (37) using (35) since $\Omega(x) = \widetilde{P}_{\mathrm{TS}}(x)$. ∎

Let us now introduce two other species, namely P_{AL} and P_{M}, of *pairs of alternated pages* and of *mixed pages*. A pair of *alternated pages* is, by definition, an unordered pair of oriented pages ($a^{\rightarrow}$-structures having only one page) of the form $\{s, \tau \cdot s\}$ with s and $\tau \cdot s$ non-isomorphic. Figure 6 a) shows a structure belonging to this species. A *mixed page* is a symmetric page having at least one alternated symmetry. Such a structure is drawn in Figure 6 b). We can then express each of these two species in terms of the other, as follows:

$$P_{\mathrm{AL}} = \Phi_2 < XB^{k-1} - (P_{\mathrm{TS}} + P_{\mathrm{M}}) >, \tag{41}$$
$$P_{\mathrm{M}} = X \cdot X^2_{=} < B^{\frac{k-2}{2}} > \cdot\, (a_{\mathrm{S}} - a_{\mathrm{TS}}), \tag{42}$$

where $\Phi_2 < F >$ represents the species of pairs of F-structures of the form $\{s, \tau \cdot s\}$ and E_+ is the species of non empty sets. At the level of ordinary generating series, we get

$$\widetilde{P}_{\mathrm{AL}}(x) = \frac{1}{2}\big(x^2\widetilde{B}^{k-1}(x^2) - \widetilde{P}_{\mathrm{TS}}(x^2) - \widetilde{P}_{\mathrm{M}}(x^2)\big), \tag{43}$$
$$\widetilde{P}_{\mathrm{M}}(x) = \Big(XX^2_{=} < B^{\frac{k-2}{2}} > \cdot\, a_{\mathrm{TS}} \cdot E_+(P_{\mathrm{AL}} + P_{\mathrm{M}})\Big)^{\sim}(x) \tag{44}$$
$$= x\widetilde{B}^{\frac{k-2}{2}}(x^2)\widetilde{a}_{\mathrm{TS}}(x)\left(\exp\left(\sum_{i\geq 1}\frac{1}{i}(\widetilde{P}_{\mathrm{AL}}(x^i) + \widetilde{P}_{\mathrm{M}}(x^i))\right) - 1\right) \tag{45}$$
$$= x\widetilde{B}^{\frac{k-2}{2}}(x^2)(\widetilde{a}_{\mathrm{S}}(x) - \widetilde{a}_{\mathrm{TS}}(x)) \tag{46}$$

Let $\widetilde{a}_{\mathrm{S}}(x)$ denote the ordinary generating series of unlabelled symmetric $a^{\rightarrow}$-structures. We have (see Figure 7)

$$\widetilde{a}_{\mathrm{S}}(x) = E(P_{\mathrm{TS}} + P_{\mathrm{AL}} + P_{\mathrm{M}})^{\sim}(x), \tag{47}$$
$$= \exp\left(\sum_{i\geq 1}\frac{1}{i}(\widetilde{P}_{\mathrm{TS}}(x^i) + \widetilde{P}_{\mathrm{AL}}(x^i) + \widetilde{P}_{\mathrm{M}}(x^i))\right). \tag{48}$$

We then deduce a recurrence for the numbers $\alpha_n = \widetilde{a}_{\mathrm{S},n}$ of symmetric k-gonal 2-trees rooted at an edge left fixed by orientation reversing, $\widetilde{P}_{\mathrm{AL},n}$ and

$\widetilde{P}_{\mathrm{M},n}$ of alternated and mixed pages, respectively, on n k-gons:

$$\alpha_n = \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{d|i} d\omega_d\right)\alpha_{n-i}, \quad \alpha_0 = 1, \tag{49}$$

$$\widetilde{P}_{\mathrm{M},n} = \sum_{i=0}^{n-1} b_{\frac{i}{2}}^{(\frac{k-2}{2})}\alpha_{n-1-i} - \widetilde{P}_{\mathrm{TS},n}, \tag{50}$$

$$\widetilde{P}_{\mathrm{AL},n} = \frac{1}{2}\left(b_{\frac{n-2}{2}}^{(k-1)} - \widetilde{P}_{\mathrm{TS},n/2} - \widetilde{P}_{\mathrm{M},n/2}\right), \tag{51}$$

where

$$\omega_k = \widetilde{P}_{\mathrm{TS},k} + \widetilde{P}_{\mathrm{AL},k} + \widetilde{P}_{\mathrm{M},k},$$

and $\widetilde{P}_{\mathrm{TS},n} = \pi_n$ is given by (38).

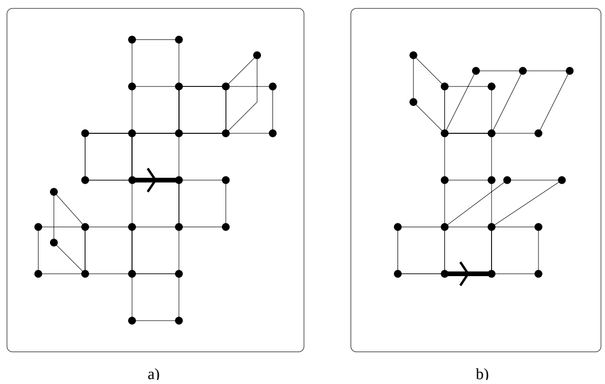


Figure 6: A pair of alternated pages and a mixed page

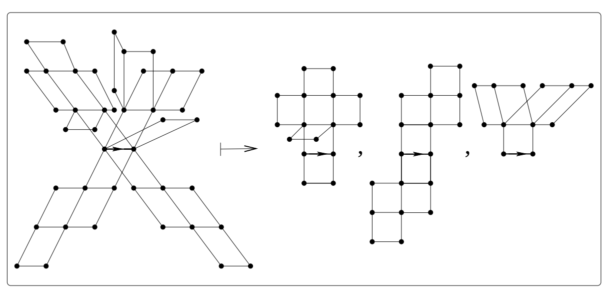


Figure 7: Decomposition of an $\widetilde{a}^{\rightarrow}$-structure fixed under τ

Proposition 10. If k is an even integer, then the number of edge rooted (unoriented) k-gonal 2-trees over n k-gons is given by

$$\widetilde{a}_n^{-} = \frac{1}{2}(b_n + \alpha_n). \tag{52}$$

Let us now turn to the species $\mathcal{A}^{\underline{\diamond}}$ of k-gonal 2-trees rooted at an edge-pointed k-gon.

Proposition 11. We have

$$\widetilde{\mathcal{A}}^{\underline{\diamond}}(x) = \frac{1}{2}\left(\widetilde{\mathcal{A}}_{o}^{\underline{\diamond}}(x) + \widetilde{\mathcal{A}}_{o,\tau}^{\underline{\diamond}}(x)\right), \tag{53}$$

where

$$\widetilde{\mathcal{A}}_{o,\tau}^{\underline{\diamond}}(x) = x\,\widetilde{\mathcal{A}}_{\mathrm{S}}^{2}(x)\widetilde{B}^{\frac{k-2}{2}}(x^2).$$

Proof. An unlabelled τ-symmetric $\mathcal{A}_{o}^{\underline{\diamond}}$-structure possesses an axis of symmetry which is, in fact, the mediatrix of the distinguished edge of the rooted polygon, and also the mediatrix of the edge facing the rooted one, see Figure 8. The two structures s and t glued on these two edges are thus symmetric, which leads to the term $(\widetilde{\mathcal{A}}_{\mathrm{S}}(x))^2$. Then, on each side of the axis, are found two $B^{\frac{k-2}{2}}$-structures α and β, which by symmetry satisfy $\beta = \tau \cdot \alpha$, contributing to the factor $\widetilde{B}^{\frac{k-2}{2}}(x^2)$. ■

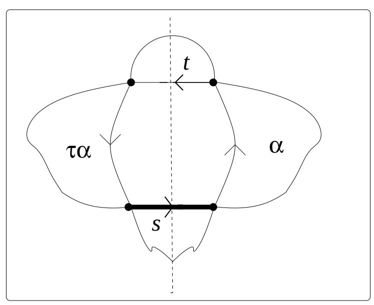


Figure 8: A τ-symmetric unlabelled $\mathcal{A}_{o}^{\underline{\diamond}}$-structures

Corollary 4. We have the following expression for the number $\widetilde{a}_{n}^{\underline{\diamond}}$ of unlabelled $\mathcal{A}^{\underline{\diamond}}$-structures,

$$\widetilde{a}_{n}^{\underline{\diamond}} = \frac{1}{2}\left(\widetilde{a}_{o,n}^{\underline{\diamond}} + \sum_{i+j=n-1} \alpha_{i}^{(2)} \cdot b_{\frac{j}{2}}^{(\frac{k-2}{2})}\right), \tag{54}$$

where $\alpha_{i}^{(2)} = [x^i]\widetilde{\mathcal{A}}_{S}^{2}(x)$. □

We proceed in a similar way for the species $\mathcal{A}^{\diamond}$, of k-gon rooted k-gonal 2-trees. Once again, we use relation (23), giving

$$\widetilde{\mathcal{A}}^{\diamond}(x) = \frac{1}{2}\left(\widetilde{\mathcal{A}}_{o}^{\diamond}(x) + \widetilde{\mathcal{A}}_{o,\tau}^{\diamond}(x)\right). \tag{55}$$

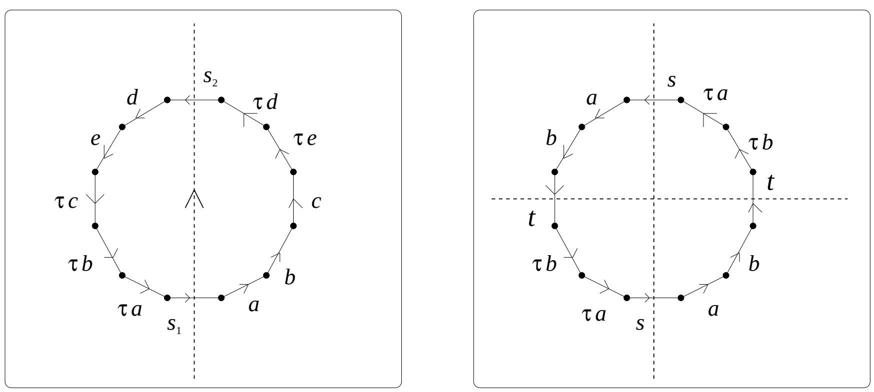
Proposition 12. Let $\widetilde{a}^{\diamond}_{o,\tau}(x)$ be the generating series of unlabelled $a^{\diamond}_{o}$-structures left fixed by orientation reversing. Then, we have

$$\widetilde{a}^{\diamond}_{o,\tau}(x) = \frac{x}{2}\widetilde{a}^{2}_{\mathrm{S}}(x)\widetilde{B}^{\frac{k-2}{2}}(x^2) + \frac{x}{2}\widetilde{B}^{\frac{k}{2}}(x^2). \tag{56}$$

Proof. Notice first that to be left fixed by orientation reversing, an $a^{\diamond}_{o}$-structure must admit at least one axis of symmetry, which can be of two kinds:

1. an axis passing through the middle of two opposite edges, or
2. an axis passing through two opposite vertices,

of the pointed polygon. The enumeration is carried out by first orienting the axis of symmetry. The first term of (56) then corresponds to a symmetry of the first kind, and the second term to a symmetry of the second kind. The structures having both symmetries are precisely those which are counted one half time in both of these terms. This is established for a general k by considering the largest power of 2, 2^m, such that $k/2^m$ is odd. We illustrate the proof in the following lines with $k = 12$; the reader will easily convince himself of the validity of this argument for any k.



a) b)

Figure 9: $\widetilde{a}^{\diamond}_{o,\tau}$-structures with an edge–edge symmetry

For $k = 12$, a general unlabelled τ-symmetric polygon-rooted oriented k-gonal 2-tree with an oriented edge–edge axis will be of the form illustrated in Figure 9 a), where s_1 and s_2 represent unlabelled a_{S}-structures, a, b, c, d and e are general unlabelled B-structures and τx represents the opposite of the B-structures x, obtained by reversing their orientation. Most of these structures are enumerated exactly by $\frac{1}{2}x\widetilde{a}^{2}_{\mathrm{S}}(x)\widetilde{B}^{5}(x^2)$. Indeed, the factor $x\widetilde{a}^{2}_{\mathrm{S}}(x)\widetilde{B}^{5}(x^2)$ is obtained in the same way as for $a^{\diamond}_{o,\tau}$-structures and the division by two is justified in the following cases:

1. $s_1 \neq s_2$ (two orientations of the axis),

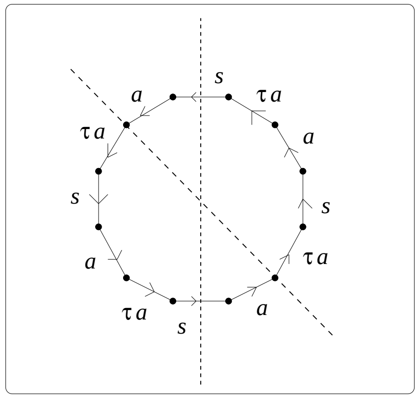
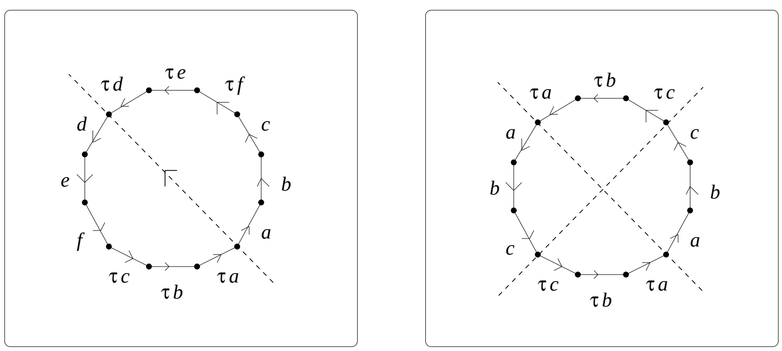


Figure 10: $\widetilde{a}^{\diamond}_{o,\tau}$-structures with edge–edge and vertex–vertex symmetries

2. $s_1 = s_2 = s$, $(a, b, c) \neq (d, e, \tau \cdot c)$ (two orientations),

3. $s_1 = s_2 = s$, $(a, b, c) = (d, e, \tau \cdot c)$, so that $c = \tau \cdot c = t \in \widetilde{a}_{\mathrm{S}}$, and either $i)$ $s \neq t$ or $ii)$ $s = t$ and $(a, b) \neq (\tau \cdot b, \tau \cdot a)$ (two choices for the symmetry axis, see Figure 9 b)),



a) b)

Figure 11: $\widetilde{a}^{\diamond}_{o,\tau}$-structure with a vertex–vertex symmetry axis

However, the structures with $s = t$ and $b = \tau \cdot a$ (see Figure 10) will occur only once and are counted only one half time in the formula. But, notice that these structures also admit a vertex–vertex symmetry axis and, as it will turn out, are also counted one half time in the second term of (56).

Similarly, an unlabelled $a^{\diamond}_{o,\tau}$-structure with an oriented vertex–vertex symmetry axis will be of the form illustrated in Figure 11 a), where a, b, $\ldots$, f are arbitrary unlabelled B-structures. Most of these terms are enumerated exactly by $\frac{1}{2}x\widetilde{B}^6(x^2)$, the division by two being justified in the following cases:

1. $(a,b,c) \neq (d,e,f)$ (two orientations of the symmetry axis),
2. $(a,b,c) = (d,e,f)$ and $(a,b,c) \neq (\tau \cdot c, \tau \cdot b, \tau \cdot a)$ (two choices for the symmetry axis, see Figure 11 b)),

However, the structures with $(a,b,c) = (d,e,f)$, $c = \tau \cdot a$ and $b = \tau \cdot b = s \in \widetilde{\mathcal{a}}_S$ appear only once and are counted one half time here. But they also have an edge-edge symmetry axis and were also counted one half time in the first term of (56) (exchange a and $\tau \cdot a$ in Figure 10). ■

The dissymmetry theorem yields, for $k \geq 4$ even,

$$\widetilde{\mathcal{a}}(x) = \frac{1}{2}\widetilde{\mathcal{a}}_o(x) + \frac{1}{2}\widetilde{\mathcal{a}}_S(x) + \frac{1}{2}\widetilde{\mathcal{a}}^{\diamond}_{o,\tau}(x) - \frac{1}{2}\widetilde{\mathcal{a}}^{\diamond}_{o,\tau}(x), \tag{57}$$

So, we have the following result.

Proposition 13. Let k be an even integer, $k \geq 4$. Then,the generating series $\widetilde{\mathcal{a}}(x)$ of unlabelled k-gonal 2-trees is given by

$$\widetilde{\mathcal{a}}(x) = \frac{1}{2}\widetilde{\mathcal{a}}_o(x) + \frac{1}{2}\widetilde{\mathcal{a}}_S(x) + \frac{x}{4}(\widetilde{B}^{\frac{k}{2}}(x^2) - \widetilde{\mathcal{a}}_S^2(x)\widetilde{B}^{\frac{k-2}{2}}(x^2)), \tag{58}$$

where $\widetilde{\mathcal{a}}_o(x)$ is given by (18) and $\widetilde{\mathcal{a}}_S(x)$ by (48). □

Corollary 5. If $k \geq 4$, is an even integer, then the number of unlabelled k-gonal 2-trees over n k-gons is given by

$$\widetilde{a}_n = \frac{1}{2}\widetilde{a}_{o,n} + \frac{1}{2}\alpha_n + \frac{1}{4}b^{(\frac{k}{2})}_{\frac{n-1}{2}} - \frac{1}{4}\sum_{i+j=n-1}\alpha_i^{(2)} \cdot b^{(\frac{k-2}{2})}_{\frac{j}{2}}, \tag{59}$$

with

$$b_l^{(m)} = [x^l]\widetilde{B}^m(x), \quad \alpha_i^{(2)} = [x^i]\widetilde{\mathcal{a}}_S^2(x).$$

5 Enumeration according to the perimeter

In this section, we are interested in the enumeration of k-gonal 2-trees according to the perimeter. The *perimeter* of a k-gonal 2-tree is the number of external edges (edges of degree at most one). In particular, if the structure s is the single edge, the perimeter is 1. In order to keep track of the perimeter, we introduce a weight function w over k-gonal 2-tree, defined by:

$$\begin{array}{rccc} w: & \mathcal{a} & \longrightarrow & \mathbb{Q}[t] \\ & s & \longmapsto & w(s) = t^{\mathrm{p}(s)}, \end{array} \tag{60}$$

where $\mathrm{p}(s)$ denotes the perimeter of the structure $s \in \mathcal{a}$. For example, the 2-tree of Figure 1 a) has perimeter 28.

5.1 A weighted version of the species B

Our first task is to determine the functional equation satisfied by the species B_w of k-gonal 2-trees pointed at an oriented edge and weighted by the perimeter counter t, with the precision that the rooted edge does not contribute to the perimeter of a B-structure except in the case of a single edge, which has perimeter 1. We have

Proposition 14. The weighted species B_w is characterized by the following functional equation

$$B_w(X) = t + E_+(XB_w^{k-1}(X)), \tag{61}$$

where E_+ is the species of non-empty sets.

Proof. The (unweighted) species B satisfies

$$B = E(XB^{k-1}) = 1 + E_+(XB^{k-1}(X)),$$

where the term 1 corresponds to the single edge. By taking into account the perimeter weight w and the fact that a single edge has weight t, we obtain (61). ∎

Note that (61) is also valid for $k = 2$. The species B_w then represents weighted edge-labelled (ordinary) rooted trees where the variables t acts as a leaf counter.

We write the generating series associated to the weighted species B_w as follows:

$$B_w(x) = B(x,t) = \sum_{\substack{n\geq 0\\ \ell\geq 1}} a_{n,\ell}^{\rightarrow} t^\ell \frac{x^n}{n!}, = \sum_{n\geq 0} a_n^{\rightarrow}(t)\frac{x^n}{n!} \tag{62}$$

$$\widetilde{B}_w(x) = \widetilde{B}(x,t) = \sum_{\substack{n\geq 0\\ \ell\geq 1}} b_{n,\ell}\, t^\ell x^n = \sum_{n\geq 0} b_n(t)x^n, \tag{63}$$

where $a_{n,\ell}^{\rightarrow}$ and $b_{n,\ell}$ are the numbers of labelled and unlabelled k-gonal 2-trees rooted at an oriented edge having n k-gons and perimeter ℓ. From equation (61), we can deduce explicit formulas for $a_n^{\rightarrow}(t)$ and $a_{n,\ell}^{\rightarrow}$ and recursive formulas for $b_n(t)$ and $b_{n,\ell}$. Notice that, because of the nature of the structures, the integer ℓ is bounded: $(k-2)n+1 \leq \ell \leq (k-1)n$.

Proposition 15. The polynomial $a_n^{\rightarrow}(t)$, giving the labelled weighted enumeration of B_w-structures over n k-gons is given by $a_0^{\rightarrow}(t) = t$ and, for $n \geq 1$,

$$a_n^{\rightarrow}(t) = \frac{n!}{m} \sum_{\ell=m-n}^{m-1} \sum_{i+j=m-\ell} (-1)^j i^n \binom{m}{\ell, i, j} t^\ell, \tag{64}$$

$$= \frac{1}{m}\sum_{i=1}^{n} \frac{m!}{(m-i)!} S(n,i) t^{m-i}, \tag{65}$$

where $m=(k-1)n+1$ is the number of edges and $S(n,j)$ denotes the Stirling numbers of the second kind, giving the number of partitions of an n-set in j blocks.

Proof. From (61), we have $B(x,t)=t+\exp(xB^{k-1}(x,t))-1$. So, we get

$$xB^{k-1}(x,t)=x(t+\exp(xB^{k-1}(x,t))-1)^{k-1}.$$

Putting $\mathcal{B}(x,t)=xB^{k-1}(x,t)$, we obtain that the series $\mathcal{B}(x,t)$ satisfies the functional equation $\mathcal{B}(x,t)=xR(\mathcal{B}(x,t))$, where $R(y)=(t+\exp(y)-1)^{k-1}$. Moreover,

$$B(x,t)=\left(\frac{\mathcal{B}(x,t)}{x}\right)^{\frac{1}{k-1}}. \tag{66}$$

The composite form of Lagrange inversion applied to equation (66) gives (64). To obtain now (65), we apply the same method but we use the following well-known relation

$$\frac{(\mathrm{e}^x-1)^j}{j!}=\sum_{n\geq j}S(n,j)\frac{x^n}{n!},$$

see [4] page 63. ■

We obtain now, in a straightforward way, expressions for $a^{\rightarrow}_{n,\ell}$. Formula (65) can also be given a Prüfer-type bijective proof.

Corollary 6. The number $a^{\rightarrow}_{n,\ell}$ of labelled B_w-structures over n k-gons and having perimeter ℓ, for $(k-2)n+1\leq\ell\leq(k-1)n$ (a weight t^ℓ), is given by

$$a^{\rightarrow}_{n,\ell} = \frac{n!}{m}\sum_{i+j=m-\ell}(-1)^j i^n\binom{m}{\ell,i,j}, \tag{67}$$

$$= \frac{(m-1)!}{\ell!}S(n,m-\ell), \tag{68}$$

where $m=(k-1)n+1$ is the number of edges. □

We notice that, when $k=3$, $\ell=n+1$ is the minimal perimeter and $a^{\rightarrow}_{n,n+1}=n!\mathbf{c_n}$, where $\mathbf{c_n}$ is the famous Catalan number, since, in this case, the B_w-structures obtained are outerplanar, see Labelle et al. [14]. These structures are the basic ones in the computation of the molecular expansion (a classification according to symmetries) of the species of outerplanar k-gonal 2-trees. For general k, $a^{\rightarrow}_{n,(k-2)n+1}=n!C_{k,n}$, where $C_{k,n}=\frac{1}{n}\binom{n(k-1)}{n-1}$ is the generalized Catalan numbers. See [16].

As in the unweighted case, we cannot obtain an explicit formula for the number $b_{n,\ell}$ as well as for the polynomial $b_n(t)$. However, we give recursive formulas.

Proposition 16. The polynomials $b_n(t)$, $n \geq 1$, satisfy the following recurrence

$$\begin{aligned} b_0(t) &= t, \\ b_n(t) &= \frac{1}{n}\left(\sum_{d|n} d \cdot b_{d-1}^{(k-1)}(t^{\frac{n}{d}}) + \sum_{i=1}^{n-1}\left(\sum_{d|i} d \cdot b_{d-1}^{(k-1)}(t^{\frac{i}{d}})\right) b_{n-i}(t)\right), \end{aligned} \tag{69}$$

where the summations are taken over integers $i, d \geq 1$, and where

$$b_n^{(k-1)}(t) = [x^n]\widetilde{B}^{k-1}(x,t) = \sum_{i_1+i_2+\cdots+i_{k-1}=n} b_{i_1}(t)b_{i_2}(t)\dots b_{i_{k-1}}(t). \tag{70}$$

Proof. We obtain recurrence (69) by taking the derivative (with respect to x) of the following expression

$$\widetilde{B}(x,t) = t + \exp\left(\sum_{i\geq 1} \frac{1}{i} x^i \widetilde{B}^{k-1}(x^i, t^i)\right) - 1,$$

obtained from (61) by passing to the ordinary generating series for unlabelled enumeration. ∎

We obtain the next proposition quite directly from the previous one.

Corollary 7. The number $b_{n,\ell}$ of unlabelled B_w-structures over n k-gons and having perimeter ℓ satisfies the following recurrence

$$b_{0,\ell} = \delta_{1,\ell}, \qquad b_{n,\ell} = \frac{1}{n}\omega_{n,\ell} + \frac{1}{n} \sum_{\substack{\nu+\mu=n \\ \nu,\mu\geq 1}} \sum_{\substack{p+q=\ell \\ p,q\geq 1}} \omega_{\nu,p} \cdot b_{\mu,q}, \tag{71}$$

where $\delta_{i,j}$ is the Kronecker symbol and

$$\omega_{n,\ell} = \sum_{d|(n,\ell)} \frac{n}{d} b^{(k-1)}_{\frac{n}{d}-1,\frac{\ell}{d}}. \tag{72}$$

□

As for the unweighted case, we can express the pointed weighted species of k-gonal 2-trees as function of the species B_w. We begin with the oriented case, which is simpler, and use it to obtain the unoriented case.

5.2 Oriented case

Let us denote by $\mathfrak{a}_w^- = (\mathfrak{a}_w)^-$, $\mathfrak{a}_w^\diamond = (\mathfrak{a}_w)^\diamond$, $\mathfrak{a}_w^{\underset{-}{\diamond}} = (\mathfrak{a}_w)^{\underset{-}{\diamond}}$, and $\mathfrak{a}_{o,w}^- = (\mathfrak{a}_{o,w})^-$, $\mathfrak{a}_{o,w}^\diamond = (\mathfrak{a}_{o,w})^\diamond$, $\mathfrak{a}_{o,w}^{\underset{-}{\diamond}} = (\mathfrak{a}_{o,w})^{\underset{-}{\diamond}}$, where w is defined by (60). Note in particular that $\mathfrak{a}_{o,w}^- \neq B_w$. The dissymetry theorem remains valid in this weighted context, for both the oriented and unoriented cases:

$$\mathfrak{a}_{o,w}^- + \mathfrak{a}_{o,w}^\diamond = \mathfrak{a}_{o,w} + \mathfrak{a}_{o,w}^{\underset{-}{\diamond}}, \tag{73}$$

$$\mathfrak{a}_w^- + \mathfrak{a}_w^\diamond = \mathfrak{a}_w + \mathfrak{a}_w^{\underset{-}{\diamond}}. \tag{74}$$

As in the unweighted case, we have to express these species in terms of the weighted species B_w. Enumeration formulas will then follow. The following proposition is quite obvious and the proof is omitted.

Proposition 17. The weighted species $\mathfrak{a}_{o,w}^{-}$, $\mathfrak{a}_{o,w}^{\diamond}$ and $\mathfrak{a}_{o,w}^{\dot\diamond}$ are characterized by

$$
\begin{aligned}
\mathfrak{a}_{o,w}^{-} &= B_w + (t-1)XB_w^{k-1}, && (75)\\
\mathfrak{a}_{o,w}^{\diamond} &= XC_k(B_w), && (76)\\
\mathfrak{a}_{o,w}^{\dot\diamond} &= XB_w^k. && (77)
\end{aligned}
$$

We then deduce easily the associated generating series of these species

$$
\mathfrak{a}_o^{-}(x,t) = B(x,t) + (t-1)xB^{k-1}(x,t) \qquad (78)
$$

and

$$
\begin{aligned}
\widetilde{\mathfrak{a}}_o^{-}(x,t) &= \widetilde{B}(x,t) + (t-1)x\widetilde{B}^{k-1}(x,t), && (79)\\
\widetilde{\mathfrak{a}}_o^{\diamond}(x,t) &= \frac{x}{k}\sum_{d|k}\phi(d)\widetilde{B}^{\frac{k}{d}}(x^d,t^d), && (80)\\
\widetilde{\mathfrak{a}}_o^{\dot\diamond}(x,t) &= x(\widetilde{B}^k(x,t) + (t-1)\widetilde{B}^{k-1}(x,t)), && (81)
\end{aligned}
$$

from which we deduce

$$
a_{o,n}^{-}(t) = n![x^n]\,\mathfrak{a}_o^{-}(x,t) = a_n^{\rightarrow}(t) + (t-1)na_{n-1}^{\rightarrow\,(k-1)}(t), \qquad (82)
$$

and, using the dissymmetry theorem,

$$
\widetilde{\mathfrak{a}}_o(x,t) = \widetilde{B}(x,t) + \frac{x}{k}\sum_{d|k}\phi(d)\widetilde{B}^{\frac{k}{d}}(x^d,t^d) - x\widetilde{B}^k(x,t) + (t-1)x\widetilde{B}^{k-1}(x,t). \qquad (83)
$$

We then get:

Proposition 18. We have, for $n \geq 2$,

$$
\begin{aligned}
a_{o,n}(t) &= \frac{a_{o,n}^{-}(t)}{m}, && (84)\\
\widetilde{a}_{o,n}(t) &= [x^n]\,\widetilde{\mathfrak{a}}_o(x,t) && (85)\\
&= b_n(t) - b_{n-1}^{(k)}(t) + \frac{1}{k}\sum_{\substack{d|k\\ d\geq 1}}\phi(d)b_{\frac{n-1}{d}}^{(\frac{k}{d})}(t^d) + (t-1)b_{n-1}^{(k-1)}(t), && (86)
\end{aligned}
$$

where $m = (k-1)n+1$ is the number of edges and $b_n^{(i)}(t)$ is defined by (70).

Corollary 8. The numbers $a_o(n,\ell)$ and $\widetilde{a}_o(n,\ell)$ of labelled and unlabelled oriented k-gonal 2-trees , over n k-gons and having perimeter ℓ are given by

$$
\begin{aligned}
a_o(n,\ell) &= \frac{1}{m}a_o^{-}(n,\ell) = \frac{1}{m}(a_{n,\ell}^{\rightarrow} + na_{n-1.\ell-1}^{\rightarrow\,(k-1)} - na_{n-1,\ell}^{\rightarrow\,(k-1)}), && (87)\\
\widetilde{a}_o(n,\ell) &= b_{n,\ell} - b_{n-1,\ell}^{(k)} + \frac{1}{k}\sum_{d|(k,\ell)}\phi(d)b_{\frac{n-1}{d},\frac{\ell}{d}}^{(\frac{k}{d})} + b_{n-1,\ell-1}^{(k-1)} - b_{n-1,\ell}^{(k-1)}. && (88)
\end{aligned}
$$

5.3 Unoriented case

As in the unweighted case, unoriented species of k-gonal 2-trees can be expressed as quotient species of the oriented ones, as follows, where notations are obvious,

$$a^{-}_{w} = \frac{a^{-}_{o,w}}{\mathbb{Z}_2}, \qquad a^{\diamond}_{w} = \frac{a^{\diamond}_{o,w}}{\mathbb{Z}_2} \qquad a^{\hat{\diamond}}_{w} = \frac{a^{\hat{\diamond}}_{o,w}}{\mathbb{Z}_2} \tag{89}$$

It is very easy to obtain the number $a_{n,\ell}$ of labelled k-gonal 2-trees over n k-gons and having a perimeter of lenght ℓ,

$$a(n,\ell) = \begin{cases} \frac{1}{2}(a_o(n,\ell+1), & \text{if } \ell = (k-1)n, \\ \frac{1}{2}a_o(n,\ell), & \text{otherwise.} \end{cases} \tag{90}$$

since the only labelled k-gonal 2-trees fixed by orientation reversal for a given perimeter and number of polygons, is the one in which each k-gon share a common edge, which has $(k-1)n$ external edges (illustrated by Figure 12). So, the polynomial $a_n(t)$, giving the weighted enumeration of labelled k-gonal 2-trees, is given by

$$a_n(t) = \sum a_{n,\ell}\, t^{\ell} \;=\; \frac{1}{2}(a_{o,n}(t) + t^{(k-1)n}). \tag{91}$$

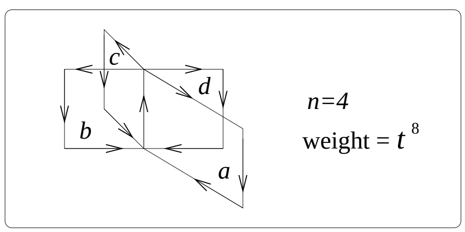


Figure 12: Labelled oriented 4-gonal 2-tree which is fixed by orientation reversion

For the unlabelled (weighted) enumeration, we have to adapt the results obtained in Section 4.2 and 4.3 to take into account the perimeter.

• k odd.

For k odd, we can easily see that the species a^{-}_{w}, $a^{\diamond}_{w}$ and $a^{\hat{\diamond}}_{w}$ satisfy the following expressions in terms of the weighted quotient species Q_w, S_w and U_w, which are adapted from Section 4.1:

$$a^{-}_{w} = Q_w(X, B_w^{\frac{k-1}{2}}), \tag{92}$$

$$a^{\diamond}_{w} = X \cdot S_w(X, B_w^{\frac{k-1}{2}}), \tag{93}$$

$$a^{\hat{\diamond}}_{w} = X \cdot U_w(X, B_w^{\frac{k-1}{2}}), \tag{94}$$

with

$$\begin{aligned}
Q_w(X,Y) &= \left(t + tXY^2 + E_{\geq 2}(XY^2)\right)/\mathbb{Z}_2, &(95)\\
S_w(X,Y) &= C_k(t + E_+(XY^2))/\mathbb{Z}_2, &(96)\\
U_w(X,Y) &= \left((t + E_+(XY^2))^k\right)/\mathbb{Z}_2, &(97)
\end{aligned}$$

where $E_{\geq 2}$ is the species of sets of cardinality at least two. The cycle index series of these species are given by:

$$\begin{aligned}
Z_{Q_w} &= \frac{1}{2}(Z_{E_w(XY^2)} + q_w), &(98)\\
Z_{S_w} &= \frac{1}{2}\left(Z_{C_k(t+E_+(XY^2))} + q_w \cdot (p_2 \circ (t + Z_{E_+(XY^2)})^{\frac{k-1}{2}})\right), &(99)\\
Z_{U_w} &= \frac{1}{2}\left(Z_{(t+E_+(XY^2))^k} + q_w \cdot (p_2 \circ (t + Z_{E_+(XY^2)})^{\frac{k-1}{2}})\right), &(100)
\end{aligned}$$

where $q_w = (t-1)(1+x_1y_2) + h \circ (x_1y_2 + p_2 \circ (x_1 \frac{y_1^2 - y_2}{2})) = q + (t-1)(1+x_1y_2)$, h being the homogeneous symmetric function, and p_i, $i \geq 1$, denotes the i^{th} power sum and $E_w(XY^2) = E(XY^2) + (t-1)(1+XY^2)$.

Another use of the dissymmetry theorem gives the ordinary generating series of unlabelled k-gonal 2-trees weighted by their perimeter:

$$\widetilde{a}(x,t) = \frac{1}{2}\left(\widetilde{a}_o(x,t) + q_w[x, \widetilde{B}^{\frac{k-1}{2}}(x,t)] + (t-1)(1 + x\widetilde{B}^{\frac{k-2}{2}}(x^2,t^2)\right), \quad (101)$$

where

$$q_w[x, \widetilde{B}^{\frac{k-1}{2}}(x,t)] := q_w(x, x^2, \ldots; \widetilde{B}^{\frac{k-1}{2}}(x,t), \widetilde{B}^{\frac{k-1}{2}}(x^2,t^2), \ldots).$$

• k even.

When k is even, it suffices to adapt all species introduced in Section 4.2 in the present weighted context. This is easily done, as follows, the index w meaning that the species are weighted according to perimeter. Note that the species $a_{\mathrm{S},w}$ is a sub weighted-species of $a_{o,w}^{-}$ by definition. We have:

$$\widetilde{a}_{\mathrm{S}}(x,t) = \left(E(P_{\mathrm{TS},w} + P_{\mathrm{M},w} + P_{\mathrm{AL},w}) + (t-1)(1 + P_{\mathrm{TS},w} + P_{\mathrm{M},w})\right)^{\sim}(x), \quad (102)$$

where

$$\begin{aligned}
a_{\mathrm{TS},w} &= t + t \cdot P_{\mathrm{TS},w} + E_{\geq 2}(P_{\mathrm{TS},w}) &(103)\\
&= (t-1)(1 + P_{\mathrm{TS},w}) + E(P_{\mathrm{TS},w}), &(104)\\
P_{\mathrm{TS},w} &= X \cdot X_{=}^2 < B^{\frac{k-2}{2}} > \cdot \left(a_{\mathrm{TS},w} + (1-t)P_{\mathrm{TS},w}\right), &(105)\\
P_{\mathrm{AL},w} &= \Phi_2 < XB_w^{k-1} - (P_{\mathrm{TS},w} + P_{\mathrm{M},w}) >, &(106)\\
P_{\mathrm{M},w} &= X \cdot X_{=}^2 < B^{\frac{k-2}{2}} > \cdot \left(a_{\mathrm{S},w} + (1-t)P_{\mathrm{M},w} - a_{\mathrm{TS},w}\right). &(107)
\end{aligned}$$

We then have

$$\widetilde{a}_{\mathrm{S}}(x,t) = \exp(\sum_{i\geq 1}\frac{1}{i}(\widetilde{P}_{\mathrm{TS}}(x^i,t^i)+\widetilde{P}_{\mathrm{M}}(x^i,t^i)+\widetilde{P}_{\mathrm{AL}}(x^i,t^i)))+(t-1)(1+\widetilde{P}_{\mathrm{TS}}(x,t)+\widetilde{P}_{\mathrm{M}}(x,t)), \tag{108}$$

where

$$\widetilde{a}_{\mathrm{TS}}(x,t) = (t-1)(1+\widetilde{P}_{\mathrm{TS}}(x,t)) + \exp\left(\sum_{i\geq 1}\frac{1}{i}\widetilde{P}_{\mathrm{TS}}(x^i,t^i)\right), \tag{109}$$

$$\widetilde{P}_{\mathrm{TS}}(x,t) = x\widetilde{B}^{\frac{k-2}{2}}(x^2,t^2)\left(\widetilde{a}_{\mathrm{TS}}(x,t)+(1-t)\widetilde{P}_{\mathrm{TS}}(x,t)\right), \tag{110}$$

$$\widetilde{P}_{\mathrm{AL}}(x,t) = \frac{1}{2}\big(x^2\widetilde{B}^{k-1}(x^2,t^2) - \widetilde{P}_{\mathrm{TS}}(x^2,t^2) - \widetilde{P}_{\mathrm{M}}(x^2,t^2)\big), \tag{111}$$

and

$$\begin{aligned}\widetilde{P}_{\mathrm{M}}(x,t) &= \left(XX_{=}^2 < B_w^{\frac{k-2}{2}} > \cdot(a_{\mathrm{TS},w} + (1-t)(1+P_{\mathrm{TS},w}))\cdot E_{+}(P_{\mathrm{AL},w}+P_{\mathrm{M},w})\right)^{\sim}(x)\\ &= x\widetilde{B}^{\frac{k-2}{2}}(x^2,t^2)\left(\widetilde{a}_{\mathrm{S}}(x,t)+(1-t)\widetilde{P}_{\mathrm{M}}(x,t)-\widetilde{a}_{\mathrm{TS}}(x,t)\right).\end{aligned} \tag{112}$$

It is then possible to compute the tilde generating functions of unlabelled structures associated to the species (89):

$$\begin{aligned}\widetilde{a}_{o,\tau}^{-}(x,t) &= \widetilde{a}_S(x,t),\\ \widetilde{a}_{o,\tau}^{\diamond}(x,t) &= x\left(\widetilde{a}_S(x,t)+(1-t)(\widetilde{P}_{\mathrm{TS}}(x,t)+\widetilde{P}_{\mathrm{M}}(x,t)\right)^2\cdot\widetilde{B}^{\frac{k-2}{2}}(x^2,t^2),\\ \widetilde{a}_{o,\tau}^{\diamond}(x,t) &= \frac{x}{2}\left(\widetilde{a}_S(x,t)+(1-t)(\widetilde{P}_{\mathrm{TS}}(x,t)+\widetilde{P}_{\mathrm{M}}(x,t)\right)^2\cdot\widetilde{B}^{\frac{k-2}{2}}(x^2,t^2)+\frac{x}{2}\widetilde{B}^{\frac{k}{2}}(x^2,t^2).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\widetilde{a}(x,t) = \frac{1}{2}\widetilde{a}_o(x,t)+\frac{1}{2}\widetilde{a}_{\mathrm{S}}(x,t)+\frac{x}{4}\widetilde{B}^{\frac{k}{2}}(x^2,t^2)\\ -\frac{x}{4}\left(\widetilde{a}_S(x,t)+(1-t)(\widetilde{P}_{\mathrm{TS}}(x,t)+\widetilde{P}_{\mathrm{M}}(x,t)\right)^2\cdot\widetilde{B}^{\frac{k-2}{2}}(x^2,t^2).\end{aligned} \tag{113}$$

6 Asymptotics

Thanks to the dissymmetry theorem and to the various combinatorial equations related to it, the asymptotic enumeration of (labelled or unlabelled) k-gonal 2-trees depends essentially on the asymptotic enumeration of B-structures where B is the auxiliary species characterized by the functional equation (5). In the labelled case, the asymptotics is trivial since we have the simple explicit formulas (9), (19) and (21). The unlabelled case is more eleborate and makes use of the funtional equation (14) satisfied by the series $\widetilde{B}(x)$.

We need first the following result, which is a consequence of the classical theorem of Bender (see [2]) and is inspired form the approach of Fowler et al. for 2-trees (see [5, 6]).

Proposition 19. Let $p = k - 1$ and $\widetilde{B}(x) = \sum b_n(p)x^n$. Then, there exist constants α_p and β_p such that

$$b_n(p) \sim \alpha_p \beta_p^n n^{-3/2}, \qquad \text{as } n \to \infty. \tag{114}$$

Moreover,

$$\alpha_p = \alpha(\xi_p) = \frac{1}{\sqrt{2\pi}} \frac{1}{p^{1+\frac{1}{p}}} \xi_p^{-\frac{1}{p}} \left(1 + \frac{p\xi_p \omega'(\xi_p)}{\omega(\xi_p)}\right)^{\frac{1}{2}} \tag{115}$$

and

$$\beta_p = \frac{1}{\xi_p}, \tag{116}$$

where ξ_p is the smallest root of the equation

$$\xi = \frac{1}{ep}\omega^{-p}(\xi), \tag{117}$$

where $\omega(x)$ is the series given by

$$\omega(x) = \mathrm{e}^{\frac{1}{2}x^2 b^p(x^2) + \frac{1}{3}x^3 b^p(x^3) + \cdots}. \tag{118}$$

Proof. Write, for simplicity, $b(x) = \widetilde{B}(x)$. Then, thanks to (14), $y = b(x)$ satisfies the relation

$$y = e^{xy^p}\omega(x), \quad \text{where} \quad \omega(x) = e^{\frac{1}{2}x^2 b^p(x^2) + \frac{1}{3}x^3 b^p(x^3) + \cdots}. \tag{119}$$

By Bender's theorem applied to the function $f(x, y) = y - e^{xy^p}\omega(x)$, we have to find a solution (ξ_p, τ_p) of the system

$$f(x, y) = 0 \qquad \text{and} \qquad f_y(x, y) = 0. \tag{120}$$

It is equivalent to say that ξ_p is solution of (117) and that $p\xi_p\tau_p^p = 1$.

Since $f_{yy}(\xi_p, \tau_p) \neq 0$, ξ_p is an algebraic singularity of degree 2 of $b(x)$ and, for x near ξ_p, we have an expression of the form

$$b(x) = \tau_{p.0} + \tau_{p.1}(1 - \frac{x}{\xi_p})^{\frac{1}{2}} + \tau_{p,2}(1 - \frac{x}{\xi_p}) + \tau_{p.3}(1 - \frac{x}{\xi_p})^{\frac{3}{2}} + \cdots \tag{121}$$

where

$$\tau_{p,0} \;=\; \tau_p \;=\; b(\xi_p) \;=\; \left(\frac{1}{p\xi_p}\right)^{\frac{1}{p}}, \tag{122}$$

$$\tau_{p,1} \;=\; -\frac{\sqrt{2}}{p^{1+\frac{1}{p}}}\xi_p^{-\frac{1}{p}} \left(1 + \frac{p\xi_p\omega'(\xi_p)}{\omega(\xi_p)}\right)^{\frac{1}{2}}, \tag{123}$$

$$\tau_{p,2} \;=\; \frac{1}{3p^{2+\frac{1}{p}}}\xi_p^{-\frac{1}{p}} \left((2p+3) - p(p-3)\frac{\xi_p\omega'(\xi_p)}{\omega(\xi_p)}\right). \tag{124}$$

The asymptotic formula (114) with α_p and β_p given by (115) and (116) then follow from the fact that the main term of the asymptotic behavior of the coefficients $b_n(p)$ of x^n in (121) depends only on the term $\tau_{p,1}(1-\frac{x}{\xi_p})^{\frac{1}{2}}$ in (121) and is given by

$$b_n(p) \sim \binom{\frac{1}{2}}{n}\tau_{p,1}(-1)^n \frac{1}{\xi_p^n} \sim \alpha_p \beta_p^n n^{-\frac{3}{2}} \quad \text{as} \quad n \to \infty. \tag{125}$$

■

Note that ξ_p is the radius of convergence of $b(x)$ and that the radius of convergence of $\omega(x)$ is $\sqrt{\xi_p}$. It can be shown that $0 < \xi_p < \sqrt{\xi_p} < 1$. This implies that numerical approximations of ξ_p, for fixed p, can be computed by iteration using (117), and a suitable truncated polynomial approximations of $b(x)$. We now state our main asymptotic result.

Proposition 20. Let $p = k-1$. Then, the number $\widetilde{a}_n$ of k-gonal 2-trees on n unlabelled k-gons satisfy

$$\widetilde{a}_n \sim \frac{1}{2}\widetilde{a}_{o,n}, \quad n \to \infty, \tag{126}$$

where $\widetilde{a}_{o,n}$ is the number of oriented k-gonal 2-trees over n unlabelled polygons. Moreover,

$$\widetilde{a}_{o,n} \sim \overline{\alpha}_p \beta_p^n n^{-5/2}, \quad n \to \infty, \tag{127}$$

where

$$\overline{\alpha_p} = 2\pi p^{1+\frac{2}{p}} \xi_p^{\frac{2}{p}} \alpha_p^3, \tag{128}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{p^{2+\frac{1}{p}}} \xi_p^{-\frac{1}{p}} \left(1 + p\frac{\omega'(\xi_p)}{\omega(\xi_p)}\right)^{\frac{3}{2}}, \tag{129}$$

and $\beta_p = \frac{1}{\xi_p}$ is the same growth as in Proposition 19.

Proof. The asymptotic formula (127) follows from the fact that the radius of convergence, ξ_p, of $\widetilde{a}(x)$, given by (31) for k odd and by (58) for k even, is equal to the radius of convergence of the dominating term $\frac{1}{2}\widetilde{a}_o(x)$. This is due to the easily checked fact that all terms in (31) and (58), except $\frac{1}{2}\widetilde{a}_o(x)$, have a radius of convergence greater or equal to $\sqrt{\xi_p} > \xi_p$. To establish (127), note first that, because of equation (18), the radius of convergence of $\widetilde{a}_o(x)$ is equal to the radius of covergence, ξ_p, of

$$b(x) - \frac{k-1}{k} x b^k(x), \tag{130}$$

where $b(x) = \widetilde{B}(x)$ and $k = p+1$. This implies that the asymptotic behavior of the coefficients $\widetilde{a}_{o,n}$ of $\widetilde{a}_o(x)$ is completely determined by that of (130). Substituting (121) into (130) and making use of (124) gives the following expansion

$$b(x) - \frac{k-1}{k} x b^k(x) = \overline{\tau}_{p,0} + \overline{\tau}_{p,1}\left(1 - \frac{x}{\xi_p}\right)^{\frac{1}{2}} + \overline{\tau}_{p,2}\left(1 - \frac{x}{\xi_p}\right) + \overline{\tau}_{p,3}\left(1 - \frac{x}{\xi_p}\right)^{\frac{3}{2}} + \cdots \tag{131}$$

where

$$\overline{\tau}_{p,0} = \frac{p}{p+1}\tau_{p,0}, \tag{132}$$

$$\overline{\tau}_{p,1} = 0, \tag{133}$$

$$\overline{\tau}_{p,2} = -\frac{1}{2}\frac{p(p+1)\tau_{p,1}^2 - 2\tau_{p,0}^2}{(p+1)\tau_{p,0}}, \tag{134}$$

$$\overline{\tau}_{p,3} = -\frac{1}{6}\frac{\tau_{p,1}(6p\tau_{p,0}\tau_{p,2} + p(p-1)\tau_{p,1}^2 - 6\tau_{p,0}^2)}{\tau_{p,0}^2}, \tag{135}$$

$$= -\frac{p}{3}\frac{\tau_{p,1}^3}{\tau_{p,0}^2}. \tag{136}$$

This implies that the dominating term for the asymptotic behavior of the coefficients $\widetilde{a}_{n,o}$ of x^n in $\widetilde{a}_o(x)$ depends only on the term $\overline{\tau}_{p,3}\left(1 - \frac{x}{\xi_p}\right)^{\frac{3}{2}}$ in (131) and is given by

$$\widetilde{a}_{n,o} \sim \binom{\frac{3}{2}}{n}\overline{\tau}_{p,3}(-1)^n \frac{1}{\xi_p^n} \sim \overline{\alpha}_p \beta_p n^{-\frac{5}{2}}, \quad \text{as } n \to \infty. \tag{137}$$

Computations making use of (136), (122) and (123), show that $\overline{\alpha}_p$ is indeed given by (128) and (129). ■

Our final result gives an explicit formula in terms of integer partitions for the common radius of convergence ξ_p of the series $\widetilde{B}(x)$, $\widetilde{a}(x)$ and $\widetilde{a}_o(x)$ from which the growth constant $\beta_p = \frac{1}{\xi_p}$ is obtained. We need the following special notations. If $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\nu)$ is a partition of an integer n in ν parts, we write $\lambda \vdash n$, $n = |\lambda|$, $\nu = l(\lambda)$, $m_i(\lambda) = |\{j : \lambda_j = i\}|$ = number of parts of size i in λ. Furthermore, we put

$$\sigma_i(\lambda) = \sum_{d|i} d m_d(\lambda), \quad \sigma_i^*(\lambda) = \sum_{d|i, d<i} d m_d(\lambda), \tag{138}$$

$$\widehat{\lambda} = 1 + |\lambda| + l(\lambda), \quad \widehat{z}(\lambda) = 2^{m_1(\lambda)} m_1(\lambda)! 3^{m_2(\lambda)} m_2(\lambda)! \ldots. \tag{139}$$

Proposition 21. We have the convergent expansion

$$\xi_p = \sum_{n=1}^{\infty} \frac{c_n}{p^n}, \tag{140}$$

where the coefficients c_n are constants, independent of p, explicitely given by

$$c_n = \sum_{\lambda \vdash n} \frac{e^{-\widehat{\lambda}}}{\widehat{\lambda}\widehat{z}(\lambda)} \prod_{i \geq 1} (\sigma_i(\lambda) - \widehat{\lambda})^{m_i(\lambda)-1} (\sigma_i^*(\lambda) - \widehat{\lambda}), \tag{141}$$

where λ runs over the set of partitions of n.

Proof. We establish the explicit formulas (140) and (141) by applying first Lagrange inversion to the equation $\xi = zR(\xi)$ where $z = \frac{1}{ep}$ and $R(t) = \omega^{-p}(t)$, to get

$$\xi_p = \xi = \sum_{n \geq 1} \gamma_n \left(\frac{1}{\mathrm{e}p}\right)^n, \quad \text{and} \quad \gamma_n = \frac{1}{n}[t^{n-1}]\omega^{-np}(t). \tag{142}$$

Next, to explicitely evaluate $\omega^{-np}(x)$, we use Labelle's version ([12]) of the Good inversion formula in the context of cycle index series as follows. We begin with

$$\begin{aligned} \omega^p(x) &= \exp(\frac{1}{2}px^2b^p(x^2) + \frac{1}{3}px^3b^p(x^3) + \cdots), & (143) \\ &= \exp(\frac{1}{2}px_2 + \frac{1}{3}px_3 + \cdots) \circ Z_{XB^p(X)}\Big|_{x_i := x^i} & (144) \end{aligned}$$

where the $\circ$ denotes the plethystic substitution. Using (7), we can then write $XB^p(X) = \frac{A(pX)}{p}$. This implies that

$$\omega^p(x) = \exp(\frac{1}{2}px_2 + \frac{1}{3}px_3 + \cdots) \circ \frac{Z_A(px_1, px_2, \ldots)}{p}\Big|_{x_i := x^i}, \tag{145}$$

and we get

$$\begin{aligned} \omega^{-np}(x) &= \exp(-\frac{n}{2}px_2 - \frac{n}{3}px_3 - \cdots) \circ \left(\frac{1}{p}Z_A(px_1, px_2, \ldots)\right)\Big|_{x_i := x^i} & (146) \\ &= \exp(-\frac{n}{2}x_2 - \frac{n}{3}x_3 - \cdots) \circ Z_A(x_1, x_2, \ldots)\Big|_{x_i := px^i}. & (147) \end{aligned}$$

Then, using Labelle's inversion formula for cycle index series, we have, for any formal cycle index series $g(x_1, x_2, \ldots)$

$$[x_1^{n_1}x_2^{n_2}\ldots]\, g \circ Z_A(x_1, x_2, \ldots) = [t_1^{n_1}t_2^{n_2}\ldots] g(t_1, t_2, \ldots) \prod_{i=1}^{\infty} (1 - t_i) \exp(n_i(t_i + \frac{1}{2}t_{2i} + \cdots)), \tag{148}$$

and

$$\prod_{j=1}^{\infty} \exp(n_j(t_j + \frac{1}{2}t_{2j} + \cdots)) = \prod_{i=1}^{\infty} \exp(\sum_{d|i} dn_d \frac{t_i}{i}). \tag{149}$$

Taking $g(x_1, x_2, \ldots) = \exp(-\frac{\nu}{2}px_2 - \frac{\nu}{3}px_3 - \cdots)$, gives, after some computations,

$$[x_1^{n_1}x_2^{n_2}\ldots]\left(\exp(-\frac{\nu}{2}x_2 - \frac{\nu}{3}x_3 - \cdots) \circ Z_A\right) =$$

$$\left\{ \begin{array}{ll} 0 & \text{if} \quad n_1 > 0, \\ \left(\dfrac{\prod_{i\geq 2}(-\nu + \sum_{d|i} dn_d)^{n_i-1}(-\nu + \sum_{d|i,d<i} dn_d)}{2^{n_2}n_2!3^{n_3}n_3!\dots} \right) & \text{if} \quad n_1 = 0. \end{array} \right. \tag{150}$$

Making the substitution $x_i := px^i$, for $i = 1, 2, 3, \dots$, gives the explicit formula

$$\omega^{-\nu p}(x) = \sum_{n\geq 0} \left(\sum_{2n_2+3n_3+\cdots=n} p^{n_2+n_3+\cdots} \frac{\prod_{i\geq 2}(-\nu + \sum_{d|i} dn_d)^{n_i-1}(-\nu + \sum_{d|i,d<i} dn_d)}{2^{n_2}n_2!3^{n_3}n_3!\dots} \right) x^n.$$

This implies, taking $\nu = n$ and using (142), that

$$\begin{aligned} \xi_p &= \sum_{n\geq 1} \frac{1}{n} \left(\sum_{2n_2+3n_3+\cdots=n-1} p^{n_2+n_3+\cdots} \frac{\prod_{i\geq 2}(1-n + \sum_{d|i} dn_d)^{n_i-1}(1-n + \sum_{d|i,d<i} dn_d)}{2^{n_2}n_2!3^{n_3}n_3!\dots} \right) \left(\frac{1}{\mathrm{e}p}\right)^n, \\ &= \sum_{n\geq 1} \frac{c_n}{p^n}, \end{aligned}$$

where the coefficients c_n, $n \geq 1$, are given by (141). ■

Table 1, in the Appendix, gives, to 20 decimal places, the constants ξ_p, α_p, $\overline{\alpha}_p$ and $\beta_p = \frac{1}{\xi_p}$ for $p = 1, \dots, 5$. Table 2 gives the exact values of the numbers $\widetilde{a}_n$, for k from 2 up to 12 and for $n = 0, 1, \dots 20$, of the number of unlabelled k-gonal 2-trees built over n k-gons.

Here are the first few values of the universal constants c_n occuring in (140), for $n = 1, \dots, 5$.

$$\begin{aligned} c_1 &= \frac{1}{\mathrm{e}} = 0.36787944117144232160, \\ c_2 &= -\frac{1}{2}\frac{1}{\mathrm{e}^3} = -0.02489353418393197149, \\ c_3 &= \frac{1}{8}\frac{1}{\mathrm{e}^5} - \frac{1}{3}\frac{1}{\mathrm{e}^4} = -0.00526296958802571004, \\ c_4 &= -\frac{1}{48}\frac{1}{\mathrm{e}^7} + \frac{1}{\mathrm{e}^6} - \frac{1}{4}\frac{1}{\mathrm{e}^5} = 0.00077526788594593923, \\ c_5 &= \frac{1}{384}\frac{1}{\mathrm{e}^9} - \frac{4}{3}\frac{1}{\mathrm{e}^8} + \frac{49}{72}\frac{1}{\mathrm{e}^7} - \frac{1}{5}\frac{1}{\mathrm{e}^6} = 0.00032212622183609932. \end{aligned} \tag{151}$$

Remark 2. The computations of this section are also valid for the case $k = 2$ ($p = 1$), corresponding to the case of classical rooted trees (*Cayley trees*) defined by the functional equation $A = XE(A)$. In this case, the growth constant $\beta = \beta_1$, in (114), is known as the Otter constant (see [17]). It is interesting to note that this constant takes the explicit form $\beta = \frac{1}{\xi_1}$, with

$$\xi_1 = \sum_{n \geq 1} c_n. \tag{152}$$

Notice also that, when $k = 3$, we recover the asymptotic results of Fowler et al. in [5, 6].

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`http://www.research.att.com/~njas/sequences`

E-mail adresses: [gilbert,lamathe,leroux]@lacim.uqam.ca

Appendix

Table 1 gives, to 20 decimal places, the constants ξ_p, α_p, $\overline{\alpha}_p$ and $\beta_p = \frac{1}{\xi_p}$ for $p = 1, \ldots, 5$.

p	ξ_p	α_p	$\overline{\alpha}_p$	β_p
1	0.338321856899	1.300312124682	1.581185475409	2.955765285652
2	0.177099522303	0.349261381742	0.349261381742	5.646542616233
3	0.119674100436	0.191997258650	0.067390781222	8.356026879296
4	0.090334539604	0.131073637349	0.034020667269	11.069962877759
5	0.072539192528	0.099178841365	0.020427915489	13.785651110085
6	0.060597948397	0.079660456931	0.013601784466	16.502208844693
7	0.052031135998	0.066517090385	0.009699566188	19.219261329064
8	0.045585869619	0.057075912245	0.007262873797	21.936622211299
9	0.040561059517	0.049970993036	0.005640546218	24.654188324989
10	0.036533820306	0.044433135893	0.004506504206	27.371897918664
11	0.033233950789	0.039996691773	0.003682863427	30.089711763681

Table 1: Numerical values of ξ_p, α_p, $\overline{\alpha}_p$ and β_p, $p = 1, \ldots, 5$

Table 2 gives the exact values of the numbers $\widetilde{a}_n$, for k from 2 up to 12 and for $n = 0, 1, \ldots 20$, of the number of unlabelled k-gonal 2-trees built over n k-gons.

Tables 3 and 4 give the polynomials $b_n(t)$, for $n = 0, 1, \ldots, 9$ and for k from 2 up to 9, of the weighted (by their perimeter) unlabelled oriented-edge-rooted k-gonal 2-trees over n k-gons.

$\boldsymbol{k = 2}$
1, 1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, 1301, 3159, 7741, 19320, 48629, 123867, 317955, 823065, 2144505
$\boldsymbol{k = 3}$
1, 1, 1, 2, 5, 12, 39, 136, 529, 2171, 9368, 41534, 188942, 874906, 4115060, 19602156, 94419351, 459183768, 2252217207, 11130545494, 55382155396
$\boldsymbol{k = 4}$
1, 1, 1, 3, 8, 32, 141, 749, 4304, 26492, 169263, 1115015, 7507211, 51466500, 358100288, 2523472751, 17978488711, 129325796854, 938234533024, 6858551493579, 50478955083341
$\boldsymbol{k = 5}$
1, 1, 1, 3, 11, 56, 359, 2597, 20386, 167819, 1429815, 12500748, 111595289, 1013544057, 9340950309, 87176935700, 822559721606, 7836316493485, 75293711520236, 728968295958626, 7105984356424859
$\boldsymbol{k = 6}$
1, 1, 1, 4, 16, 103, 799, 7286, 71094, 729974, 7743818, 84307887, 937002302, 10595117272, 121568251909, 1412555701804, 16594126114458, 196829590326284, 2354703777373055, 28385225424840078, 344524656398655124
$\boldsymbol{k = 7}$
1, 1, 1, 4, 20, 158, 1539, 16970, 199879, 2460350, 31266165, 407461893, 5420228329, 73352481577, 1007312969202, 14008437540003, 196963172193733, 2796235114720116, 40038505601111596, 577693117173844307, 8392528734991449808
$\boldsymbol{k = 8}$
1, 1, 1, 5, 26, 245, 2737, 35291, 483819, 6937913, 102666626, 1558022255, 24133790815, 380320794122, 6081804068869, 98490990290897, 1612634990857755, 26660840123167203, 444560998431678554, 7469779489114328514, 126375763235359105446
$\boldsymbol{k = 9}$
1, 1, 1, 5, 32, 343, 4505, 66603, 1045335, 17115162, 289107854, 5007144433, 88516438360, 1591949961503, 29053438148676, 536972307386326, 10034276171127780, 189331187319203010, 3603141751525175854, 69097496637591215442, 1334213677527481808220
$\boldsymbol{k = 10}$
1, 1, 1, 6, 39, 482, 7053, 117399, 2070289, 38097139, 723169329, 14074851642, 279609377638, 5651139037570, 115901006038377, 2407291353219949, 50553753543016719, 1071971262516091572, 22926544048209731554, 494103705426160765546, 10722146465907412669810
$\boldsymbol{k = 11}$
1, 1, 1, 6, 46, 636, 10527, 194997, 3823327, 78118107, 1646300388, 35570427615, 784467060622, 17601062294302, 400750115756742, 9240636709048733, 215435023547580882, 5071520482516388865, 120417032326341878672, 2881134828445365441407, 69410468220307148620226
$\boldsymbol{k = 12}$
1, 1, 1, 7, 55, 840, 15189, 309607, 6671842, 149850849, 3471296793, 82442359291, 1998559329142, 49290785442796, 1233639304644946, 31268489727956101, 801335133177932829, 20736286803363051714, 541224489038545084067, 14234799536039481373552, 376974819516101224941091

Table 2: Values of $\widetilde{a}_n$ for $k = 2, \ldots, 12$ and $n = 0, \ldots, 20$

$k = 2$

$t,$
$t,$
$t + t^2,$
$t + 2\,t^2 + t^3,$
$t + 4\,t^2 + 3\,t^3 + t^4,$
$t + 6\,t^2 + 8\,t^3 + 4\,t^4 + t^5,$
$t + 9\,t^2 + 18\,t^3 + 14\,t^4 + 5\,t^5 + t^6,$
$t + 12\,t^2 + 35\,t^3 + 39\,t^4 + 21\,t^5 + 6\,t^6 + t^7,$
$t + 16\,t^2 + 62\,t^3 + 97\,t^4 + 72\,t^5 + 30\,t^6 + 7\,t^7 + t^8,$
$t + 20\,t^2 + 103\,t^3 + 212\,t^4 + 214\,t^5 + 120\,t^6 + 40\,t^7 + 8\,t^8 + t^9$

$k = 3$

t
t^2
$2\,t^3 + t^4$
$5\,t^4 + 4\,t^5 + t^6$
$14\,t^5 + 18\,t^6 + 6\,t^7 + t^8$
$42\,t^6 + 72\,t^7 + 37\,t^8 + 8\,t^9 + t^{10}$
$132\,t^7 + 291\,t^8 + 204\,t^9 + 64\,t^{10} + 10\,t^{11} + t^{12}$
$429\,t^8 + 1152\,t^9 + 1048\,t^{10} + 438\,t^{11} + 97\,t^{12} + 12\,t^{13} + t^{14}$
$1430\,t^9 + 4558\,t^{10} + 5128\,t^{11} + 2757\,t^{12} + 804\,t^{13} + 138\,t^{14} + 14\,t^{15} + t^{16}$
$4862\,t^{10} + 17944\,t^{11} + 24249\,t^{12} + 16108\,t^{13} + 5981\,t^{14} + 1332\,t^{15} + 185\,t^{16} + 16\,t^{17} + t^{18}$

$k = 4$

t
t^3
$3\,t^5 + t^6$
$12\,t^7 + 6\,t^8 + t^9$
$55\,t^9 + 42\,t^{10} + 9\,t^{11} + t^{12}$
$273\,t^{11} + 274\,t^{12} + 87\,t^{13} + 12\,t^{14} + t^{15}$
$1428\,t^{13} + 1806\,t^{14} + 767\,t^{15} + 150\,t^{16} + 15\,t^{17} + t^{18}$
$7752\,t^{15} + 11820\,t^{16} + 6387\,t^{17} + 1641\,t^{18} + 228\,t^{19} + 18\,t^{20} + t^{21}$
$43263\,t^{17} + 77440\,t^{18} + 51078\,t^{19} + 16614\,t^{20} + 3006\,t^{21} + 324\,t^{22} + 21\,t^{23} + t^{24}$
$246675\,t^{19} + 507246\,t^{20} + 396905\,t^{21} + 157638\,t^{22} + 35847\,t^{23} + 4972\,t^{24} + 435\,t^{25} + 24\,t^{26} + t^{27}$

$k = 5$

t
t^4
$4\,t^7 + t^8$
$22\,t^{10} + 8\,t^{11} + t^{12}$
$140\,t^{13} + 76\,t^{14} + 12\,t^{15} + t^{16}$
$969\,t^{16} + 688\,t^{17} + 158\,t^{18} + 16\,t^{19} + t^{20}$
$7084\,t^{19} + 6290\,t^{20} + 1916\,t^{21} + 272\,t^{22} + 20\,t^{23} + t^{24}$
$53820\,t^{22} + 57376\,t^{23} + 22064\,t^{24} + 4092\,t^{25} + 414\,t^{26} + 24\,t^{27} + t^{28}$
$420732\,t^{25} + 524412\,t^{26} + 244840\,t^{27} + 57113\,t^{28} + 7488\,t^{29} + 588\,t^{30} + 28\,t^{31} + t^{32}$
$3362260\,t^{28} + 4799568\,t^{29} + 2645854\,t^{30} + 749908\,t^{31} + 122908\,t^{32} + 12376\,t^{33} + 790\,t^{34} + 32\,t^{35} + t^{36}$

Table 3: Polynomials $b_n(t)$ for $k = 2, 3, 4, 5$ and $n = 0, \dots 9$

$k = 6$

t

t^5

$5\,t^9 + t^{10}$

$35\,t^{13} + 10\,t^{14} + t^{15}$

$285\,t^{17} + 120\,t^{18} + 15\,t^{19} + t^{20}$

$2530\,t^{21} + 1390\,t^{22} + 250\,t^{23} + 20\,t^{24} + t^{25}$

$23751\,t^{25} + 16255\,t^{26} + 3860\,t^{27} + 430\,t^{28} + 25\,t29 + t^{30}$

$231880\,t^{29} + 190106\,t^{30} + 56755\,t^{31} + 8235\,t^{32} + 655\,t^{33} + 30\,t^{34} + t^{35}$

$2330445\,t^{33} + 2229120\,t^{34} + 805621\,t^{35} + 146510\,t^{36} + 15060\,t^{37} + 930\,t^{38} + 35\,t^{39} + t^{40}$

$23950355\,t^{37} + 26193570\,t^{38} + 11149900\,t^{39} + 2457081\,t^{40} + 314810\,t^{41} + 24880\,t^{42} + 1250\,t^{43} + 40\,t^{44} + t^{45}$

$k = 7$

t

t^6

$6\,t^{11} + t^{12}$

$51\,t^{16} + 12\,t^{17} + t^{18}$

$506\,t^{21} + 174\,t^{22} + 18\,t^{23} + t^{24}$

$5481\,t^{26} + 2456\,t^{27} + 363\,t^{28} + 24\,t^{29} + t^{30}$

$62832\,t^{31} + 34989\,t^{32} + 6808\,t^{33} + 624\,t^{34} + 30\,t^{35} + t^{36}$

$749398\,t^{36} + 499188\,t^{37} + 121800\,t^{38} + 14514\,t^{39} + 951\,t^{40} + 36\,t^{41} + t^{42}$

$9203634\,t^{41} + 7143466\,t^{42} + 2106138\,t^{43} + 313872\,t^{44} + 26532\,t^{45} + 1350\,t^{46} + 42\,t^{47} + t^{48}$

$115607310\,t^{46} + 102489288\,t^{47} + 35536296\,t^{48} + 6406278\,t^{49} + 673749\,t^{50} + 43820\,t^{51} + 1815\,t^{52} + 48\,t^{53} + t^{54}$

$k = 8$

t

t^7

$7\,t^{13} + t^{14}$

$70\,t^{19} + 14\,t^{20} + t^{21}$

$819\,t^{25} + 238\,t^{26} + 21\,t^{27} + t^{28}$

$10472\,t^{31} + 3962\,t^{32} + 497\,t^{33} + 28\,t^{34} + t^{35}$

$141778\,t^{37} + 66556\,t^{38} + 10969\,t^{39} + 854\,t^{40} + 35\,t^{41} + t^{42}$

$1997688\,t^{43} + 1120658\,t^{44} + 231203\,t^{45} + 23373\,t^{46} + 1302\,t^{47} + 42\,t^{48} + t^{49}$

$28989675\,t^{49} + 18932368\,t^{50} + 4713849\,t^{51} + 595077\,t^{52} + 42714\,t^{53} + 1848\,t^{54} + 49\,t^{55} + t^{56}$

$430321633\,t^{55} + 320771256\,t^{56} + 93827895\,t^{57} + 14311479\,t^{58} + 1276471\,t^{59} + 70532\,t^{60} + 2485\,t^{61} + 56\,t^{62} + t^{63}$

$k = 9$

t

t^8

$8\,t^{15} + t^{16}$

$92\,t^{22} + 16\,t^{23} + t^{24}$

$1240\,t^{29} + 312\,t^{30} + 24\,t^{31} + t^{32}$

$18278\,t^{36} + 5984\,t^{37} + 652\,t^{38} + 32\,t^{39} + t^{40}$

$285384\,t^{43} + 115796\,t^{44} + 16552\,t^{45} + 1120\,t^{46} + 40\,t^{47} + t^{48}$

$4638348\,t^{50} + 2247376\,t^{51} + 401632\,t^{52} + 35256\,t^{53} + 1708\,t^{54} + 48\,t^{55} + t^{56}$

$77652024\,t^{57} + 43772920\,t^{58} + 9432184\,t^{59} + 1032814\,t^{60} + 64416\,t^{61} + 2424\,t^{62} + 56\,t^{63} + t^{64}$

$1329890705\,t^{64} + 855243648\,t^{65} + 216340024\,t^{66} + 28597424\,t^{67} + 2214272\,t^{68} + 106352\,t^{69} + 3260\,t^{70} + 64\,t^{71} + t^{72}$

Table 4: Polynomials $b_n(t)$ for $k = 6, 7, 8, 9$ and $n = 0, \ldots, 9$

$\boldsymbol{k=2}$

t

t^2

t^2

$t^2 + t^3$

$t^2 + t^3 + t^4$

$t^2 + 2\,t^3 + 2\,t^4 + t^5$

$t^2 + 3\,t^3 + 4\,t^4 + 2\,t^5 + t^6$

$t^2 + 4\,t^3 + 8\,t^4 + 6\,t^5 + 3\,t^6 + t^7$

$t^2 + 5\,t^3 + 14\,t^4 + 14\,t^5 + 9\,t^6 + 3\,t^7 + t^8$

$t^2 + 7\,t^3 + 23\,t^4 + 32\,t^5 + 26\,t^6 + 12\,t^7 + 4\,t^8 + t^9$

$t^2 + 8\,t^3 + 36\,t^4 + 64\,t^5 + 66\,t^6 + 39\,t^7 + 16\,t^8 + 4\,t^9 + t^{10}$

$\boldsymbol{k=3}$

t

t^3

t^4

$t^5 + t^6$

$3\,t^6 + t^7 + t^8$

$4\,t^7 + 5\,t^8 + 2\,t^9 + t^{10}$

$12\,t^8 + 14\,t^9 + 10\,t^{10} + 2\,t^{11} + t^{12}$

$27\,t^9 + 53\,t^{10} + 37\,t^{11} + 15\,t^{12} + 3\,t^{13} + t^{14}$

$82\,t^{10} + 179\,t^{11} + 171\,t^{12} + 71\,t^{13} + 22\,t^{14} + 3\,t^{15} + t^{16}$

$228\,t^{11} + 664\,t^{12} + 716\,t^{13} + 401\,t^{14} + 128\,t^{15} + 29\,t^{16} + 4\,t^{17} + t^{18}$

$733\,t^{12} + 2386\,t^{13} + 3128\,t^{14} + 2051\,t^{15} + 825\,t^{16} + 201\,t^{17} + 39\,t^{18} + 4\,t^{19} + t^{20}$

$\boldsymbol{k=4}$

t

t^4

t^6

$2\,t^8 + t^9$

$7\,t^{10} + 3\,t^{11} + t^{12}$

$25\,t^{12} + 18\,t^{13} + 5\,t^{14} + t^{15}$

$108\,t^{14} + 101\,t^{15} + 36\,t^{16} + 6\,t^{17} + t^{18}$

$492\,t^{16} + 588\,t^{17} + 259\,t^{18} + 58\,t^{19} + 8\,t^{20} + t^{21}$

$2431\,t^{18} + 3471\,t^{19} + 1887\,t^{20} + 519\,t^{21} + 87\,t^{22} + 9\,t^{23} + t^{24}$

$12371\,t^{20} + 20834\,t^{21} + 13521\,t^{22} + 4569\,t^{23} + 921\,t^{24} + 120\,t^{25} + 11\,t^{26} + t^{27}$

$65169\,t^{22} + 125976\,t^{23} + 96096\,t^{24} + 38730\,t^{25} + 9411\,t^{26} + 1474\,t^{27} + 160\,t^{28} + 12\,t^{29} + t^{30}$

$\boldsymbol{k=5}$

t

t^5

t^8

$2\,t^{11} + t^{12}$

$8\,t^{14} + 2\,t^{15} + t^{16}$

$33\,t^{17} + 18\,t^{18} + 4\,t^{19} + t^{20}$

$194\,t^{20} + 124\,t^{21} + 36\,t^{22} + 4\,t^{23} + t^{24}$

$1196\,t^{23} + 1014\,t^{24} + 324\,t^{25} + 56\,t^{26} + 6\,t^{27} + t^{28}$

$8196\,t^{26} + 8226\,t^{27} + 3233\,t^{28} + 640\,t^{29} + 84\,t^{30} + 6\,t^{31} + t^{32}$

$58140\,t^{29} + 68780\,t^{30} + 31846\,t^{31} + 7787\,t^{32} + 1143\,t^{33} + 114\,t^{34} + 8\,t^{35} + t^{36}$

$427975\,t^{32} + 579266\,t^{33} + 313832\,t^{34} + 90742$[illegible]$\,t^{35} + 16019\,t^{36} + 1820\,t^{37} + 152\,t^{38} + 8\,t^{39} + t^{40}$

Table 5: Coefficients of $\widetilde{a}_o(x,t)$ for $k = 2,3,4,5$ and $n = 0,\ldots,10$

$\boldsymbol{k = 6}$

t

t^{6}

t^{10}

$3\,t^{14} + t^{15}$

$19\,t^{18} + 5\,t^{19} + t^{20}$

$118\,t^{22} + 50\,t^{23} + 8\,t^{24} + t^{25}$

$931\,t^{26} + 495\,t^{27} + 100\,t^{28} + 10\,t^{29} + t^{30}$

$7756\,t^{30} + 5110\,t^{31} + 1266\,t^{32} + 164\,t^{33} + 13\,t^{34} + t^{35}$

$68685\,t^{34} + 53801\,t^{35} + 16275\,t^{36} + 2560\,t^{37} + 245\,t^{38} + 15\,t^{39} + t^{40}$

$630465\,t^{38} + 575535\,t^{39} + 206954\,t^{40} + 39445\,t^{41} + 4529\,t^{42} + 340\,t^{43} + 18\,t^{44} + t^{45}$

$5966610\,t^{42} + 6224520\,t^{43} + 2611405\,t^{44} + 589676\,t^{45} + 81145\,t^{46} + 7285\,t^{47} + 454\,t^{48} + 20\,t^{49} + t^{50}$

$\boldsymbol{k = 7}$

t

t^{7}

t^{12}

$3\,t^{17} + t^{18}$

$16\,t^{22} + 3\,t^{23} + t^{24}$

$112\,t^{27} + 39\,t^{28} + 6\,t^{29} + t^{30}$

$1020\,t^{32} + 434\,t^{33} + 78\,t^{34} + 6\,t^{35} + t^{36}$

$10222\,t^{37} + 5487\,t^{38} + 1127\,t^{39} + 124\,t^{40} + 9\,t^{41} + t^{42}$

$109947\,t^{42} + 70053\,t^{43} + 17436\,t^{44} + 2247\,t^{45} + 186\,t^{46} + 9\,t^{47} + t^{48}$

$1230840\,t^{47} + 914103\,t^{48} + 268995\,t^{49} + 42144\,t^{50} + 4000\,t^{51} + 255\,t^{52} + 12\,t^{53} + t^{54}$

$14218671\,t^{52} + 12057540\,t^{53} + 4131929\,t^{54} + 764623\,t^{55} + 86652\,t^{56} + 6397\,t^{57} + 340\,t^{58} + 12\,t^{59} + t^{60}$

$\boldsymbol{k = 8}$

t

t^{8}

t^{14}

$4\,t^{20} + t^{21}$

$35\,t^{26} + 7\,t^{27} + t^{28}$

$332\,t^{32} + 98\,t^{33} + 11\,t^{34} + t^{35}$

$3766\,t^{38} + 1393\,t^{39} + 196\,t^{40} + 14\,t^{41} + t^{42}$

$45448\,t^{44} + 20650\,t^{45} + 3561\,t^{46} + 322\,t^{47} + 18\,t^{48} + t^{49}$

$580203\,t^{50} + 312739\,t^{51} + 65590\,t^{52} + 7217\,t^{53} + 483\,t^{54} + 21\,t^{55} + t^{56}$

$7684881\,t^{56} + 4813130\,t^{57} + 1197467\,t^{58} + 158928\,t^{59} + 12762\,t^{60} + 672\,t^{61} + 25\,t^{62} + t^{63}$

$104898024\,t^{62} + 74961328\,t^{63} + 21701960\,t^{64} + 3403708\,t^{65} + 326760\,t^{66} + 20552\,t^{67} + 896\,t^{68} + 28\,t^{69} + t^{70}$

$\boldsymbol{k = 9}$

t

t^{9}

t^{16}

$4\,t^{23} + t^{24}$

$27\,t^{30} + 4\,t^{31} + t^{32}$

$266\,t^{37} + 68\,t^{38} + 8\,t^{39} + t^{40}$

$3312\,t^{44} + 1048\,t^{45} + 136\,t^{46} + 8\,t^{47} + t^{48}$

$45711\,t^{51} + 17948\,t^{52} + 2712\,t^{53} + 219\,t^{54} + 12\,t^{55} + t^{56}$

$670344\,t^{58} + 312276\,t^{59} + 56942\,t^{60} + 5432\,t^{61} + 328\,t^{62} + 12\,t^{63} + t^{64}$

$10233201\,t^{65} + 5539348\,t^{66} + 1194736\,t^{67} + 137754\,t^{68} + 9654\,t^{69} + 452\,t^{70} + 16\,t^{71} + t^{72}$

$161055618\,t^{72} + 99432684\,t^{73} + 24928832\,t^{74} + 3391482\,t^{75} + 283146\,t^{76} + 15472\,t^{77} + 603\,t^{78} + 16\,t^{79} + t^{80}$

Table 6: Coefficients of $\widetilde{a}_o(x,t)$ for $k = 6, 7, 8, 9$ and $n = 0, \ldots, 10$

$k = 2$

t
t^2
t^2
$t^2 + t^3$
$t^2 + t^3 + t^4$
$t^2 + 2t^3 + 2t^4 + t^5$
$t^2 + 3t^3 + 4t^4 + 2t^5 + t^6$
$t^2 + 4t^3 + 8t^4 + 6t^5 + 3t^6 + t^7$
$t^2 + 5t^3 + 14t^4 + 14t^5 + 9t^6 + 3t^7 + t^8$
$t^2 + 7t^3 + 23t^4 + 32t^5 + 26t^6 + 12t^7 + 4t^8 + t^9$
$t^2 + 8t^3 + 36t^4 + 64t^5 + 66t^6 + 39t^7 + 16t^8 + 4t^9 + t^{10}$

$k = 3$

t
t^3
t^4
$t^5 + t^6$
$4t^6 + 2t^7 + t^8$
$6t^7 + 8t^8 + 3t^9 + t^{10}$
$19t^8 + 28t^9 + 16t^{10} + 4t^{11} + t^{12}$
$49t^9 + 100t^{10} + 70t^{11} + 26t^{12} + 5t^{13} + t^{14}$
$150t^{10} + 358t^{11} + 325t^{12} + 142t^{13} + 38t^{14} + 6t^{15} + t^{16}$
$442t^{11} + 1309t^{12} + 1414t^{13} + 783t^{14} + 250t^{15} + 52t^{16} + 7t^{17} + t^{18}$
$1424t^{12} + 4772t^{13} + 6186t^{14} + 4102t^{15} + 1615t^{16} + 402t^{17} + 70t^{18} + 8t^{19} + t^{20}$

$k = 4$

t
t^4
t^6
$2t^8 + t^9$
$5t^{10} + 2t^{11} + t^{12}$
$16t^{12} + 11t^{13} + 4t^{14} + t^{15}$
$60t^{14} + 54t^{15} + 22t^{16} + 4t^{17} + t^{18}$
$261t^{16} + 305t^{17} + 142t^{18} + 34t^{19} + 6t^{20} + t^{21}$
$1243t^{18} + 1755t^{19} + 975t^{20} + 273t^{21} + 51t^{22} + 6t^{23} + t^{24}$
$6257t^{20} + 10478t^{21} + 6853t^{22} + 2336t^{23} + 490t^{24} + 69t^{25} + 8t^{26} + t^{27}$
$32721t^{22} + 63100t^{23} + 48271t^{24} + 19497t^{25} + 4803t^{26} + 770t^{27} + 92t^{28} + 8t^{29} + t^{30}$

$k = 5$

t
t^5
t^8
$2t^{11} + t^{12}$
$12t^{14} + 4t^{15} + t^{16}$
$57t^{17} + 32t^{18} + 6t^{19} + t^{20}$
$366t^{20} + 248t^{21} + 64t^{22} + 8t^{23} + t^{24}$
$2340t^{23} + 2002t^{24} + 630t^{25} + 104t^{26} + 10t^{27} + t^{28}$
$16252t^{26} + 16452t^{27} + 6393t^{28} + 1280t^{29} + 156t^{30} + 12t^{31} + t^{32}$
$115940t^{29} + 137378t^{30} + 63516t^{31} + 15493t^{32} + 2259t^{33} + 216t^{34} + 14t^{35} + t^{36}$
$854981t^{32} + 1158532t^{33} + 626996t^{34} + 181484t^{35} + 31887t^{36} + 3640t^{37} + 288t^{38} +$
$16t^{39} + t^{40}$

Table 7: Coefficients of $\widetilde{a}(x,t)$ for $k = 2, 3, 4, 5$ and $n = 0, \ldots, 10$

$k = 6$

t

t^{6}

t^{10}

$3\,t^{14} + t^{15}$

$12\,t^{18} + 3\,t^{19} + t^{20}$

$68\,t^{22} + 28\,t^{23} + 6\,t^{24} + t^{25}$

$483\,t^{26} + 253\,t^{27} + 56\,t^{28} + 6\,t^{29} + t^{30}$

$3946\,t^{30} + 2582\,t^{31} + 659\,t^{32} + 89\,t^{33} + 9\,t^{34} + t^{35}$

$34485\,t^{34} + 26953\,t^{35} + 8213\,t^{36} + 1300\,t^{37} + 133\,t^{38} + 9\,t^{39} + t^{40}$

$315810\,t^{38} + 288021\,t^{39} + 103799\,t^{40} + 19831\,t^{41} + 2318\,t^{42} + 182\,t^{43} + 12\,t^{44} + t^{45}$

$2984570\,t^{42} + 3112780\,t^{43} + 1306605\,t^{44} + 295143\,t^{45} + 40775\,t^{46} + 3689\,t^{47} + 243\,t^{48} + 12\,t^{49} + t^{50}$

$k = 7$

t

t^{7}

t^{12}

$3\,t^{17} + t^{18}$

$26\,t^{22} + 6\,t^{23} + t^{24}$

$203\,t^{27} + 72\,t^{28} + 9\,t^{29} + t^{30}$

$41989\,t^{32} + 868\,t^{33} + 144\,t^{34} + 12\,t^{35} + t^{36}$

$20254\,t^{37} + 10914\,t^{38} + 2212\,t^{39} + 236\,t^{40} + 15\,t^{41} + t^{42}$

$219388\,t^{42} + 140106\,t^{43} + 34704\,t^{44} + 4494\,t^{45} + 354\,t^{46} + 18\,t^{47} + t^{48}$

$2459730\,t^{47} + 1827555\,t^{48} + 537357\,t^{49} + 84102\,t^{50} + 7937\,t^{51} + 492\,t^{52} + 21\,t^{53} + t^{54}$

$28431861\,t^{52} + 24115080\,t^{53} + 8261473\,t^{54} + 1529246\,t^{55} + 172956\,t^{56} + 12794\,t^{57} + 656\,t^{58} + 24\,t^{59} + t^{60}$

$k = 8$

t

t^{8}

t^{14}

$4\,t^{20} + t^{21}$

$21\,t^{26} + 4\,t^{27} + t^{28}$

$183\,t^{32} + 53\,t^{33} + 8\,t^{34} + t^{35}$

$1918\,t^{38} + 704\,t^{39} + 106\,t^{40} + 8\,t^{41} + t^{42}$

$22908\,t^{44} + 10375\,t^{45} + 1825\,t^{46} + 170\,t^{47} + 12\,t^{48} + t^{49}$

$290511\,t^{50} + 156471\,t^{51} + 32934\,t^{52} + 3635\,t^{53} + 255\,t^{54} + 12\,t^{55} + t^{56}$

$3844688\,t^{56} + 2407227\,t^{57} + 599513\,t^{58} + 79651\,t^{59} + 6466\,t^{60} + 351\,t^{61} + 16\,t^{62} + t^{63}$

$52454248\,t^{62} + 37482092\,t^{63} + 10853332\,t^{64} + 1702405\,t^{65} + 163728\,t^{66} + 10336\,t^{67} + 468\,t^{68} + 16\,t^{69} + t^{70}$

$k = 9$

t

t^{9}

t^{16}

$4\,t^{23} + t^{24}$

$46\,t^{30} + 8\,t^{31} + t^{32}$

$494\,t^{37} + 128\,t^{38} + 12\,t^{39} + t^{40}$

$6532\,t^{44} + 2096\,t^{45} + 256\,t^{46} + 16\,t^{47} + t^{48}$

$90954\,t^{51} + 35788\,t^{52} + 5348\,t^{53} + 422\,t^{54} + 20\,t^{55} + t^{56}$

$1339448\,t^{58} + 624552\,t^{59} + 113582\,t^{60} + 10864\,t^{61} + 632\,t^{62} + 24\,t^{63} + t^{64}$

$20459857\,t^{65} + 11077108\,t^{66} + 2387924\,t^{67} +$ 3[illegible]75174 $t^{68} + 19194\,t^{69} + 880\,t^{70} + 28\,t^{71} + t^{72}$

$322092958\,t^{72} + 198865368\,t^{73} + 49851852\,t^{74} + 6782964\,t^{75} + 565666\,t^{76} + 30944\,t^{77} + 1174\,t^{78} + 32\,t^{79} + t^{80}$

Table 8: Coefficients of $\widetilde{a}(x,t)$ for $k = 6, 7, 8, 9$ and $n = 0, \ldots, 10$