COUNTING UNROOTED MAPS ON THE PLANE

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Abstract. A planar map is a 2-cell embedding of a connected planar graph, loops and parallel edges allowed, on the sphere. A *plane* map is a planar map with a distinguished outside ("infinite") face. An unrooted map is an equivalence class of maps under orientation-preserving homeomorphism, and a rooted map is a map with a distinguished oriented edge. Previously we obtained formulae for the number of unrooted planar n-edge maps of various classes, including all maps, non-separable maps, eulerian maps and loopless maps. In this article, using the same technique we obtain closed formulae for counting unrooted plane maps of all these classes and their duals. The corresponding formulae for rooted maps are known to be all sum-free; the formulae that we obtain for unrooted maps contain only a sum over the divisors of n. We count also unrooted two-vertex plane maps.

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0. Introduction

0.1. A *map* is a 2-cell embedding of an undirected connected graph, loops and parallel edges allowed, on an unbounded surface. If the surface is a sphere, then the map is a *planar map*; if the surface is an infinite plane, then the map is a *plane map* and one of its faces is distinguished as the *outside face*. Given any planar map or plane map, the number v of its vertices, the number n of its edges and the number f of its faces satisfy Euler's formula:

$$v + f = n + 2. \tag{0.1}$$

A *dart* of a map on an orientable surface is a half-edge or edge-end. A *homeomorphism* between two maps on orientable surfaces is a bicontinuous bijection between their embedding surfaces that takes the vertices, edges and faces of one map into the vertices, edges and faces of the other; in the case of plane maps, it also takes the outside face of one map into the outside face of the other. An *isomorphism* between two maps on orientable surfaces is an equivalence class of orientation-preserving homeomorphisms, where two homeomorphisms are considered equivalent if they induce the same bijection between the darts of one map and the darts of the other, and an *automorphism* of a map is an isomorphism between the map and itself. A map is *rooted* by distinguishing a dart, called the *root*. The only automorphism of a map that fixes the root is the trivial automorphism that fixes all its darts [16]. The following proposition is easily proved from the above definitions:

Proposition 0.1. *Distinguishing each of the f faces of a rooted planar map gives rise to f distinct rooted plane maps.* □

An *isthmus* is an edge that is incident on both sides to the same face; thus, an isthmus is the face-vertex dual of a loop. The *valency* of a vertex is the number of edges incident to it, loops counting twice. The *valency* of a face is the number of edges on its boundary, isthmuses counting twice.

0.2. An effective method of counting rooted planar n-edge maps was developed by Tutte in the 1960s [16]. For several important classes of maps, in particular, the classes of arbitrary, non-separable, loopless, eulerian and unicursal maps (see below), the corresponding enumerative formulae turned out to be sum-free and remarkably simple.

An *unrooted* map is an isomorphism class of maps. A general method of counting unrooted planar maps was developed by the first-named author in 1978 and published in [6]. It is based on the consideration of rotational automorphisms of maps and their corresponding quotient maps. In particular, a formula for the number of all unrooted planar maps with n edges was obtained. Other classes of unrooted planar maps that have been counted by number of edges (mainly with the aid of the above-mentioned method) are non-separable [10], eulerian (all vertices of even valency) and unicursal (exactly two vertices of odd valency) [11], loopless [12], one-face (trees) [17] and two-face [1]. In all these cases, the corresponding formulae contain a summation only over the divisors of n. Moreover, for the majority of these classes, the second-named author [19] has obtained simple formulae for counting maps by two parameters: numbers of vertices and faces.

In the present paper we are concerned with the related problem of counting unrooted plane maps. At first sight, it is simpler than counting unrooted planar maps since one still can apply the general enumeration scheme and the set of symmetries of plane maps is more restrictive. In particular, the techniques of [19] can be easily adapted to counting unrooted plane maps by numbers of vertices and faces: in the corresponding formulae for planar maps, one needs only to modify slightly a factor in the summands that correspond to rotations around axes that intersect one or two faces and to nullify the other summands. Summing these quantities over v and f which satisfy (0.1) gives rise to the number of unrooted plane maps with n edges. However the problem becomes more challenging if we look for a more efficient one-parametric solution, provided such a solution exists. The available simple one-parametric formulae for unrooted planar n-edge maps of several types (including the above-mentioned ones) suggest the existence of similar, or simpler, formulae (with the summation only over the divisors of n) for unrooted plane maps of the same types. The present research confirms such a general prediction with, however, one notable exception. We conjecture more specifically (although informally) that an efficient enumeration (in the above-mentioned sense) of unrooted plane n-edge maps is possible whenever such an enumeration holds for unrooted *planar* n-edge maps and also when there exists a sum-free formula for *rooted* plane n-edge maps, where (as adopted throughout this paper) the root may or may not be incident to the distinguished outside face.

0.3. The class of all planar maps and of non-separable planar maps are *self-dual*: for every map in the class, its face-vertex dual is also in the class. The remaining classes listed above are not self-dual; the corresponding dual classes are bipartite (all faces of even valency), dual-unicursal (exactly two faces of odd valency), isthmusless and two-vertex maps. By face-vertex duality, there are exactly as many rooted or unrooted n-edge maps in a non-self-dual class as there are in its dual class.

For a class of planar maps $\mathcal{M}$, let $M'(n), M^+(n), M^+_{\mathsf{f}}(n)$ and $M^+_{\mathsf{v}}(n)$ stand for the number of, respectively, rooted maps, unrooted maps, unrooted maps with a distinguished face and unrooted maps with a distinguished vertex in $\mathcal{M}$ with n edges. Obviously, $M^+_{\mathsf{f}}(n)$ is the number of unrooted *plane* maps in $\mathcal{M}$, whereas $M^+_{\mathsf{v}}(n)$ is the number of unrooted *plane* maps in the dual class $\mathcal{M}^*$. We also consider the number rooted maps with a distinguished face $M'_{\mathsf{f}}(n)$ or vertex $M'_{\mathsf{v}}(n)$. For a self-dual class it follows from (0.1) that

$$M'_{\mathsf{f}}(n) = M'_{\mathsf{v}}(n) = \frac{n+2}{2} M'(n). \tag{0.2}$$

However, if $\mathcal{M}$ is not self-dual, an efficient calculation of these quantities is not a trivial problem. In two cases under consideration, namely for eulerian and loopless maps (and, naturally, for their duals), it was in fact solved successfully in [4]. Using these results, together with the recent enumeration of unrooted eulerian and loopless planar maps, we found simple enumerative formulae for unrooted eulerian and loopless plane n-edge maps. In contrast, in the case of unicursal maps (in spite of the fact that they are closely related to eulerian maps), simple enumerative formulae are known only for rooted and unrooted planar maps.

0.4. In Section 1 we find a general relationship between the quantities $M^+(n), M^+_{\mathsf{f}}(n)$ and $M^+_{\mathsf{v}}(n)$. The formula for $(M^+_{\mathsf{f}}(n) + M^+_{\mathsf{v}}(n))/2 - M^+(n)$, or for $M^+_{\mathsf{f}}(n) - M^+(n)$ if the class is self-dual, turns out to be considerably simpler than any of ones for the three quantities separately, so that if any two of them are known (either one of them in the case of a self-dual class), then the remaining one is easy to calculate. The general formulae are then applied to obtain closed formulae for the number of all plane maps and non-separable plane maps (Section 2), eulerian and bipartite plane maps (Section 3), loopless and isthmusless plane maps (Section 4) and two-vertex plane maps (Section 5). Finally we obtain a formula for rooted unicursal plane n-edge maps. Numerical tables are given in an appendix.

For uniformity and convenience, we represent our results in two equivalent forms: explicitly in terms of integer-fold binomials $\binom{mn}{n}$ and reductively in terms of the corresponding rooted enumerators. The former is motivated by the recent research [9], which showed certain advantages of such a representation of enumerative formulae. The latter looks neater sometimes.

The present results are heavily based not only on the general enumerative scheme and our previous counting formulae [10, 11, 12, 4] as was pointed out above, but also on some details of their proofs.

There is an alternative better-known two-parametric approach where plane maps are specified by n and the size (valency) m of the outside face. A related idea is to consider rooted plane maps subject to the restriction that the root is incident to the outside face. Sometimes, this approach results in quite efficient formulae for rooted and unrooted two-parametric plane maps (cf., e.g., [2]). It is generally unclear, however, when and how in this way one can get rid of the parameter m.

1. **Plane vs. planar**

1.1. The following general scheme for counting unrooted planar n-edge maps comes from [6] (or [8]), to which the reader is referred for additional details (cf. also [12]).

By Burnside's lemma,

$$M^+(n) = \frac{1}{2n}\sum_{\rho} \mathrm{fix}(n,\rho), \tag{1.1}$$

where $\mathrm{fix}(n,\rho) = \mathrm{fix}(n,\rho,\mathcal{M})$ is the number of rooted maps of $\mathcal{M}$ fixed by ρ - that is, for which ρ is an automorphism of the unrooted version of the map - and ρ runs over all the permutations of the $2n$ darts that can be an automorphism of an n-edge map (in particular, such a permutation has to consist of independent cycles of equal length [6]).

The identity permutation, id, which fixes all the darts, is an automorphism of every n-edge map; so the contribution of this permutation to (1.1) is

$$\mathrm{fix}(n,id) = M'(n). \tag{1.2}$$

Any non-trivial automorphism ρ of a planar map can be represented geometrically as a rotation of the sphere about an axis that intersects the map in two *elements* (vertices, edges or faces) called *axial elements*. A rooted map Γ fixed by an automorphism ρ of order $p \geq 2$ (where ρ is not assumed to fix the root) can be represented as p isomorphic copies of a rooted map Δ, called the *quotient map* of Γ with respect to ρ and denoted by Γ/ρ. To each non-axial element of Δ there correspond p non-axial elements of Γ and to each of the two axial elements of Δ there corresponds a single axial element of Γ whose valency is p times the number of darts incident to the corresponding axial element of Δ. An axial edge of Δ has only one dart; so to make Δ a map we complete this half-edge with a vertex of valency 1 called a *singular vertex*; the single dart contained by the singular vertex was added along with the vertex and is therefore not the root. Given a rooted map Δ with at most two singular vertices, two elements chosen to be axial (which are either vertices or faces and must include all the singular vertices) and a non-trivial automorphism ρ (which must be of order 2 if Δ contains at least one singular vertex), there is a unique rooted map Γ (the *lifting* of Δ) such that $\Delta = \Gamma/\rho$; so $\mathrm{fix}(n,\rho)$ is equal to the number of rooted maps that are the quotient maps of some rooted n-edge map with respect to ρ.

For $i = 0, 1$ and 2, let $Q'_i(n) = Q'_i(n,\mathcal{M})$ be the number of rooted quotient maps with i singular vertices of all the rooted n-edge maps of a given class $\mathcal{M}$. If $i = 1$, then the quotient map has $(n+1)/2$ edges, so that n must be odd. If $i = 2$, then the quotient map has $(n+2)/2$ edges, so that n must be even. Substituting these values and (1.2) into (1.1) we obtain the following general formula:

$$M^+(n) = \frac{1}{2n}\left[M'(n) + Q'_0(n) + \begin{cases} Q'_1(n) & \text{if } n \text{ is odd} \\ Q'_2(n) & \text{if } n \text{ is even} \end{cases}\right]. \tag{1.3}$$

Now we consider the number $M^+_{\mathsf f}(n)$ of unrooted n-edge maps of a given class $\mathcal M$ with a distinguished outside face. The analogue of (1.1) is

$$M^+_{\mathsf f}(n) = \frac{1}{2n}\sum_{\rho}{}' \operatorname{fix}(n,\rho), \tag{1.4}$$

where $\sum'$ indicates that ρ must also fix the distinguished face.

Suppose that ρ is the trivial automorphism. The analogue of (1.2) is the number $M'_{\mathsf f}(n)$ of rooted n-edge maps of the same class with a distinguished face. By Proposition 0.1 and formula (0.1),

$$M'_{\mathsf f}(n) = \sum_{v+f=n+2} fM'(v,f), \tag{1.5}$$

where $M'(v,f)$ is the number of rooted maps of that class with v vertices and f faces. Clearly this number is also equal to the total number of faces in all the rooted n-edge maps of that class (cf. [4]).

For $i = 0, 1$ and 2, let $Q'_{\mathsf f,i}(n) = Q'_{\mathsf f,i}(n,\mathcal M)$ be the number of rooted quotient maps with i singular vertices of all the rooted n-edge maps of the given class under non-trivial automorphisms one of whose axial elements is the distinguished face. If $i = 1$, then as in the case of planar maps n must be odd. If $i = 2$, then as before n must be even; also, since both of the axial elements are the singular vertices, neither of them can be the distinguished face, so that $Q'_{\mathsf f,2}(n) = 0$. The analogue of (1.3) is thus

$$M^+_{\mathsf f}(n) = \frac{1}{2n}\left[M'_{\mathsf f}(n) + Q'_{\mathsf f,0}(n) + \begin{cases} Q'_{\mathsf f,1}(n) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}\end{cases}\right]. \tag{1.6}$$

1.2. The number of unrooted n-edge maps with a distinguished face in the dual class $\mathcal M^*$ is equal to the number $M^+_{\mathsf v}(n)$ of unrooted n-edge maps in the primal class $\mathcal M$ with a distinguished vertex. For $i = 0, 1$ and 2, let $Q'_{\mathsf v,i}(n) = Q'_{\mathsf v,i}(n,\mathcal M)$ be the number of rooted quotient maps with i singular vertices of all the rooted n-edge maps of the given class $\mathcal M$ under non-trivial automorphisms one of whose axial elements is the distinguished vertex. Using the face-vertex dual of Proposition 0.1 we obtain the following analogue of (1.6):

$$M^+_{\mathsf v}(n) = \frac{1}{2n}\left[M'_{\mathsf v}(n) + Q'_{\mathsf v,0}(n) + \begin{cases} Q'_{\mathsf v,1}(n) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}\end{cases}\right], \tag{1.7}$$

where $M'_{\mathsf v}(n)$ is the number of rooted maps with a distinguished vertex:

$$M'_{\mathsf v}(n) = \sum_{v+f=n+2} vM'(v,f). \tag{1.5$'$}$$

It follows directly from (1.5), (1.5$'$) and Euler's formula (0.1) that

$$M'_{\mathsf f}(n) + M'_{\mathsf v}(n) = (n+2)M'(n). \tag{1.8}$$

Proposition 1.1. *For any class of maps $\mathcal M$,*

$$Q'_{\mathsf f,0}(n) + Q'_{\mathsf v,0}(n) = 2Q'_0(n). \tag{1.9}$$

Proof. The number $Q'_0(n)$ is the total number of ways of choosing an unordered pair of axial elements, each of which may be either a face or a vertex, in all the rooted planar maps that are the quotient maps with no singular vertices of some rooted planar n-edge map of the given class. Since the maps are rooted, all unordered pairs of elements are distinct. If the axial elements are now labelled north and south, the pairs of axial elements are now ordered, increasing the number of pairs to $2Q'_0(n)$. If the north axial element is a vertex, then it can be declared the distinguished vertex, and the total number ways of distinguishing a vertex, making it the north axial element and then choosing the south axial element is

$Q'_{\mathsf{v},0}(n)$. If the north axial element is a face, then it can be declared the distinguished face, and the total number of ways of distinguishing a face, making it the north axial element and then choosing the south axial element is $Q'_{\mathsf{f},0}(n)$. Then (1.9) follows from the fact that the north axial element must be either a vertex or a face. □

Proposition 1.2. *For any class of maps $\mathcal{M}$,*

$$Q'_{\mathsf{f},1}(n) + Q'_{\mathsf{v},1}(n) = Q'_1(n). \tag{1.10}$$

Proof. The number $Q'_1(n)$ is the total number of ways of choosing the axial element that isn't the singular vertex in all the rooted planar maps that are the quotient maps with one singular vertex of some rooted planar n-edge map of the given class. If the non-singular axial element is a vertex, then it can be declared the distinguished vertex, and the total number of ways of distinguishing a vertex is $Q'_{\mathsf{v},1}(n)$. If the non-singular element is a face, then it can be declared the distinguished face, and the total number of ways of distinguishing a face is $Q'_{\mathsf{f},1}(n)$. Then (1.10) follows from the same fact as (1.9). □

Proposition 1.3. *For any class of maps $\mathcal{M}$,*

$$M^+_{\mathsf{f}}(n) + M^+_{\mathsf{v}}(n) = \frac{1}{2n}\left[(n+2)M'(n) + 2Q'_0(n) + \begin{cases} Q'_1(n) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}\right]. \tag{1.11}$$

Proof. This formula follows directly from (1.6) – (1.10). □

Proposition 1.4. *For any self-dual class of maps $\mathcal{M} = \mathcal{M}^*$,*

$$2M^+_{\mathsf{f}}(n) = \frac{1}{2n}\left[(n+2)M'(n) + 2Q'_0(n) + \begin{cases} Q'_1(n) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}\right]. \tag{1.12}$$

Proof. This formula follows directly from (1.8), (1.11) and (0.2). □

1.3. Comparing (1.11) and (1.12) with (1.3) we see that we can eliminate $Q'_0(n)$ by subtracting twice formula (1.3) from either (1.11) or (1.12). The respective formulae are the following.

Theorem 1.5. *For any class of planar maps $\mathcal{M}$,*

$$M^+_{\mathsf{f}}(n) + M^+_{\mathsf{v}}(n) = 2M^+(n) + \frac{1}{2}M'(n) - \begin{cases} \frac{1}{2n}Q'_1(n) & \text{if } n \text{ is odd} \\ \frac{1}{n}Q'_2(n) & \text{if } n \text{ is even.} \end{cases} \tag{1.13}$$

If $\mathcal{M}$ is self-dual, then

$$M^+_{\mathsf{f}}(n) = M^+(n) + \frac{1}{4}M'(n) - \begin{cases} \frac{1}{4n}Q'_1(n) & \text{if } n \text{ is odd} \\ \frac{1}{2n}Q'_2(n) & \text{if } n \text{ is even.} \end{cases} \tag{1.14}$$

□

Note that the left-hand side quantity of (1.13) may be interpreted as the number $\widehat{M}^+(n)$ of unrooted maps in $\mathcal{M}$ with one face *or* vertex distinguished (and fixed by automorphisms).

Generally, since $Q'_0(n)$ contains a sum over divisors of n whereas $Q'_1(n)$ is a single term, eliminating $Q'_0(n)$ rather than $Q'_1(n)$ leads to a formula that is more elegant and computationally more efficient provided that we have a table of values of $M^+(n)$.

2. **Arbitrary and non-separable maps**

2.1. For the class of arbitrary planar maps $\mathcal{A}$, the number of unrooted n-edge maps is given by the following formula [6]:

$$A^+(n) = \frac{1}{2n}\left[A'(n) + \sum_{\substack{k<n\\ k|n}} \phi\left(\frac{n}{k}\right)\binom{k+2}{2} A'(k)\right] + \begin{cases} \frac{n+3}{4} A'\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \\ \frac{n-1}{4} A'\left(\frac{n-2}{2}\right) & \text{if } n \text{ is even,} \end{cases} \tag{2.1}$$

where $\phi(n)$ is the Euler totient function and $A'(n)$, the number of all rooted planar n-edge maps, is given by the following formula [16]:

$$A'(n) = \frac{2\cdot 3^n (2n)!}{n!\,(n+2)!} = \frac{2\cdot 3^n}{(n+1)(n+2)}\binom{2n}{n}, \qquad n \geq 0. \tag{2.2}$$

Since the class of all planar n-edge maps is self-dual, we obtain immediately from (1.12) and (2.1) the following formula for the number $A^+_{\mathsf{f}}(n)$ of all unrooted plane n-edge maps:

Proposition 2.1. *For all unrooted plane maps,*

$$A^+_{\mathsf{f}}(n) = \frac{1}{2n}\left[\frac{n+2}{2} A'(n) + \sum_{\substack{k<n\\ k|n}} \phi\left(\frac{n}{k}\right)\binom{k+2}{2} A'(k)\right] + \begin{cases} \frac{n+3}{8} A'\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{2.3}$$

Equivalently, from (2.2) *we obtain*

$$A^+_{\mathsf{f}}(n) = \frac{1}{2n}\left[\frac{3^n}{n+1}\binom{2n}{n} + \sum_{\substack{k<n\\ k|n}} \phi\left(\frac{n}{k}\right)3^k\binom{2k}{k}\right] + \begin{cases} \frac{3^{(n-1)/2}}{n+1}\binom{n-1}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{2.3$'$}$$

□

In order to work only with integers when substituting into these formulae (and similar subsequent ones), the last term should be multiplied by $2n$ and inserted into the brackets. In the discussion that follows these formulae, the expressions "first term" and "second term" refer to the first and second term, respectively, between the brackets.

From (1.14) we obtain the following simple expression for $A^+_{\mathsf{f}}(n)$ in terms of $A^+(n)$:

Corollary 2.2.

$$A^+_{\mathsf{f}}(n) = A^+(n) + \frac{1}{4}A'(n) - \begin{cases} \frac{n+3}{8} A'\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \\ \frac{n-1}{4} A'\left(\frac{n-2}{2}\right) & \text{if } n \text{ is even.} \end{cases} \tag{2.4}$$

□

This formula was announced without proof in [7] and (with minor misprints) in [8].

2.2. For the class of non-separable planar maps $\mathcal{B}$, the number of unrooted n-edge maps is given by the following formula [10]:

$$B^+(n) = \frac{1}{2n}\left[B'(n) + \sum_{\substack{k<n\\ k|n}} \phi\left(\frac{n}{k}\right)\binom{3k-1}{2} B'(k)\right] + \begin{cases} \frac{n+1}{4} B'\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd} \\ \frac{3n-4}{16} B'\left(\frac{n}{2}\right) & \text{if } n \text{ is even,} \end{cases} \tag{2.5}$$

where $B'(n)$, the number of rooted non-separable planar n-edge maps, is given by the following formula [16]:

$$B'(n) = \frac{2(3n-3)!}{n!\,(2n-1)!} = \frac{4}{3(3n-2)(3n-1)}\binom{3n}{n}, \qquad n \geq 1. \tag{2.6}$$

Since this class too is self-dual, we obtain from (1.12) and (2.5) the following formula for the number $B^+_{\mathsf{f}}(n)$ of unrooted non-separable plane n-edge maps:

Proposition 2.3. *For unrooted non-separable plane maps,*

$$B^+_{\mathsf{f}}(n) = \frac{1}{2n}\left[\frac{n+2}{2}B'(n) + \sum_{\substack{k<n\\ k|n}} \phi\left(\frac{n}{k}\right)\binom{3k-1}{2}B'(k)\right] + \begin{cases} \frac{n+1}{8}\,B'\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.}\end{cases} \tag{2.7}$$

Equivalently, from (2.6) *we obtain*

$$\begin{aligned} B^+_{\mathsf{f}}(n) &= \frac{1}{3n}\left[\frac{(n+2)}{(3n-2)(3n-1)}\binom{3n}{n} + \sum_{k<n,\, k|n} \phi\left(\frac{n}{k}\right)\binom{3k}{k}\right] \\ &+ \begin{cases} \frac{2(n+1)}{3(3n-1)(3n+1)}\binom{3\frac{n+1}{2}}{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.}\end{cases} \end{aligned} \tag{2.7'}$$

□

Now, from (1.14) we obtain the following simple expression for $B^+_{\mathsf{f}}(n)$ in terms of $B^+(n)$:

Corollary 2.4.

$$B^+_{\mathsf{f}}(n) = B^+(n) + \frac{1}{4}B'(n) - \begin{cases} \frac{n+1}{8}\,B'\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd} \\ \frac{3n-4}{16}\,B'\left(\frac{n}{2}\right) & \text{if } n \text{ is even.}\end{cases} \tag{2.8}$$

□

The numerical values of the functions $A'_{\mathsf{f}}(n), A^+_{\mathsf{f}}(n), B'_{\mathsf{f}}(n)$ and $B^+_{\mathsf{f}}(n)$ up to $n = 20$ are given in Table 1 in the Appendix. Note that the values of $B^+_{\mathsf{f}}(n)$ for $n = 2, \ldots, 7$ were calculated by Brown [2, Table III] via a formula containing a complicated multiple sum. This is the sequence A000087 in Sloane's Encyclopedia [15].

3. **Eulerian and bipartite maps**

3.1. For the class of eulerian planar maps $\mathcal{E}$, the number of rooted n-edge maps is [18]

$$E'(n) = \frac{3\cdot 2^{n-1}(2n)!}{n!\,(n+2)!} = \frac{3\cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n}, \qquad n \geq 1. \tag{3.1}$$

From (3.1) and a result of [8] the authors showed in [11] that the number of unrooted eulerian planar maps with n edges is given by the following formula:

$$\begin{aligned} E^+(n) &= \frac{1}{2n}\left[\frac{3\cdot 2^n}{2(n+1)(n+2)}\binom{2n}{n} + 3\sum_{k<n,\, k|n} \phi\left(\frac{n}{k}\right)2^{k-2}\binom{2k}{k}\right] \\ &+ \begin{cases} \frac{2^{(n-1)/2}}{n+1}\binom{n-1}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ \frac{1}{n}\sum_{k|\frac{n}{2}} \phi\left(\frac{n}{k}\right)2^{k-3}\binom{2k}{k} + \frac{2^{(n-4)/2}}{n+2}\binom{n}{\frac{n}{2}} & \text{if } n \text{ is even.}\end{cases} \end{aligned} \tag{3.2}$$

The first term on the right-hand side of (3.2) is contributed by the trivial automorphism, the second term by the non-trivial automorphisms under which the quotient maps are eulerian, the term for odd n by quotient maps with one singular vertex, the first term for even n by unicursal quotient maps with no singular vertices and the remaining term by quotient maps with two singular vertices.

In a slightly more convenient form,

$$E^+(n) = \frac{1}{2n}\left[\frac{3\cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n} + \sum_{k<n,\,k|n}\phi\Big(\frac{n}{k}\Big)\Big(\delta\Big(\frac{n}{k}\Big)+2\Big)2^{k-2}\binom{2k}{k}\right] + \begin{cases} \dfrac{2^{(n-1)/2}}{n+1}\dbinom{n-1}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ \dfrac{2^{(n-4)/2}}{n+2}\dbinom{n}{\frac{n}{2}} & \text{if } n \text{ is even,} \end{cases} \tag{3.2$'$}$$

where

$$\delta(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases} \tag{3.3}$$

Suppose now that a face is distinguished and called the north axial element. The following proposition was proved in [4].

Proposition 3.1. *The number of rooted eulerian planar maps with n edges and a distinguished vertex is given by the formula*

$$E'_{\mathsf{v}}(n) = \frac{n+2}{3}\,E'(n) = \frac{2^{n-1}}{n+1}\binom{2n}{n}. \tag{3.4}$$

□

By (1.8) and (3.4), the number of rooted eulerian planar maps with n edges and a distinguished face is $E'_{\mathsf{f}}(n) = (n+2)E'(n) - E'_{\mathsf{v}}(n) = (n+2)E'(n) - (n+2)E'(n)/3$. By (3.1) we have

$$E'_{\mathsf{f}}(n) = \frac{2(n+2)}{3}\,E'(n) = \frac{2^n}{n+1}\binom{2n}{n}. \tag{3.5}$$

For each value of k in the second term of the right-hand side of (3.2), the factor of $\phi(n/k)$ is equal to $E'(k)$ multiplied by the number of choices of axial pairs, which is $(k+2)(k+1)/2$ because any one of the $k+2$ vertices and faces can be chosen as an axial element, and the axial elements are not distinguished. But now we are distinguishing a face and calling it the north axial element. If a given quotient map has v vertices, then it has $f = k+2-v$ faces. Since the north axial element must be the distinguished face, there are $k+2-v$ ways of distinguishing the face and calling it the north axial element. The south axial element can then be chosen from any of the $k+1$ vertices and other faces, so that the number of choices of ordered axial pairs is $(k+1)(k+2-v)$. By an argument similar to the derivation of (3.5), we replace the factor $\frac{(k+1)(k+2)}{2}E'(k)$ in the second term on the right-hand side of (3.2) by $(k+1)(k+2)E'(k) - (k+1)(k+2)E'(k)/3$ to obtain $\sum_{k<n,\,k|n}\phi\big(\frac{n}{k}\big)2^{k-1}\binom{2k}{k}$.

The remaining terms on the right-hand side of (3.2) are all contributed by quotient maps whose axial elements are vertices, either singular or non-singular; so none of these terms contribute to the number of eulerian maps with a distinguished (axial) face; so

$$Q'_{\mathsf{f},0}(n) = \sum_{k<n,\,k|n}\phi\Big(\frac{n}{k}\Big)2^{k-1}\binom{2k}{k} \tag{3.6}$$

and $Q'_{\mathsf{f},1}(n) = 0$. Substituting these values into (1.6) we obtain

Theorem 3.2. *The number of unrooted eulerian plane maps with n edges is given by*

$$E^+_{\mathsf{f}}(n) = \frac{1}{2n}\left[\frac{2^n}{n+1}\binom{2n}{n} + \sum_{k<n,\, k|n} \phi\Big(\frac{n}{k}\Big) 2^k \binom{2k}{k}\right]. \tag{3.7}$$

Equivalently, from (3.5) *we obtain*

$$E^+_{\mathsf{f}}(n) = \frac{1}{3n}\left[(n+2)E'(n) + 2\sum_{k<n,\, k|n} \phi\Big(\frac{n}{k}\Big)\binom{k+2}{2} E'(k)\right]. \tag{3.7$'$}$$

□

Rather unexpectedly, the right-hand side of (3.7) does not contain an additional term for odd n.

3.2. It is well known that a planar map is eulerian if and only if its dual is bipartite (see, e.g., [18]). From (1.11), (3.1), (3.2) and (3.7) we obtain

Corollary 3.3. *The number of unrooted bipartite plane maps with n edges is given by*

$$E^+_{\mathsf{v}}(n) = \frac{1}{2n}\left[\frac{2^{n-1}}{n+1}\binom{2n}{n} + \sum_{\substack{k<n\\ k|n}} \phi\Big(\frac{n}{k}\Big)\delta\Big(\frac{n}{k}\Big)2^{k-1}\binom{2k}{k}\right] + \begin{cases} \dfrac{2^{(n-1)/2}}{n+1}\dbinom{n-1}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even,} \end{cases} \tag{3.8}$$

where $\delta(n)$ is the function defined by (3.3)*. Equivalently for $n \geq 2$,*

$$\begin{aligned} E^+_{\mathsf{v}}(n) &= \frac{1}{6n}\left[(n+2)E'(n) + 2\sum_{k<n,\, k|n} \phi\Big(\frac{n}{k}\Big)\delta\Big(\frac{n}{k}\Big)\binom{k+2}{2}E'(k)\right] \\ &+ \begin{cases} \dfrac{n+3}{6}\, E'\Big(\dfrac{n-1}{2}\Big) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned} \tag{3.8$'$}$$

□

From (1.13), (3.1) and (3.2$'$) we obtain the following identity: for $n \geq 2$,

$$E^+_{\mathsf{f}}(n) + E^+_{\mathsf{v}}(n) = 2E^+(n) + \frac{1}{2}E'(n) - \begin{cases} \dfrac{n+3}{6}\, E'\Big(\dfrac{n-1}{2}\Big) & \text{if } n \text{ is odd} \\ \dfrac{n+4}{12}\, E'\Big(\dfrac{n}{2}\Big) & \text{if } n \text{ is even.} \end{cases} \tag{3.9}$$

Initial values of the functions $E'_{\mathsf{f}}(n), E^+_{\mathsf{f}}(n), E'_{\mathsf{v}}(n)$ and $E^+_{\mathsf{v}}(n)$ are given in Table 2 in the Appendix.

4. **Loopless and isthmusless maps**

This section contains the most difficult results of the present paper.

4.1. For the class of loopless planar maps $\mathcal{L}$, the number of rooted maps with n edges is [20]

$$L'(n) = \frac{2(4n+1)!}{(n+1)!\,(3n+2)!} = \frac{2(4n+1)}{(n+1)(3n+1)(3n+2)}\binom{4n}{n}, \qquad n \geq 0. \tag{4.1}$$

It was shown by the authors in [12] that the number of unrooted loopless planar maps with n edges is given by

$$L^+(n) = \frac{1}{2n}\left[\frac{2(4n+1)}{(n+1)(3n+1)(3n+2)}\binom{4n}{n} + \sum_{k<n,\, k|n} \phi\left(\frac{n}{k}\right)\binom{4k}{k}\right] + \begin{cases} \dfrac{1}{n+1}\dbinom{2n}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ \dfrac{1}{2n}\dbinom{2n}{\frac{n-2}{2}} & \text{if } n \text{ is even.} \end{cases} \qquad (4.2)$$

The first term on the right-hand side of (4.2) is contributed by the trivial automorphism, the second term by the non-trivial automorphisms under which the quotient maps are loopless, the term for odd n by quotient maps with one singular vertex and the term for even n by quotient maps with two singular vertices.

Suppose now that a face is distinguished and called the north axial element. The following propositions were proved in [4].

Proposition 4.1. *The number of rooted loopless planar maps with n edges and a distinguished face is*

$$L'_{\mathsf{f}}(n) = \frac{1}{3n+1}\binom{4n}{n} = \frac{(n+1)(3n+2)}{2(4n+1)}\, L'(n). \qquad (4.3)$$

□

Proposition 4.2. *The number of rooted loopless planar maps with n edges and a distinguished vertex is*

$$L'_{\mathsf{v}}(n) = \frac{5n^2+13n+2}{(n+1)(3n+1)(3n+2)}\binom{4n}{n} = \frac{5n^2+13n+2}{2(4n+1)}\, L'(n). \qquad (4.4)$$

□

We now evaluate the remaining terms in (1.6) specialized to loopless maps, the distinguished-face analogue of (4.2), by following the argument used in [12], modifying it wherever necessary to account for the distinguished face. The quotient map of a loopless map under a non-trivial automorphism is either a loopless map or a nested sequence of loopless maps $M_1, \ldots, M_k$ with each pair of adjacent components of the sequence separated by a loop. In the latter case, one axial element is in the extremal components M_1 with a edges and the other axial element is in the other extremal component M_k with b edges; also, an axial element is not allowed to be the vertex in its component incident to the loop separating that component from the adjacent one in the sequence. We suppose for the moment that the quotient map has n edges; later we will substitute the appropriate number of edges into the enumeration formula for rooted quotient maps we obtain below.

Suppose that the quotient map has no singular vertices.

If the quotient map is not loopless, then the number of such maps *without a distinguished face and with the axial elements not distinguished from each other* is given by formula (4.5), which is a corrected version of formula (16) of [12] in which it is not assumed that $a \geq b$ (it is (4.5) which leads to the enumeration formula obtained in [12]).

$$n \sum_{\substack{a,b\geq 0 \\ a+b\leq n}} (a+1)L'(a)(b+1)L'(b) \cdot [x^{n-(a+b)-1}]\,(1+z), \qquad (4.5)$$

where

$$z = x(1+z)^4 \qquad (4.6)$$

and $[x^i]f(x)$ means the coefficient of x^i in the power series expansion of the function $f(x)$.

We modify (4.5) so as to account for the fact that the axial elements are now distinguished from one another and that one of them is the distinguished face. Distinguishing the axial elements multiplies (4.5) by 2. Without loss of generality we call the extremal component containing the distinguished face M_k, which has b edges. In (4.5), the factor $(b+1)L'(b)$ was obtained by taking all the b-edge rooted loopless maps and choosing any of the $b+1$ faces or vertices except the forbidden one (incident to the loop) to be the axial element. Instead, we choose any face of M_k to be the axial element; so that $(b+1)L'(b)$ must be replaced by the total number of faces in all the rooted loopless maps with b edges, which is given by (4.3) with n replaced by b.

We reproduce formula (17) of [12] (with a replaced by b) as (4.7):

$$\sum_{b=0}^{\infty}(b+1)L'(b)\,x^b = (1+z)^2. \tag{4.7}$$

The analogous formula that must replace (4.7) is given in the following proposition.

Proposition 4.3.

$$\sum_{b=0}^{\infty} \frac{1}{3b+1}\binom{4b}{b}\,x^b = (1+z). \tag{4.8}$$

Proof. The formula for Lagrange inversion (see, e.g., [5] for a combinatorial proof) in the special case when $z = xg(z)$ (instead of the general case $z = a + xg(z)$) can be simplified to

$$[x^0]f(z) = f(0); \qquad [x^n]f(z) = \frac{1}{n}[z^{n-1}]\left(f'(z)(g(z))^n\right) \tag{4.9}$$

for all $n \geq 1$. Here $g(z)$ is given by (4.6) as $(1+z)^4$. Equating the coefficient of x^n in the left-hand side of (4.8) with (4.9) we find that $f'(z) = 1$, so that $f(z) = z + C$, where C is some constant. Since the coefficient of x^0 in the left-hand side of (4,8) is 1, by the first equation of (4.9) we have $f(0) = 1$, whence we obtain (4.8). □

Comparing (4.7) with (4.8) and recalling that we must multiply by 2, we see that instead of formula (18) of [12], which is evaluated from (4.5) and is equal to $n[x^{n-1}](1+z)^5$, we must use

$$2n\,[x^{n-1}]\,(1+z)^4. \tag{4.10}$$

Applying Lagrange inversion to (4.10) we obtain the formula

$$8\,\frac{n}{n-1}\,[z^{n-2}]\,(1+z)^{4n-1} = \frac{2n}{3n+1}\binom{4n}{n}. \tag{4.11}$$

If the quotient map is loopless, then the north axial element is the distinguished face and the south axial element can be any of the other $n+1$ vertices and faces; so the number of these quotient maps is given by

$$\frac{n+1}{3n+1}\binom{4n}{n}. \tag{4.12}$$

Adding (4.11) to (4.12) we obtain the total number of n-edge quotient maps with no singular vertices of loopless maps, given by formula (4.13):

$$\binom{4n}{n}. \tag{4.13}$$

Now if the automorphism is of order n/k, then there are $\phi(n/k)$ such automorphisms and the quotient map will have k edges, so that (see formula (1.6))

$$Q'_{\mathsf{f},0}(n) = \sum_{k<n,\,k|n} \phi\Big(\frac{n}{k}\Big)\binom{4k}{k}, \tag{4.14}$$

which is also equal to $Q'_0(n)$, the second term on the right-hand side of (4.2).

Remark. A bijective proof of this equality would be interesting.

Suppose now that the quotient map has one singular vertex. Again for the moment we suppose the quotient map to have n edges.

Suppose the quotient map is not loopless. Then the number of such maps *without a distinguished face but with the south axial element being the distinguished face* is given by (4.15), which is formula (23) of [12]:

$$\frac{2n-1}{2}\sum_{\substack{a,b\geq 0\\ a+b\leq n-1}}(2a+1)L'(a)(b+1)L'(b)\cdot[x^{n-(a+b)-2}]\,(1+z). \tag{4.15}$$

Since the axial elements are already distinguished, to account for the distinguished face we need not multiply by 2; we just replace $(b+1)L'(b)$ by (4.3) with n replaced by b, which means that instead of simplifying (4.15) to

$$\frac{2n-1}{2}\,[x^{n-2}]\,(1+z)^6, \tag{4.16}$$

which is formula (25) of [12], we obtain

$$\frac{2n-1}{2}\,[x^{n-2}]\,(1+z)^5. \tag{4.17}$$

Applying Lagrange inversion to (4.17) and simplifying, we obtain the expression

$$\frac{5(2n-1)(4n-4)!}{(n-2)!\,(3n-1)!}. \tag{4.18}$$

Suppose the quotient map is loopless. Without the distinguished face, the number of such maps is given by formula (4.19), which is formula (28) of [12]:

$$n(2n-1)L'(n-1). \tag{4.19}$$

The factor n represents the number of choices of the north axial element. To account for the distinguished face, we replace $nL'(n-1)$ by (4.3) with n replaced by $n-1$ and obtain

$$\frac{(2n-1)(4n-4)!}{(n-1)!\,(3n-2)!}. \tag{4.20}$$

Adding (4.20) to (4.18) we obtain the total number of n-edge quotient maps with one singular vertex of loopless maps, given by formula (4.21):

$$\binom{4n-2}{n-1}. \tag{4.21}$$

Now the quotient map of an n-edge map will have not n edges but $(n+1)/2$. By replacing n by $(n+1)/2$ in (4.21), we find that

$$Q'_{\mathsf{f},1}(n) = \binom{2n}{\frac{n-1}{2}}. \tag{4.22}$$

Substituting from (4.3), (4.14) and (4.22) into (1.6), we obtain

Theorem 4.4. *The number $L^+_{\mathsf{f}}(n)$ of unrooted loopless plane maps with n edges is*

$$L^+_{\mathsf{f}}(n) = \frac{1}{2n}\left[\frac{1}{3n+1}\binom{4n}{n} + \sum_{\substack{k<n\\ k|n}}\phi\left(\frac{n}{k}\right)\binom{4k}{k}\right] + \begin{cases}\dfrac{1}{2n}\dbinom{2n}{\frac{n-1}{2}} & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even.}\end{cases} \tag{4.23}$$

Equivalently, from (4.1) *we obtain*

$$L_{\mathsf{f}}^{+}(n) = \frac{1}{2n}\left[\frac{(n+1)(3n+2)}{2(4n+1)}\,L'(n) + \sum_{\substack{k<n\\k|n}} \phi\left(\frac{n}{k}\right)\frac{(k+1)(3k+1)(3k+2)}{2(4k+1)}\,L'(k)\right] + \begin{cases} \dfrac{n+1}{4}\,L'\left(\dfrac{n-1}{2}\right) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{4.23$'$}$$

and from (4.3) *we obtain the following more compact version:*

$$L_{\mathsf{f}}^{+}(n) = \frac{1}{2n}\left[L_{\mathsf{f}}'(n) + \sum_{\substack{k<n\\k|n}} \phi\left(\frac{n}{k}\right)(3k+1)\,L_{\mathsf{f}}'(k)\right] + \begin{cases} \dfrac{n+1}{4}\,L'\left(\dfrac{n-1}{2}\right) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{4.23$''$}$$

□

4.2. To count the number $L_{\mathsf{v}}^{+}(n)$ of unrooted isthmusless plane maps with n edges, we use formula (1.11) with the letter M replaced by L everywhere. Now $L_{\mathsf{f}}^{+}(n)$ is given by (4.21), $L'(n)$ by (4.1), and the remaining terms of (1.11) are the corresponding terms of (4.2). Making these substitutions we obtain

Corollary 4.5. *The number of unrooted isthmusless plane maps with n edges is*

$$L_{\mathsf{v}}^{+}(n) = \frac{1}{2n}\left[\frac{5n^2+13n+2}{(n+1)(3n+1)(3n+2)}\binom{4n}{n} + \sum_{k<n,\, k|n} \phi\left(\frac{n}{k}\right)\binom{4k}{k}\right] + \begin{cases} \dfrac{n-1}{2n(n+1)}\dbinom{2n}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{4.24}$$

□

From (1.13) (or, instead, from (4.23) and (4.24)) and (4.2) we obtain the following identity:

$$L_{\mathsf{f}}^{+}(n) + L_{\mathsf{v}}^{+}(n) = 2L^{+}(n) + \frac{1}{2}L'(n) - \begin{cases} \dfrac{1}{n+1}\dbinom{2n}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ \dfrac{1}{n}\dbinom{2n}{\frac{n-2}{2}} & \text{if } n \text{ is even.} \end{cases} \tag{4.25}$$

Initial values of the functions $L_{\mathsf{f}}'(n), L_{\mathsf{f}}^{+}(n), L_{\mathsf{v}}'(n)$ and $L_{\mathsf{v}}^{+}(n)$ are given in Table 3 in the Appendix.

5. Two-face, two-vertex and unicursal maps

5.1. There are other classes of maps, aside from the ones treated above, for which unrooted enumeration in the plane can be easily obtained by a slight modification of the methods we designed for the sphere. These include triangular (or, dually, trivalent) maps, which we leave as an open problem, two-face maps, which were treated in [1], and two-vertex maps, which we treat below.

From [1] we have the following three formulae (taken, in order, from formulae (16), (78) and (15)).

The number of rooted two-face planar n-edge maps is given by

$$T'(n) = 2^{2n-1} - \binom{2n-1}{n-1}, \qquad n \geq 1. \tag{5.1}$$

The number of unrooted two-face planar n-edge maps is given by

$$T^+(n) = \frac{1}{2n}\left[T'(n) + \sum_{k<n,\,k|n} \phi\left(\frac{n}{k}\right) T'(k)\right] + \frac{1}{2}\binom{n-1}{\lfloor \frac{n}{2} \rfloor}. \tag{5.2}$$

The number of unrooted two-face plane n-edge maps is given by

$$T^+_{\mathsf{f}}(n) = \frac{1}{n}\left[T'(n) + \sum_{k<n,\,k|n} \phi\left(\frac{n}{k}\right) T'(k)\right]. \tag{5.3}$$

Remark. $T^+_{\mathsf{f}}(n)$ generates the sequence A060404 [15]. The latter is also known as the enumerator of cycles of objects, where the individual objects are enumerated by the Catalan numbers. Two-face plane maps can be easily interpreted in such a form. The generating function of this sequences is the following (see loc. cit.):

$$\sum_{n=1}^{\infty} T^+_{\mathsf{f}}(n)\, x^n = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log\bigl(1 - f(x^k)\bigr), \tag{5.4}$$

where $f(x) = \bigl(1 - \sqrt{1-4x}\,\bigr)/2x - 1$ is the generating function for the Catalan numbers.

To calculate $T^+_{\mathsf{v}}(n)$ from (5.1) – (5.3) and (1.11) we have to know $Q'_0(n)$ and $Q'_1(n)$. These numbers are contained in [1]; we recalculate them independently. The second term on the right-hand side of (5.2) is the total contribution made by the non-trivial automorphisms that fix both faces. The axial elements are the two faces and cannot be singular vertices, so that $Q'_1(n)$ is equal to the last term in (5.2) when n is odd. But $Q'_0(n)$ is greater than the second term on the right-hand side of (5.2) because some of the quotient maps with no singular vertices are contributed by automorphisms that switch the two faces. In this case, the quotient map is a rooted plane tree with $n/2$ edges and, therefore, $n/2+1$ vertices. Both the axial elements are vertices; so the contribution of the face-switching automorphisms to $Q'_0(n)$ is given by the number of rooted plane trees with $n/2$ edges (which is the Catalan number with index $n/2$ [3]) multiplied by the number of unordered pairs of vertices chosen from among $n/2+1$. It follows that

$$Q'_0(n) = \sum_{k<n,\,k|n} \phi\left(\frac{n}{k}\right) T'(k) + \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{n}{2}\dbinom{n-1}{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases} \tag{5.5}$$

Substituting from (5.1), (5.3), (5.5) and the term of (5.2) that is equal to $Q'_1(n)$ into (1.11), we obtain

Proposition 5.1. *The number $T^+_{\mathsf{v}}(n)$ of unrooted two-vertex plane maps is given by*

$$2T^+_{\mathsf{v}}(n) = 2^{2n-1} - \binom{2n-1}{n-1} + \binom{n-1}{\lfloor \frac{n}{2} \rfloor} \tag{5.6}$$

□

Now $Q'_2(n)$ is equal to the last term on the right-hand side of (5.2) for even n minus the corresponding term in (5.5). Substituting for $Q'_1(n)$ and $Q'_2(n)$ into (1.13) we obtain

$$T^+_{\mathsf{f}}(n) + T^+_{\mathsf{v}}(n) = 2T^+(n) + \frac{1}{2}T'(n) - \frac{1}{2}\binom{n-1}{\lfloor \frac{n}{2} \rfloor}. \tag{5.7}$$

Initial values of the functions $T'_{\mathsf{f}}(n), T^+_{\mathsf{f}}(n), T'_{\mathsf{v}}(n)$ and $T^+_{\mathsf{v}}(n)$ are given in Table 4 in the Appendix.

It is no coincidence that (5.6) contains no sum over the divisors of n; a non-trivial automorphism that fixes a vertex must exchange the two faces and thus be of order 2.

If (5.3) had not been available, it would have been easier to derive (5.6) directly and then obtain (5.3) from (5.6) using (1.13) instead of the other way around.

5.2. Not all classes of maps yield closed-form formulae for rooted enumeration in the plane even if they do so on the sphere. To illustrate this point, we compare the enumeration of rooted unicursal planar maps done by us in [11] with the enumeration of rooted unicursal plane maps which we do below.

A map is called *unicursal* if exactly two of its vertices are of odd valency. The number $U'(n)$ of rooted unicursal planar maps with n edges was shown in [11] to be equal to

$$U'(n) = [x^{n-1}] \sum_{v=2}^{n+1} \frac{n!}{(v-2)!\,(n-v+2)!}\,(1-4x)^{-1}\Big[\frac{(1-4x)^{-1/2}-1}{2}\Big]^{v-2}. \tag{5.8}$$

Setting $z := x(z+1)^2$ and using Lagrange inversion, we evaluated (5.8) as

$$U'(n) = 2\frac{(2n-1)!}{(n+1)!\,(n-1)!}\sum_{i=0}^{n-2}\binom{n-2}{i} + \frac{(2n)!}{(n+1)!\,(n-1)!}\sum_{i=0}^{n-2}\binom{n-2}{i}, \tag{5.9}$$

which we simplified to

$$U'(n) = 2^{n-2}\binom{2n}{n}, \qquad n \geq 1. \tag{5.10}$$

The number $U'_{\mathsf{f}}(n)$ of rooted unicursal plane maps with n edges is found by multiplying each term in the sum of (5.8) by the number $n-v+2$ of faces:

$$U'_{\mathsf{f}}(n) = [x^{n-1}] \sum_{v=2}^{n+1} \frac{n!}{(v-2)!\,(n-v+1)!}\,(1-4x)^{-1}\Big[\frac{(1-4x)^{-1/2}-1}{2}\Big]^{v-2}. \tag{5.11}$$

Setting $z := x(z+1)^2$ and using Lagrange inversion, we evaluate (5.11) as

$$U'_{\mathsf{f}}(n) = n\binom{2n}{n}\sum_{i=0}^{n-2}\binom{n-2}{i}\Big(\frac{1}{n+1+i} + \frac{n}{n+2+i}\Big). \tag{5.12}$$

This formula is valid only for $n \geq 2$; $U'_{\mathsf{f}}(1) = 1$ because there is only one rooted unicursal map with one edge and it has one face.

A map is called *dual-unicursal* if exactly two of its faces are of odd valency. The number $U'_{\mathsf{v}}(n)$ of rooted dual-unicursal plane maps is determined by formula (1.8):

$$U'_{\mathsf{f}}(n) + U'_{\mathsf{v}}(n) = (n+2)U'(n). \tag{5.13}$$

Initial values of the functions $U'_{\mathsf{f}}(n)$ and $U'_{\mathsf{v}}(n)$ are given in Table 5 in the Appendix.

Unlike the sums in (5.9), the sums in (5.12) do not seem to simplify; in particular, Maple evaluated them in terms of hypergeometric functions. An interesting problem, which we leave open, is to find a closed-form formula for $U'_{\mathsf{f}}(n)$ or to prove that none exists; the familiar WZ-method (see, e.g., [13, Ch. 7] and [14, Sect. 3.7]) could probably be applied. A general challenging open problem would be to find a systematic method for deciding whether a closed-form formula exists for the number of rooted planar or plane n-edge maps of a given class by examining the maps themselves instead of the result of an analytical calculation such as Lagrange inversion.

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Appendix: Numerical tables

TABLE 1. The numbers of arbitrary and non-separable plane maps (rooted and unrooted)

n	$A'_{\mathsf{f}}(n)$	$A^{+}_{\mathsf{f}}(n)$	$B'_{\mathsf{f}}(n)$	$B^{+}_{\mathsf{f}}(n)$
1	3	2	3	2
2	18	6	2	1
3	135	26	5	2
4	1134	150	18	4
5	10206	1032	77	10
6	96228	8074	364	37
7	938223	67086	1836	138
8	9382230	586752	9690	628
9	95698746	5317226	52877	2972
10	991787004	49592424	296010	14903
11	10413763542	473357994	1690845	76994
12	110546105292	4606116310	9817080	409594
13	1184422556700	45554761836	57769740	2222628
14	12791763612360	456848968518	343806368	12281570
15	139110429284415	4637014782748	2065802056	68864086
16	1522031755700070	47563495004742	12515350122	391120036
17	16742349312700770	492422043299964	76367432013	2246122574
18	185047018719324300	5140194991046122	468922828150	13025721601
19	2054021907784499730	54053208147441474	2895381678735	76194378042
20	22887672686741568420	572191817441284272	17966214519330	449155863868

TABLE 2. The numbers of eulerian and bipartite plane maps (rooted and unrooted)

n	$E'_{\mathsf{f}}(n)$	$E^{+}_{\mathsf{f}}(n)$	$E'_{\mathsf{v}}(n)$	$E^{+}_{\mathsf{v}}(n)$
1	2	1	1	1
2	8	3	4	2
3	40	8	20	5
4	224	32	112	18
5	1344	136	672	72
6	8448	722	4224	368
7	54912	3924	27456	1982
8	366080	22954	183040	11514
9	2489344	138316	1244672	69270
10	17199104	860364	8599552	430384
11	120393728	5472444	60196864	2736894
12	852017152	35503288	426008576	17752884
13	6085836800	234070648	3042918400	117039548
14	43818024960	1564945158	21909012480	782480424
15	317680680960	10589356592	158840340480	5294705752
16	2317200261120	72412611194	1158600130560	36206357114
17	16992801914880	499788291616	8496400957440	249894328848
18	125210119372800	3478059566250	62605059686400	1739030128872
19	926554883358720	24383023246284	463277441679360	12191512867814
20	6882979133521920	172074483068320	3441489566760960	86037243899240

Table 3. The numbers of loopless and isthmusless plane maps (rooted and unrooted)

n	$L'_{\mathsf{f}}(n)$	$L^+_{\mathsf{f}}(n)$	$L'_{\mathsf{v}}(n)$	$L^+_{\mathsf{v}}(n)$
1	1	1	2	1
2	4	2	8	3
3	22	6	43	9
4	140	22	268	38
5	969	103	1824	187
6	7084	614	13156	1120
7	53820	3872	98865	7083
8	420732	26414	765948	47990
9	3362260	186988	6075256	337676
10	27343888	1367976	49094708	2455517
11	225568798	10254326	402801425	18310155
12	1882933364	78461338	3346590068	139447034
13	15875338990	610598818	28099903160	1080773098
14	134993766600	4821248244	238079915640	8502896424
15	1156393243320	38546510368	2032914717645	67763884363
16	9969937491420	311560875422	17476713955548	546147639926
17	86445222719724	2542507084588	151143219598008	4445389286380
18	753310723010608	20925300483992	1314045772469632	36501274080076
19	6594154339031800	173530381632724	11478299163026540	302060508150976
20	57956002331347120	1448900079476152	100688538612524720	2517213486505592

Table 4. The numbers of two-face and two-vertex plane maps (rooted and unrooted)

n	$T'_{\mathsf{f}}(n)$	$T^{+}_{\mathsf{f}}(n)$	$T'_{\mathsf{v}}(n)$	$T^{+}_{\mathsf{v}}(n)$
1	2	1	1	1
2	10	3	10	3
3	44	8	66	12
4	186	25	372	48
5	772	78	1930	196
6	3172	270	9516	798
7	12952	926	45332	3248
8	52666	3305	210664	13184
9	213524	11868	960858	53416
10	863820	43232	4319100	216018
11	3488872	158586	19188796	872344
12	14073060	586530	84438360	3518496
13	56708264	2181088	368603716	14177528
14	228318856	8154710	1598231992	57080572
15	918624304	30620868	6889682280	229657792
16	3693886906	115435625	29551095248	923474944
17	14846262964	436654794	126193235194	3711572176
18	59644341436	1656793374	536799072924	14911097514
19	239532643144	6303490610	2275560109868	59883185096
20	961665098956	24041649128	9616650989560	240416320928

TABLE 5. The numbers of rooted unicursal and dual-unicursal plane maps

n	$U'_{\mathsf{f}}(n)$	$U'_{\mathsf{v}}(n)$
1	1	2
2	10	14
3	93	107
4	836	844
5	7355	6757
6	63750	54522
7	546553	441863
8	4646920	3589880
9	39250935	29206025
10	329789450	237780982
11	2758868981	1936486411
12	22995369996	15771410420
13	191074697203	128431734797
14	1583463268366	1045618229234
15	13092015636465	8510270668815
16	108024564809744	69241255165936
17	889730213085167	563154350637073
18	7316434446188562	4578526894227438
19	60078376613838829	37209886138826771
20	492692533579612180	302291556342169580