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WHAT MAKES AN EXAMPLE EXEMPLARY?: PEDAGOGICAL & RESEARCH ISSUES IN TRANSITIONS FROM NUMBERS TO NUMBER THEORY

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ABSTRACT

Worked examples have always been a component of teaching mathematics, and rehearsing techniques on suites of exercises has almost always been seen as an essential part of learning mathematics. Why then has there been but limited success? The case is put forward that in order for examples to be seen as being exemplary, learners need to appreciate the generality which is being particularised. But how are they to come to appreciate this generality other than through examples? I suggest that when learners begin to see through a few particulars to a generality, they are beginning to formulate a theory, for the word *theory* means literally *a way of seeing.* When they can look at a problem and see it as representative of a class of problems, then they are giving evidence of having a theory.

INTRODUCTION

Worked examples have been used in teaching mathematics since the earliest of historical records, at least. In order for such examples to be useful, learners must see them as exemplifying something. If that 'something' is, for example (sic) 'the mystery of mathematics', 'the impossibility of doing something similar myself', or 'the ridiculousness of all this stuff', then learners are unlikely to make mathematical progress. If on the other hand the 'something' is a class of problems and a collection of techniques and ways of thinking, then the worked examples have served at least some of their purpose. Looking through a particular and seeing generality is a form of and a building block for, generating a theory. When a learner has constructed their own 'theory', their own seeing of how 'to do' a whole class of problems, then real learning is taking place. In the language of Sinclair (this forum), learners participate in an aesthetic of efficiency and compactness afforded by their awareness of generality, or what the examples are exemplifying. Rowland (this forum) provides further examples of techniques for supporting learners in seeing the general through the generic particular. See also Mason & Pimm (1984).

The fact that text authors have, throughout the ages, inserted worked examples and only in some periods tried to start with general rules before illustrating their use in particular, suggests that in most generations teachers have been aware of the difficulty of proceeding from the general to the particular (at least unless very carefully guided: see Davydov 1990).

METHOD OF WORKING

My method of enquiry is to identify phenomena I wish to study, and to seek examples within my own experience. I then construct task-exercises to offer to others to see if they recognise, or can be directed to recognise, what I find myself noticing. Through refinement and adjustment of task-exercises in the light of experience and of reading relevant literature, I both extend my own awarenesses, and offer others experiences which may highlight or even awaken sensitivities and awarenesses for them. These sensitivities and awarenesses may inform their future practice. As task-exercises are developed and shared, actions which exploit what is noticed for the benefit of learners are incorporated. My method does not attempt to capture or cover the experience of readers. Rather it aims to make contact with that experience, perhaps challenging interpretations, perhaps pointing to other features not previously noticed. The data of this method are the experiences generated, the sensitivities to notice which are enhanced. If you recognise at least something of what I am talking about as a result of having worked on the task-exercises, then you may be stimulated to look out for similar experiences in the future, and over time, begin to act upon what you notice. Validity in this method lies in you finding your actions being informed in the future, not in what I say (Mason 2002).

EXAMPLES

Consider the following task:

> *Find the greatest common divisor of 84 and 90. Find their least common multiple as well.*

You can imagine a whole page of these sorts of questions, and if you do, then you have a sense of a class of tasks. But do learners also have a sense of this class, even when they have worked through the page? Could they make up for themselves versions which 'showed possible difficulties that can arise in some cases' or which 'illustrate what things can happen'? Are they aware of two approaches (factoring and the Euclidean algorithm), and do they have criteria for choosing one over another? If such tasks are augmented a little:

Multiply your two answers together and compare the product with 84 x 90

something more is suggested, which can be pushed further into the realm of generality with something like

Might this always happen? or Will this always happen? or even, Can you find two numbers for which this does not happen?

The indefinite pronoun 'this' may invite learners to clarify what the 'this' is, or may leave them helpless, depending on their past experiences with these sorts of question. To prompt learners to think of factoring, the task

Find a number with exactly 13 factors

makes use of a largish number (13) which invites simplification (try 3 factors, try 1 factor!) and then re-generalisation beyond the 13, with extensions concerning characterising or describing the class of all numbers with a given odd number of factors, and finding the smallest such number. There are also extensions to even numbers for the bold. This is typical of the approach developed in the 60s and 70s in the U.K. (Banwell *et al* 1972) in that learners are invited to undertake a task in which they make choices in order to simplify on the way to re-complexifying for themselves.

Finding a number with exactly 13 factors is not a single task but an entry point into a whole domain of tasks which include finding numbers with a given number of factors, how many smaller numbers less than itself are relatively prime, among others (see Banwell *et al* 1972 p37, 102-3). Such a task domain includes various variants in presentation as well as in content. It is of pedagogic use only if it becomes a vehicle for learners to use their natural powers to imagine and to express, to generalise and to particularise, to conjecture and to convince, and if their attention is drawn to the fact of these powers and to ways of refining and honing them. For example, although an observer might say that they detected the emergence of a theory about the structure of numbers as products of primes, and about the parity of the number of divisors, participants might be wholly unaware that that is what they were doing. Their attention is likely to be confined to trying to sort out their ideas and to justify their conjectures. Yet what they are doing is what mathematicians do. Becoming aware of the emergence and articulation of a 'theory' can both inspire and support the development of learner's mathematical self esteem.

Another collection of tasks which promote number factoring are obtained by 'undoing' the first task:

> *Find a pair of numbers whose* gcd *is 6; find another pair; and another … leading to the learner deciding spontaneously to classify or characterise all such pairs. Similarly, find a pair of numbers whose* lcm *is 1260, leading to, how many different pairs can have a specified* lcm*?*

Participants are not only developing a theory about the structure of pairs of numbers with a given *lcm*,. They are also experiencing the use of their power to organise and characterise in a mathematical context, much as mathematician are wont to do. In other words, by being immersed in such tasks learners are likely to develop a 'theory' of what doing mathematics is like, a theory which would be very different from a theory developed as a result of only attending lectures and doing routine exercises (Watson & Mason 1998).

A further advantage of refining and honing their powers to think mathematically is that learners are less likely to be caught by, or to persist in, spontaneous theories such as that 'all functions are montonic': if $x > y$ then it is likely that $f(x) > f(y)$ (Zazkis 1999), or that all functions are linear: e.g. $\sin(A + B) = \sin(A + B)$ and $\ln(A + B) = \ln A + \ln B$. These are all too common manifestations, even by learners who when questioned directly know that they are false generalisations. The problem is perhaps that learners are entirely unaware that they have these theories, and they are not in the habit of testing theories and looking for counter-examples. In the midst of a complex problem, they simply do not have sufficient free attention to monitor what they are doing.

The temptation of authors and teachers is to lay out examples, perhaps as tasks, and to expect learners to build on the experience of a succession of tasks in order to become aware of or to experience that succession. But as Kant effectively points out in his *Critique of Pure Reason*: a succession of experiences does not add up to an experience of that succession.

There is documented evidence of this in many different contexts. For example, in a professional development session I offered the following sequence of statements, with lots of pausing so that participants would see that they were supposed to check each equation in turn, and to attend to the patterns between equations:

$1 + 2 = 3 \qquad 4 + 5 + 6 = 7 + 8 \qquad 9 + 10 + 11 + 12 = 13 + 14 + 15$

'We' (meaning I wrote what some of the participants said) had written down two more rows. I asked for an expression of generality. One person suggested

$n + (n + 1) = \ldots$ but was then a bit stuck.

It transpired that his attention was entirely on the first equation. The others were not seen as part of 'the experience to be generalised'. Rowland (2000) met the same thing with pre-service primary teachers. Asked to check that

$3 + 4 + 5 = 3 \text{ x } 4$, $8 + 9 + 10 = 3 \text{ x } 9$ and $29 + 30 + 31 = 3 \text{ x } 30$

and then to write down a statement in words generalising these three examples, many wrote nothing or nothing that could be construed, and some wrote a false generalisation such as

three consecutive numbers added together equals the product of the first two

achieved by attending solely to the first 'example'.

It is quite likely that unfamiliarity with being asked to express a generality produced a tunnel vision effect, so that attention became focused on but one instance. The fact that the two threes play different roles could be overlooked, which would make the conjectured generality at least understandable. In the case of the sequence above, progress was made by asking people to chant the equations out loud but with emphasis first on the first number, then in a second pass, on the last.

Watson (2000) has pointed to the phenomenon of 'reading with the grain' and the necessity of 'reading across the grain' in order to experience structure. Thus in the Tunja sequences which I developed in order to promote simultaneous work on factoring quadratics and on multiplication of negative numbers (Mason 1999, 2001), I have found as expected, that non-mathematical audiences are perfectly capable of working with the grain, that is of following a pattern which is closely related to counting numbers and perhaps square numbers. For example

$3 \text{ x } 5 = 4^2 - 1$ $4 \text{ x } 6 = 5^2 - 1$ $5 \text{ x } 7 = 6^2 - 1$

can be extended 'downwards' to more equations by observing the counting-numbers in sequence. Being directed to read across the grain, that is to relate both sides of each 'equation', leads to the realisation that the symbolic expressions must be the same, somehow. This is one of the necessary awarenesses that make algebraic manipulations meaningful: different looking expressions can nevertheless express the same thing, so there ought to be a way to get from one expression to the other simply by manipulating symbols. The sequence can also be pushed backwards to reveal necessary facts about multiplication of and by negatives, given that we want the 'pattern' to continue.

As is well known, specific patterns of questions can lead learners to unexpected meta-generalisations. For example, being asked to express a pattern as a general formula can lead them to a pattern of behaviour which avoids the intended inner-task (Tahta 1980) and exercises only the outer-task namely, to find a formula which fits. As mathematicians well know, though intelligence testers seem yet to discover, no sequence, even the Tunja or the Consecutive Sums, uniquely defines its next term. There must be some source for a pattern which is agreed. Thus the sequence

1, 2, 4, 8, …

can have many different fifth terms (indeed, any fifth term, but see Sloane 1973 for examples of sequences which count things, and which begin this way). If the sequence is counting some aspect of a sequence of pictures, such as the number of regions of a circle formed by 0, 1, 2, 3, … chords, or the number of regions of space formed by 0, 1, 2, 3, … planes in general position (Polya 1965), then it is essential to have a statement of that generality before embarking on trying to find a general formula.

But even where learners are frequently engaged in formula-finding, the whole exercise can turn into 'train-spotting' (Hewitt 1992) rather than productive mathematical thinking which exercises and refines the power to generalise.

Consider then some further options for extension.

The act of finding the *gcd* and *lcm* of two numbers can be seen as functions, but this requires the learner to step out of immediate action-experience, and to contemplate the whole. Having achieved some measure of competence with these calculations, the calculations themselves can be seen as objects. This is the domain of reification, when a process is also experienced as an object. In Mason *et al* (1985) this was used to characterise one of the major steps in appreciating algebra, when expressions like $3x + 4$ come to be seen both as a specification of a calculation process, and as the object resulting from that process, and this dual nature then becomes the essence of algebraic expressions which replace numbers as the objects to be manipulated. Sfard (1991, 1992, 1994) developed the notion of reification while (Gray & Tall 1994) used the term *procept* to indicate the evolution of a concept from carrying out a process as a theorem-in-action (Vergnaud 1981) to seeing the process as an object in itself. Notation for the process helps enormously, for once something is named, it comes into psychological existence..

One way to stimulate learners to experience calculations as processes is some variant of

Tell an absent friend how to calculate the gcd *and* lcm *of a pair of numbers.*

Program a machine to find the gcd *and* lcm *of a pair of numbers.*

> *I am going to be given a pair of numbers, but I can't tell you what they are at the moment (or I have a friend who has a pair ...). Please tell me how to calculate their* gcd *and* lcm*.*

Suddenly what seemed almost frighteningly open becomes bounded. A theory might just be possible. Notice that I do not provide my 'answers' nor even my theories in the sense of ways of seeing. For once theories are published, pedagogic value leaks away.

FINAL COMMENTS

Terms such as *investigative teaching* (and its variants such as *discovery learning*) provoke extreme reactions in many audiences, while *lecturing* and *starting from the abstract* provoke similar reactions in different audiences. Neither reactions are helpful as they are based on emotive associations with general labels, rather than precise details of pedagogic strategies. When teaching that is even marginally effective is examined closely, aspects of both pedagogic stances, of starting from the particular and the concrete and starting from the general and the complex will be found to have value. Strict adherence to one format is likely to foster the pedagogic theory that 'this is always how things are done', whereas variation in approach is more likely to broaden learners' views of what mathematics is about, what questions it addresses, and what methods it employs (Watson & Mason 1998). Above all, the most important theory we want learners to construct is that they do actually possess the requisite powers to do mathematics and to think mathematically. Then they can make an informed choice as to whether to develop and make use of those powers within mathematics in the future.

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