ENUMERATION OF SOLID 2-TREES ACCORDING TO EDGE NUMBER AND EDGE DEGREE DISTRIBUTION

MICHEL BOUSQUET AND CÉDRIC LAMATHE

Abstract. The goal of this paper is to enumerate solid 2-trees according to the number of edges (or triangles) and also according to the edge degree distribution. We first enumerate oriented solid 2-trees using the general methods of the theory of species. In order to obtain non oriented enumeration formulas, we use quotient species which consists in a specialization of Pólya theory.

Résumé. Le but de cet article est d'obtenir l'énumération des 2-arbres solides selon le nombre d'arêtes (ou de triangles) ainsi que selon la distribution des degrés des arêtes. Nous obtenons d'abord le dénombrement des 2-arbres solides orientés en utilisant les méthodes de la théorie des espèces. Pour obtenir le dénombrement des 2-arbres solides non orientés, nous utilisons la notion d'espèce quotient qui provient d'une spécialisation de la théorie de Pólya.

1. Introduction

Definition 1. Let $\mathcal{E}$ be a non-empty finite set of n elements called *edges*. A *2-tree* is either a single edge (if $n = 1$) or a non-empty subset $\mathcal{T} \subseteq \mathcal{P}_3(\mathcal{E})$ whose elements are called *triangles*, satisfying the following conditions:

(1) For every pair $\{a, b\} = \{\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}\}$ of distinct elements of $\mathcal{T}$, we have $|a \cap b| \leq 1$, which means that two distinct triangles share at most one edge.

(2) For every ordered pair $(a, b) = (\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\})$ of distinct elements of $\mathcal{T}$, there is a unique sequence $(t_0 = a, t_1, t_2, \ldots, t_k = b)$ such that for $i = 0, 1, \ldots, k-1$, we have $t_i \in \mathcal{T}$ and $|t_i \cap t_{i+1}| = 1$, which means that each pair of consecutive triangles in this sequence share exactly one edge.

An edge e and a triangle t are *incident* to each other if $e \in t$. The *degree* of an edge is the number of triangles which are incident to that edge. The *edge degree distribution* of a 2-tree having more than one edge is described by a vector $\vec{n} = (n_1, n_2, \ldots)$, where n_i is the number of edges of degree i. We denote by $\mathrm{Supp}(\vec{n})$ the *support* of $\vec{n}$ which is the set of indices i such that $n_i \neq 0$. Figure 1 shows a 2-tree having 11 edges, 5 triangles and edge degree distribution given by $\vec{n} = (8, 2, 1)$.

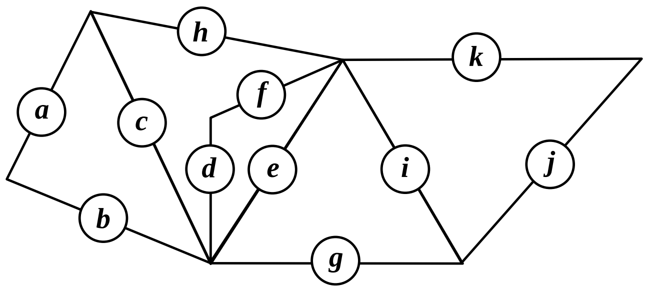

Figure 1. A 2-tree on $\mathcal{E} = \{a, b, c, d, e, f, g, h, i, j, k\}$.

Several classes of 2-trees have been studied before. Beineke and Pippert enumerate some k-dimensional trees labelled at vertices in [1]. In [10], Harary and Palmer count unlabelled 2-trees. For the enumeration of plane 2-trees, see [18], and for a classification according to symmetries of plane and planar 2-trees, see [13]. In [8, 9], Fowler et al. worked on general 2-trees and give asymptotical results. More recently, Labelle et al. [14, 15] generalize the results of Fowler et al. to the larger familly of k-gonal 2-trees and the same authors [16] propose a weighted enumeration of the same 2-trees according to the perimeter of the structures. We also mention the works of Kloks in [11, 12] about partial biconnected 2-trees. Here, we consider a new class of 2-trees, that is, *solid* 2-trees, *i.e.*, 2-trees embedded in three-dimensional space. This paper is the full version of a paper presented at the 14th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC) July, 8-12, 2002, Melbourne, Australia (see [5]).

The first result gives a sufficient and necessary condition on edges to ensure the existence of a 2-tree.

Lemma 1. Let m, n be two nonnegative integers and $\vec{n} = (n_1, n_2, \ldots)$, an infinite vector of nonnegative integers. Then,

(1) There exists a 2-tree having m triangles and n edges if and only if $n = 2m + 1$.
(2) There exists a 2-tree having n edges, m triangles and $\vec{n}$ as edge degree distribution if and only if

$$\sum_i n_i = n \quad \text{and} \quad \sum_i i n_i = 3m. \tag{1}$$

Proof. Item 1 is quite obvious as the reader can check. For item 2, the condition $\sum_i n_i = n$ is straightforward. Concerning the relation $\sum_i i n_i = 3m$, it suffices to observe that the left-hand side counts the total degree of the structure, while, in the right-hand side, each triangle contributes for three units in the total degree. ∎

We say that $\vec{n} = (n_1, n_2, \ldots)$ is a *coherent* (or *valid*) edge degree distribution if condition (1) is satisfied.

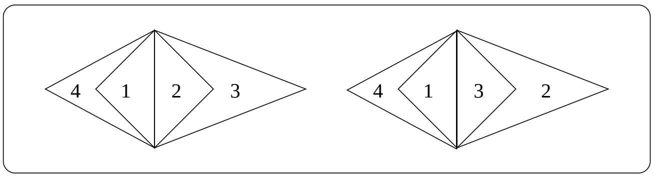

FIGURE 2. Two distinct solid 2-trees but the same 2-tree.

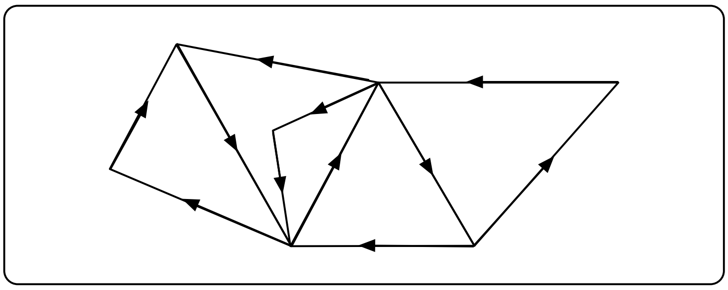

FIGURE 3. A well oriented 2-tree.

A *solid* 2-tree can be viewed topologically as a 2-tree in which the faces of the triangles cannot interpenetrate themselves. As a consequence, there is a cyclic configuration of triangles around every edge. Figure 2 shows an example of two different solid 2-trees which are in fact the same 2-tree. Indeed, the cyclic order on labels 1, 2, 3, 4 given to the triangles for the two 2-trees are different. A *well oriented* solid 2-tree is obtained from a solid 2-tree in the following way: first, pick any triangle and give a cyclic orientation on its edges; then, each triangle adjacent to the first triangle inherits a cyclic orientation (see Figure 3). This process is repeated until all edges receive an orientation.

By the arborescent nature of the structure, there will be no conflict (the orientation of each edge is always well defined). Figure 3 shows an example of a well oriented 2-tree.

The species of non-oriented and well oriented solid 2-trees are denoted respectively by $\mathcal{A}$ and $\mathcal{A}_o$. For details about species, see [2]. In order to analyze these two species, the following auxiliary species are used:

- The species of *triangles* X: a single triangle is denoted by X;
- The species of *edges* Y: a single edge is denoted by Y;
- The species L of *lists* or *linear orders*;
- The species C and C_3, respectively of *oriented cycles* and of *oriented cycles of length* 3;
- The species $\mathcal{A}^-$ and $\mathcal{A}_o^-$, respectively of non oriented and well oriented solid 2-trees *rooted at an edge*;
- The species $\mathcal{A}^\triangle$ and $\mathcal{A}_o^\triangle$, respectively of non oriented and well oriented solid 2-trees *rooted at a triangle*;
- The species $\mathcal{A}^{\underline{\triangle}}$ and $\mathcal{A}_o^{\underline{\triangle}}$, respectively of non oriented and well oriented solid 2-trees *rooted at a triangle having itself one of its edges distinguished*;
- Finally, the species $\mathcal{B}$ of *planted* oriented solid 2-trees. To obtain a planted solid 2-tree from a solid 2-tree, pick any edge and break the circular order of triangles around it by fixing a starting triangle, keeping the order induced by the cycle of triangles. Equivalently, we can consider that the cyclic order of triangles around the distinguished edge contains a virtual triangle (denoted $*$ in Figure 4) indicating the first triangle of the linear order. A $\mathcal{B}$-structure then consists of an oriented root edge Y incident to a linear order (L-structure) of triangles X, each of which has its two remaining sides which are themselves $\mathcal{B}$-structures. Therefore, the species $\mathcal{B}$ satisfies the following combinatorial equation:

$$\mathcal{B}(X,Y) = YL(X\mathcal{B}^2(X,Y)), \tag{2}$$

as illustrated by Figure 4. It is important to remark that the species $\mathcal{B}$ is not isomorphic to the species $\mathcal{A}_o^-$ of edge-rooted oriented solid 2-trees.

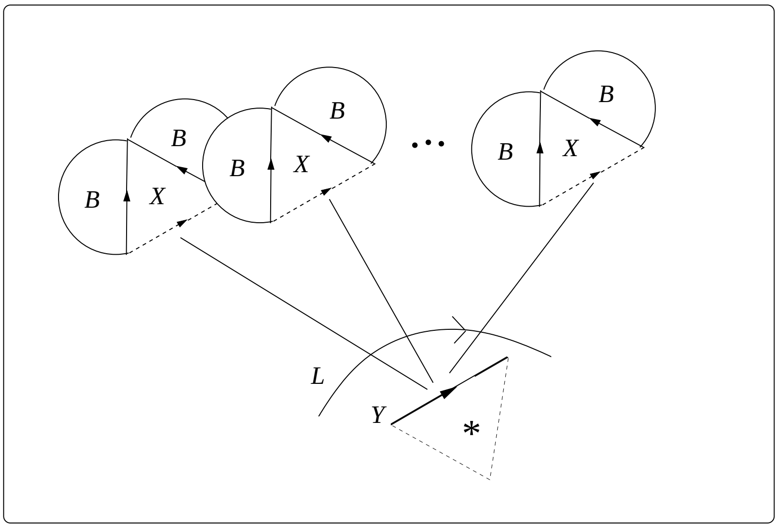

FIGURE 4. A $\mathcal{B}$-structure.

Observe that $\mathcal{B}$ has been defined as a *two-sort* species where the sorts are X and Y. Since the numbers of edges n and of triangles m are linked by the relation $n = 2m+1$, as stated in Lemma 1, equation (2) above can either be expressed as a one sort species in X alone by setting $Y := 1$, or in

Y alone, by setting $X := 1$ respectively, giving the two following equations:

$$\mathcal{B}(X,1) = L(X\mathcal{B}^2(X,1)), \tag{3}$$

$$\mathcal{B}(1,Y) = YL(\mathcal{B}^2(1,Y)). \tag{4}$$

Recall that setting $X := 1$ in a two sort species $F(X,Y)$ essentially means unlabelling the elements of sort X (keeping the labels of points of sort Y). The second form in equation (4) is more suitable for the use of Lagrange inversion formula. Therefore, the species Y of edges is employed as the base singleton species to make our computations and we rather use the shorter form $\mathcal{B}(Y) = YL(\mathcal{B}^2(Y))$ for (4). Hence, the structures are labelled at edges. However, some results will be more concise when expressed as a function of the number m of triangles.

In this paper, we make an extensive use of Lagrange inversion formula (see [2]). Let $A(y)$ and $R(y)$ be formal series satisfying $A(y) = yR(A(y))$ and $R(0) = 0$. If F is another formal series, then

$$[y^n]F(A(y)) = \frac{1}{n}[t^{n-1}]F'(t)R^n(t), \tag{5}$$

where $[y^n]F(A(y))$ denotes the coefficient of y^n in $F(A(y))$.

Another main tool used in this paper is the following dissymmetry theorem which has been proved in [8, 9]. Note that in their paper, the authors made a proof for general 2-trees but obviously, the proof is also valid for both well oriented and non oriented solid 2-trees.

Theorem 1. The species $\mathcal{A}_o$ and $\mathcal{A}$, respectively of well oriented and (non oriented) solid 2-trees, satisfy the following relations:

$$\mathcal{A}_o^{-} + \mathcal{A}_o^{\triangle} = \mathcal{A}_o + \mathcal{A}_o^{\underline{\triangle}}, \tag{6}$$

and

$$\mathcal{A}^{-} + \mathcal{A}^{\triangle} = \mathcal{A} + \mathcal{A}^{\underline{\triangle}}. \tag{7}$$

□

To each species F, we associate two series: the exponential generating series of labelled structures $F(x)$ and the ordinary generating series of unlabelled structures $\widetilde{F}(x)$, defined as follows:

$$F(x) = \sum_{n\geq 0} |F[n]| \frac{x^n}{n!}, \tag{8}$$

$$\widetilde{F}(x) = \sum_{n\geq 0} |\widetilde{F}[n]| \, x^n, \tag{9}$$

where $|F[n]|$ and $|\widetilde{F}[n]|$ are respectively the numbers of labelled and unlabelled F-structures over n elements. Note that we use the notation $|\widetilde{F}[n]|$ instead of $|F[n]/\mathbb{S}_n|$ (see [2]), where $\mathbb{S}_n$ is the symmetric group of order n, to lighten the notations.

2. Well oriented solid 2-trees

We begin this section by expressing the species appearing in the dissymmetry theorem (oriented case) in terms of the species $\mathcal{B}$.

Proposition 1. The species $\mathcal{A}_o^{-}$, $\mathcal{A}_o^{\triangle}$ and $\mathcal{A}_o^{\underline{\triangle}}$ satisfy the following isomorphisms of species:

$$\mathcal{A}_o^{-}(Y) = Y + YC(\mathcal{B}^2(Y)), \tag{10}$$

$$\mathcal{A}_o^{\triangle}(Y) = C_3(\mathcal{B}(Y)), \tag{11}$$

$$\mathcal{A}_o^{\underline{\triangle}}(Y) = \mathcal{B}(Y)^3. \tag{12}$$

Proof. Let us begin with relation (10). The term Y corresponds to the case of a single rooted edge. In the general case, as illustrated by Figure 5 a), by convention with the right-hand rule, we define a cyclic order over the triangles glued around the oriented root-edge. Next, each triangle in this cyclic configuration possesses, on its two remaining oriented edges, two $\mathcal{B}$-structures, leading to the expression $YC(\mathcal{B}^2(Y))$. For (11), it suffices to remark that, since the structures are well oriented, there is a cyclic order of length three around the edges of the root triangle (see Figure 5 b)). These edges being oriented, we can attach $\mathcal{B}$-structures on them, giving quite directly (11). We obtain (12) in a very similar way (see Figure 5 c)). ■

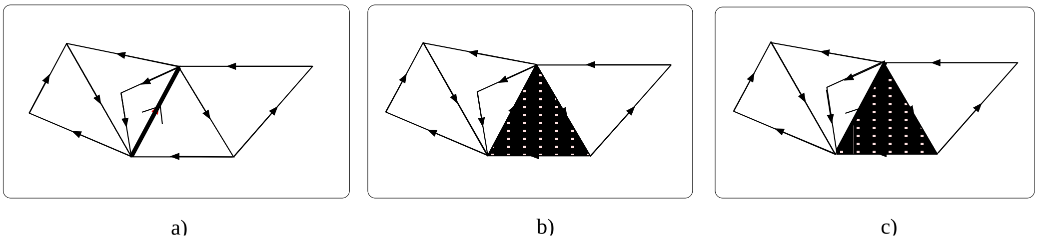

FIGURE 5. Illustration of equations (10), (11) and (12).

2.1. Enumeration according to the number of edges.

In this section, we obtain the labelled and unlabelled enumeration of oriented solid 2-trees according to the number n of edges. We also give formulas in terms of the number m of triangles.

• Labelled case

Let $\mathcal{A}_o[n]$ be the set of oriented edge-labelled solid 2-trees over n edges. We similarly define $\mathcal{A}_o^-[n]$, $\mathcal{A}_o^\triangle[n]$ and $\mathcal{A}_o^{\underline{\triangle}}[n]$. Our first task is to determine $|\mathcal{A}_o^-[n]|$, the cardinality of the set $\mathcal{A}_o^-[n]$, that is, the number of $\mathcal{A}_o^-$-structures over n labelled edges. By applying Lagrange inversion with $F(t) = C(t^2) = -\ln(1-t^2)$ and $R(t) = L(t^2) = (1-t^2)^{-1}$, we find, for $n > 1$,

$$\begin{aligned}[y^n]\mathcal{A}_o^-(y) &= [y^{n-1}]C(\mathcal{B}^2(y)) \\ &= \frac{2}{3(n-1)}\binom{3(n-1)/2}{n-1}.\end{aligned}$$

Hence, we have

$$|\mathcal{A}_o^-[n]| = n![y^n]\mathcal{A}_o^-(y) = \frac{2}{3}n(n-2)!\binom{\frac{3(n-1)}{2}}{n-1}, \quad \text{for } n > 1. \tag{13}$$

Note that, when a solid 2-tree over n edges is labelled, we have n different choices for the root-edge. Therefore,

$$n|\mathcal{A}_o[n]| = |\mathcal{A}_o^-[n]|,$$

and next proposition follows.

Proposition 2. The number $|\mathcal{A}_o[n]|$ of well oriented edge-labelled solid 2-trees over n edges is given by

$$|\mathcal{A}_o[n]| = \frac{2}{3}(n-2)!\binom{\frac{3(n-1)}{2}}{n-1}, \quad n > 1. \tag{14}$$

□

Note that if we express equation (14) as a function of m, the number of triangles, we obtain

$$|\mathcal{A}_{o,t}[m]| = \frac{(m-1)!}{3}\frac{1}{2m+1}\binom{3m}{m}, \quad m \geq 2, \tag{15}$$

where the index t in $|\mathcal{A}_{o,t}[m]|$ means that the structures are labelled at triangles instead of edges. To obtain this formula, it suffices to notice that

$$|\mathcal{A}_{o,t}[m]| = \frac{m!}{n!}|\mathcal{A}_o[n]|$$

since the automorphisms groups with respect to the labellings at edges or at triangles are isomorphic, and this, independently of the considered solid 2-tree.

The integer sequences defined respectively by relations (13), (14) and (15) are listed in the on-line Encyclopedia of Integer Sequences [19] as sequence numbers A081321, A082787 and A076151.

• Unlabelled case

We first need to compute the ordinary generating series $\widetilde{\mathcal{A}_o^-}(y)$ of unlabelled $\mathcal{A}_o^-$-structures. In order to accomplish this, we use the following property.

Theorem 2. ([2]) Let F and G be two species. Then, we have

$$(F(G))^{\sim}(x) = Z_F(\widetilde{G}(x), \widetilde{G}(x^2), \widetilde{G}(x^3), \ldots), \tag{16}$$

where the *cycle index series* Z_F of a species is defined by

$$Z_F(x_1, x_2, \ldots) = \sum_{k\geq 0} \frac{1}{k!} \sum_{\sigma\in\mathbb{S}_k} \mathrm{fix}F[\sigma] x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \cdots, \tag{17}$$

where $\mathbb{S}_k$ is the symmetric group of order k, σ_i, the number of cycles of length i in the permutation $\sigma \in \mathbb{S}_k$ and $\mathrm{fix}F[\sigma]$, the number of F-structures left fixed under the relabelling induced by σ. □

For example, if $F = C$, the species of oriented cycles, we have

$$Z_C(x_1, x_2, \ldots) = \sum_{k\geq 1} \frac{\phi(k)}{k} \ln\left(\frac{1}{1 - x_k}\right), \tag{18}$$

where ϕ is the Euler's totient function. Now, applying this to the species $\mathcal{A}_o^- = Y + YC(\mathcal{B}^2)$, we get

$$\begin{aligned} \widetilde{\mathcal{A}_o^-}(y) &= y + yZ_C(\widetilde{\mathcal{B}}^2(y), \widetilde{\mathcal{B}}^2(y^2), \widetilde{\mathcal{B}}^2(y^3), \ldots) \\ &= y + y\sum_{k\geq 1} \frac{\phi(k)}{k} \ln\left(\frac{1}{1 - \widetilde{\mathcal{B}}^2(y^k)}\right). \end{aligned}$$

Since the species $\mathcal{B}$ is asymmetric (there are exactly $n!$ labelled structures for each unlabelled structures or equivalently, the stabilizer of each $\mathcal{B}$-structure is trivial), we have $\widetilde{\mathcal{B}}(y) = \mathcal{B}(y)$. Hence, for $n > 1$,

$$\begin{aligned} |\widetilde{\mathcal{A}_o^-}[n]| &= [y^n]\widetilde{\mathcal{A}_o^-}(y) \\ &= [y^{n-1}]\sum_{k\geq 1} \frac{\phi(k)}{k} \ln\left(\frac{1}{1 - \mathcal{B}^2(y^k)}\right). \end{aligned}$$

But, using the fact that $[y^n]H(y^k) = [y^{n/k}]H(y)$ and Lagrange inversion,

$$\begin{aligned} [y^{n-1}] \ln\left(\frac{1}{1 - \mathcal{B}^2(y^k)}\right) &= \frac{2k}{n-1}[t^{\frac{n-1}{k}-2}](1 - t^2)^{-\frac{n-1}{k}-1} \\ &= \frac{2k}{3(n-1)}\binom{3(n-1)/2k}{(n-1)/k}. \end{aligned}$$

Obviously, k must divide $n - 1$ and $(n - 1)/k$ must be even. Letting $d = (n - 1)/k$, we finally get

$$|\widetilde{\mathcal{A}_o^-}[n]| = \frac{2}{3(n-1)} \sum_d \phi(\tfrac{n-1}{d})\binom{3d/2}{d}, \tag{19}$$

the sum being taken over all even divisors d of $n - 1$. The sequence defined by equation (19) is listed in [19] under number A082936, and is known to enumerate necklaces composed by n white beads and $2n$ black beads. To show this bijectively, it suffices to remark that an oriented solid 2-tree rooted at an edge can be mapped to a plane bicolored rooted tree where white node corresponds to triangle and black nodes to edge. Several classical bijection then exist between plane trees and necklaces.

To compute $|\widetilde{\mathcal{A}}_o^{\triangle}[n]|$, we use equation (11) and the fact that

$$Z_{C_3}(y_1, y_2, \ldots) = \frac{1}{3}(y_1^3 + 2y_3).$$

We have

$$[y^n]\mathcal{B}^3(y) = \frac{1}{n}\binom{3(n-1)/2}{n-1},$$

and

$$[y^n]\mathcal{B}(y^3) = [y^{n/3}]\mathcal{B}(y) = \frac{3}{n}\binom{(n-3)/2}{n/3-1},$$

so that,

$$|\widetilde{\mathcal{A}}_o^{\triangle}[n]| = \frac{1}{3n}\binom{\frac{3(n-1)}{2}}{n-1} + \frac{2}{n}\chi(3|n)\binom{\frac{(n-3)}{2}}{\frac{n}{3}-1}, \tag{20}$$

where $\chi(3|n) = 1$ if 3 divides n and 0 otherwise. It can be easily shown, by a very similar way, that

$$|\widetilde{\mathcal{A}}_o^{\triangle}[n]| = \frac{1}{n}\binom{\frac{3(n-1)}{2}}{n-1}. \tag{21}$$

So, by virtue of the dissymmetry theorem (6), we get the following result:

Proposition 3. The number of unlabelled well oriented solid 2-trees over n edges is given by

$$|\widetilde{\mathcal{A}}_o[n]| = \frac{2}{3(n-1)}\sum_d \phi\left(\frac{n-1}{d}\right)\binom{3d/2}{d} + \chi(3|n)\frac{2}{n}\binom{\frac{n-3}{2}}{\frac{n}{3}-1} - \frac{2}{3n}\binom{\frac{3(n-1)}{2}}{n-1}, \tag{22}$$

the sum being taken over all even divisors d of $n-1$. □

We can also write $|\widetilde{\mathcal{A}}_{o,t}[m]|$ as a function of the number m of triangles, as follows:

$$|\widetilde{\mathcal{A}}_{o,t}[m]| = \frac{1}{3m}\sum_{d|m}\phi\left(\frac{m}{d}\right)\binom{3d}{d} + \chi(3|2m+1)\frac{2}{2m+1}\binom{m-1}{\frac{2m-2}{3}} - \frac{2}{3(2m+1)}\binom{3m}{m}.$$

Note that this expression also counts the number of unlabelled plane triangular (3-gonal) cacti on m triangles (see [3], formulas (88)–(91) with $m = 3$). These cacti, built with triangles, are embedded in the plane and the vertices of the triangles are oriented according to the plane orientation. The sequence of these numbers is known as sequence A054423 in the on-line Encyclopedia of Integers Sequences ([19]). There is a quite direct bijection between plane triangular cacti and oriented solid 2-trees. The correspondence is illustrated by Figure 6. To obtain a plane 3-gonal cactus from

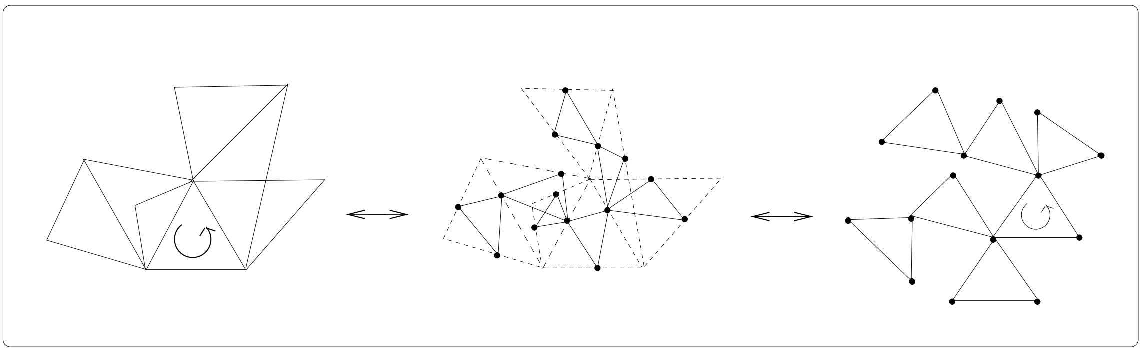

FIGURE 6. Bijection between oriented solid 2-trees and plane triangular cacti.

an oriented solid 2-tree, construct the dual of each triangle by putting vertices on edges of each triangle, and joining vertices belonging to the same triangle (see Figure 6). Preserving the cyclic order around each edge gives a plane 3-gonal cactus (using the plane orientation). This construction closely resembles the one of the line-graph of a solid 2-tree.

2.2. **Enumeration according to edge degree distribution.**

For enumeration according to edge degree distribution, we follow the approach and the notations of Labelle and Leroux [17] for plane trees enumeration according to their degree distribution. Consider $r = (r_1, r_2, r_3, \ldots)$ an infinite vector of formal variables. Recall that $\mathcal{A}[n]$ is the set of solid 2-trees over n edges. In order to keep track of the edge degree distribution, we introduce, for a given integer n, the following weight function (see [17]):

$$\begin{array}{rccc} w: & \mathcal{A}[n] & \longrightarrow & \mathbb{Q}[r_1, r_2, \ldots] \\ & s & \longmapsto & w(s) \end{array} \tag{23}$$

where $\mathbb{Q}[r_1, r_2, \ldots]$ is the ring of polynomials over the field of rational numbers $\mathbb{Q}$ in the variables $r_1, r_2, \ldots$, and where the weight of a given $\mathcal{A}$-structure s is defined by $w(s) = r_1^{n_1} r_2^{n_2} \cdots$, where n_i is the number of edges of degree i in the structure s.

We use the subscript "$_w$" to indicate that a species is weighted according to the weight function w. For instance, we denote by $\mathcal{A}_{o,w}$ and $\mathcal{A}_w$ the weighted versions of the species $\mathcal{A}_o$ and $\mathcal{A}$. We write $\mathcal{A}_w^- = (\mathcal{A}_w)^-$, $\mathcal{A}_w^{\vartriangle} = (\mathcal{A}_w)^{\vartriangle}$, $\mathcal{A}_w^{\underline{\vartriangle}} = (\mathcal{A}_w)^{\underline{\vartriangle}}$, and $\mathcal{A}_{o,w}^- = (\mathcal{A}_{o,w})^-$, $\mathcal{A}_{o,w}^{\vartriangle} = (\mathcal{A}_{o,w})^{\vartriangle}$, $\mathcal{A}_{o,w}^{\underline{\vartriangle}} = (\mathcal{A}_{o,w})^{\underline{\vartriangle}}$. Furthermore, the dissymmetry theorem remains valid in the present weighted context in both oriented and non-oriented cases:

$$\mathcal{A}_{o,w}^- + \mathcal{A}_{o,w}^{\vartriangle} = \mathcal{A}_{o,w} + \mathcal{A}_{o,w}^{\underline{\vartriangle}}, \tag{24}$$

$$\mathcal{A}_w^- + \mathcal{A}_w^{\vartriangle} = \mathcal{A}_w + \mathcal{A}_w^{\underline{\vartriangle}}. \tag{25}$$

Equations (2), (10), (11) and (12) have the following weighted versions:

$$\mathcal{B}_r(Y) = Y L_{r'}(\mathcal{B}_r^2(Y)), \tag{26}$$

and

$$\mathcal{A}_{o,w}^-(Y) = Y + Y C_r(\mathcal{B}_r^2(Y)), \tag{27}$$

$$\mathcal{A}_{o,w}^{\vartriangle}(Y) = C_3(\mathcal{B}_r(Y)), \tag{28}$$

$$\mathcal{A}_{o,w}^{\underline{\vartriangle}}(Y) = \mathcal{B}_r^3(Y), \tag{29}$$

where

$$C_r = \sum_{i \geq 1} C_{i,r_i} = C_{1,r_1} + C_{2,r_2} + C_{3,r_3} + \cdots$$

is the weighted species of oriented cycles such that a cycle of length i has the weight r_i (denoted C_{i,r_i}), and its derivative

$$L_{r'} = (C_r)' = \sum_{i \geq 0} L_{i,r_{i+1}} = 1_{r_1} + Y_{r_2} + (Y^2)_{r_3} + \cdots$$

which is the species of lists where a list of length i has the weight r_{i+1}. We recall that the derivative C' of the species of oriented cycles is isomorphic to the species L of linear orders since the data of an oriented cycle on an augmented set $U + \{*\}$ is equivalent to give a linear order on the elements of the set U; see [2, 17]. Moreover, it is well known that these species have the following generating series of labelled structures (see [2, 17]):

$$C_r(y) = r_1 y + r_2 \frac{y^2}{2} + r_3 \frac{y^3}{3} + \cdots$$

and

$$L_{r'}(y) = (C_r(y))' = r_1 + r_2 y + r_3 y^2 + \cdots .$$

Notice that, if the planted edge of a $\mathcal{B}$-structure has degree k, a weight of r_{k+1} is associated to take into account the virtual triangle used to build the structure. Then, the single edge, view as a $\mathcal{B}$-structure, receives the weight r_1. It is coherent with the definition and the use of the $\mathcal{B}$-structures, since these structures are always glued on an edge of another triangle. This also explains the reason why there is no extraneous factor of r_3 from the species C_3 in the functional equation characterizing the species $\mathcal{A}_{o,w}^{\vartriangle}$. It is also important to note that the species $\mathcal{B}_r$ and $\mathcal{A}_{o,w}^-$ are still not isomorphic.

Let $\vec{n} = (n_1, n_2, n_3, \ldots)$ be a vector of nonnegative integers. Recall that, from Lemma 1, there exists a 2-tree having a total of n edges and n_i edges of degree i, $i \geq 1$, if and only if the following relations are satisfied:

$$\sum_i n_i = n \quad \text{and} \quad \sum_i i n_i = 3\binom{n-1}{2}. \tag{30}$$

Let us begin the weighted enumeration by the labelled case.

• **Labelled case**

Let $\vec{n}$ be a coherent vector in the sense of Lemma 1 (satisfying (30)). Then, the number $|\mathcal{A}_o^-[\vec{n}]|$ of well oriented edge-rooted labelled solid 2-trees having $\vec{n}$ as edge degree distribution, is given by

$$|\mathcal{A}_o^-[\vec{n}]| = n![r_1^{n_1} r_2^{n_2} \cdots][y^n]\mathcal{A}_{o,w}^-(y). \tag{31}$$

We have

$$\begin{aligned}
[y^n]\mathcal{A}_{o,w}^-(y) &= \frac{1}{n-1}[t^{n-2}]\frac{d}{dt}(C_r(t^2)) \cdot L_{r'}^{n-1}(t^2) \\
&= \frac{2}{n-1}[t^{n-3}](r_1 + r_2 t^2 + r_3 t^4 + \cdots)^n \\
&= \frac{2}{n-1}[t^{n-3}] \sum_{\ell_1+\ell_2+\cdots=n} \binom{n}{\ell_1, \ell_2, \ldots} r_1^{\ell_1} r_2^{\ell_2} \cdots t^{2\ell_2+4\ell_3+6\ell_4+\cdots}.
\end{aligned}$$

Finally, we obtain

$$[y^n]\mathcal{A}_{o,w}^-(y) = \frac{2}{n-1} \sum_{\ell_1, \ell_2, \ldots} \binom{n}{\ell_1, \ell_2, \ldots} r_1^{\ell_1} r_2^{\ell_2} \cdots,$$

the sum being taken over all vectors $(\ell_1, \ell_2, \ldots)$ satisfying

$$\sum_i \ell_i = n \quad \text{and} \quad \sum_i 2(i-1)\ell_i = n-3.$$

We note that this condition is equivalent to relation (30). Hence, using (31), we have

$$|\mathcal{A}_o^-[\vec{n}]| = 2n(n-2)!\binom{n}{n_1, n_2, \ldots}. \tag{32}$$

As in the unweighted case, we have

$$|\mathcal{A}_o^-[\vec{n}]| = n|\mathcal{A}_o[\vec{n}]|,$$

and we get the following result.

Proposition 4. Let $\vec{n}$ be a coherent edge degree distribution. Then, the number of labelled oriented solid 2-trees having $\vec{n}$ as edge degree distribution, $|\mathcal{A}_o[\vec{n}]|$, is given by

$$|\mathcal{A}_o[\vec{n}]| = 2(n-2)!\binom{n}{n_1, n_2, \ldots}. \tag{33}$$

□

We now give the unlabelled weighted enumeration.

• **Unlabelled case**

Let $\vec{n} = (n_1, n_2, \ldots)$ be a coherent edge degree distribution. In order to compute the number $|\widetilde{\mathcal{A}}_o^-[\vec{n}]|$ of unlabelled $\mathcal{A}_o^-$-structures having $\vec{n}$ as edge degree distribution, we use the weighted version of Theorem 2.

Theorem 3. ([2]) Given two weighted species F_w and G_v, the generating series $\widetilde{H}(y)$ of unlabelled H-structures, where $H = F_w(G_v)$, is given by

$$\widetilde{H}(y) = Z_{F_w}(\widetilde{G}_v(y), \widetilde{G}_{v^2}(y^2), \widetilde{G}_{v^3}(y^3), \ldots), \tag{34}$$

with $G_{v^k}(y^k) = p_k \circ G_v(y)$, where p_k denotes the k^{th} power sum and for all structure s, $v^k(s) = (v(s))^k$. □

In the present case, we have $\mathcal{A}_{o,w}^- = Y + YC_r(\mathcal{B}_r^2)$, and since the species $\mathcal{B}$ is asymmetric, that is $\widetilde{\mathcal{B}}_r(y) = \mathcal{B}_r(y)$,

$$|\widetilde{\mathcal{A}}_o^-[\vec{n}]| = [r_1^{n_1} r_2^{n_2} \cdots][y^{n-1}]\, Z_{C_r}(\mathcal{B}_r^2(y), \mathcal{B}_{r^2}^2(y^2), \mathcal{B}_{r^3}^2(y^3), \ldots). \tag{35}$$

But, the cycle index series of the weighted species C_r, $Z_{C_r}(y_1, y_2, \ldots)$, can be expressed as the following sum:

$$Z_{C_r}(y_1, y_2, \ldots) = \sum_{k\geq 1} \frac{r_k}{k} \sum_{d|k} \phi(d) y_d^{k/d}. \tag{36}$$

Roughly speaking, the integer k represents the degree of the root-edge in the $\mathcal{A}_o^-$-structure. Hence, k may only belong to $\mathrm{Supp}(\vec{n})$, the *support* of $\vec{n}$, which consists in the set of integers $i \geq 1$ such that $n_i \neq 0$. Thus, we have, for $n > 1$,

$$|\widetilde{\mathcal{A}}_o^-[\vec{n}]| = [r_1^{n_1} r_2^{n_2} \cdots][y^{n-1}] \sum_{k\in \mathrm{Supp}(\vec{n})} \frac{r_k}{k} \sum_{d|k} \phi(d) \mathcal{B}_{r^d}^{2k/d}(y^d). \tag{37}$$

We first compute

$$[y^{n-1}]\mathcal{B}_{r^d}^{2k/d}(y^d) = [y^{(n-1)/d}]\mathcal{B}_{r^d}^{2k/d}(y).$$

Using Lagrange inversion, we get the following result, which will be usefull during later computations:

Lemma 2. We have,

$$[y^m]\mathcal{B}_{r^d}^{\ell}(y) = \frac{\ell}{m} \sum_{\ell_1, \ell_2, \ldots} \binom{m}{\ell_1, \ell_2, \ldots} r_1^{d\ell_1} r_2^{d\ell_2} \cdots, \tag{38}$$

where the ℓ_i's satisfy $\sum_i \ell_i = m$ and $\sum_i 2(i-1)\ell_i = m - \ell$. □

Now, letting $m = (n-1)/d$ and $\ell = 2k/d$ in the previous lemma, we obtain

$$|\widetilde{\mathcal{A}}_o^-[\vec{n}]| = [r_1^{n_1} r_2^{n_2} \cdots] \frac{2}{n-1} \sum_{k\in \mathrm{Supp}(\vec{n})} \sum_{d|k} \phi(d) \sum_{\ell_1, \ell_2, \ldots} \binom{(n-1)/d}{\ell_1, \ell_2, \ldots} r_1^{d\ell_1} r_2^{d\ell_2} \cdots r_k^{d\ell_k+1} \cdots. \tag{39}$$

Finally, we proved the following proposition:

Proposition 5. Let $\vec{n}$ be a coherent edge degree distribution. Then, the number $|\widetilde{\mathcal{A}}_o^-[\vec{n}]|$ of unlabelled oriented solid 2-trees pointed at an edge and having $\vec{n}$ as edge degree distribution is given by

$$|\widetilde{\mathcal{A}}_o^-[\vec{n}]| = \frac{2}{n-1} \sum_{k\in \mathrm{Supp}(\vec{n})} \sum_{d|\{k, \vec{n}-\delta_k\}} \phi(d) \binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}}, \tag{40}$$

where $\frac{\vec{n}-\delta_k}{d} = (\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_k-1}{d}, \ldots)$, for $d \geq 1$,

$$\binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}} = \binom{\frac{n-1}{d}}{n_1/d, n_2/d, \ldots, (n_k-1)/d, \ldots},$$

and $d|\{k, \vec{n}-\delta_k\}$ means that the integer d must divide k and all the components of the vector $\vec{n}-\delta_k$. □

Let $|\widetilde{\mathcal{A}}_o^{\triangle}[\vec{n}]|$ and $|\widetilde{\mathcal{A}}_o^{\underline{\triangle}}[\vec{n}]|$ be the numbers of unlabelled oriented solid 2-trees pointed respectively at a triangle and at a triangle rooted itself at one of its edges and having $\vec{n}$ as edge degree distribution. Next proposition gives explicit formulas for these numbers.

Proposition 6. Let $\vec{n}$ be a coherent edge degree distribution. Then, the numbers $|\widetilde{\mathcal{A}}_o^{\triangle}[\vec{n}]|$ and $|\widetilde{\mathcal{A}}_o^{\triangle}[\vec{n}]|$ are given by

$$|\widetilde{\mathcal{A}}_o^{\triangle}[\vec{n}]| = \frac{1}{n}\binom{n}{n_1, n_2, \ldots} + 2\frac{\chi(3|\vec{n})}{n}\binom{n/3}{n_1/3, n_2/3, \ldots}, \tag{41}$$

$$|\widetilde{\mathcal{A}}_o^{\triangle}[\vec{n}]| = \frac{3}{n}\binom{n}{n_1, n_2, \ldots}, \tag{42}$$

where

$$\chi(3|\vec{n}) = \begin{cases} 1, & \text{if all the components of } \vec{n} \text{ are multiples of 3,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let us start with $|\widetilde{\mathcal{A}}_o^{\triangle}[\vec{n}]|$. We have

$$\begin{aligned} |\widetilde{\mathcal{A}}_o^{\triangle}[\vec{n}]| &= [r_1^{n_1} r_2^{n_2} \cdots][y^n]\widetilde{\mathcal{A}}_{o,w}^{\triangle}(y) \\ &= [r_1^{n_1} r_2^{n_2} \cdots][y^n] Z_{C_3}(\widetilde{\mathcal{B}}_r(y)\widetilde{\mathcal{B}}_{r^2}(y^2), \ldots). \end{aligned}$$

Since $Z_{C_3}(y_1, y_2, \ldots) = (y_1^3 + 2y_3)/3$, and $\widetilde{\mathcal{B}}_r(y) = \mathcal{B}_r(y)$,

$$|\widetilde{\mathcal{A}}_o^{\triangle}[\vec{n}]| = \frac{1}{3}[r_1^{n_1} r_2^{n_2} \cdots][y^n]\left(\mathcal{B}_r^3(y) + 2\mathcal{B}_{r^3}(y^3)\right). \tag{43}$$

From equation (38) in Lemma 2, letting $m = n$, $\ell = 3$ and $d = 1$, we get

$$[y^n]\mathcal{B}_r^3(y) = \frac{3}{n}\sum_{\ell_1, \ell_2, \ldots}\binom{n}{\ell_1, \ell_2, \ldots} r_1^{\ell_1} r_2^{\ell_2} \cdots, \tag{44}$$

where the ℓ_i's satisfy $\sum_i \ell_i = n$ and $\sum_i 2(i-1)\ell_i = n - 3$. Now letting $m = n/3$, $\ell = 1$ and $d = 3$ in (38), we obtain

$$[y^n]\mathcal{B}_{r^3}(y^3) = [y^{n/3}]\mathcal{B}_{r^3}(y) = \frac{3}{n}\sum_{\ell_1, \ell_2, \ldots}\binom{n/3}{\ell_1, \ell_2, \ldots} r_1^{3\ell_1} r_2^{3\ell_2} \cdots, \tag{45}$$

where the ℓ_i's satisfy $\sum_i \ell_i = n/3$ and $\sum_i 2(i-1)\ell_i = n/3 - 1$. Now letting $\ell_i = n_i$ in (44) and $\ell_i = n_i/3$ in (45), we get equation (41). We obtain (42) in a very similar way, details are left to the reader. ∎

Finally, using the dissymmetry theorem (6), we obtain the final result of this section.

Proposition 7. Let $\vec{n}$ be a coherent edge degree distribution. Then, the number $|\widetilde{\mathcal{A}}_o[\vec{n}]|$ of unlabelled oriented solid 2-trees having $\vec{n}$ as edge degree distribution is given by

$$|\widetilde{\mathcal{A}}_o[\vec{n}]| = \frac{2}{n-1}\sum_{k\in\mathrm{Supp}(\vec{n})}\sum_{d|\{k, \vec{n}-\delta_k\}}\phi(d)\binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}} + 2\frac{\chi(3|\vec{n})}{n}\binom{\frac{n}{3}}{\frac{n_1}{3}, \frac{n_2}{3}, \ldots} - \frac{2}{n}\binom{n}{n_1, n_2, \ldots}, \tag{46}$$

where

$$\chi(3|\vec{n}) = \begin{cases} 1, & \text{if all the components of } \vec{n} \text{ are multiples of 3,} \\ 0, & \text{otherwise,} \end{cases}$$

$$\frac{\vec{n}-\delta_k}{d} = \left(\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_k - 1}{d}, \ldots\right), \quad \text{for } d \geq 1,$$

and

$$\binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}} = \binom{\frac{n-1}{d}}{\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_k-1}{d}, \ldots}.$$

□

3. Non-oriented solid 2-trees

In order to compute the numbers of labelled and unlabelled solid 2-trees, we use Burnside's Lemma with the group $\mathbb{Z}_2 = \{Id, \tau\}$, where the action of τ is to reverse the orientation of the structures. This involves the notion of quotient species (see [4]).

3.1. Enumeration according to the number of edges.

As in the unweighted case, we begin with the labelled and unlabelled enumeration according to the number of edges.

• Labelled case

The labelled case is particularly simple since every labelled oriented 2-tree has exactly two possible orientations except the structure consisting of a single oriented edge. Hence, we have:

Proposition 8. The number $|\mathcal{A}[n]|$ of edge-labelled solid 2-trees over n edges is given by

$$|\mathcal{A}[n]| = \begin{cases} \frac{1}{2}|\mathcal{A}_o[n]|, & \text{if } n > 1, \\ 1, & \text{if } n = 1. \end{cases} \tag{47}$$

□

Of course, the same argument remains valid for all other pointed structures discussed in the previous section.

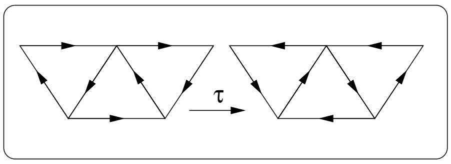

FIGURE 7. An unlabelled oriented 2-tree invariant under the action of τ.

• Unlabelled case

In the unlabelled case, the action of τ is not so trivial. Figure 7 shows an oriented 2-tree which is left fixed under the action of τ. Let $\mathcal{A}^-$ be the species of (unoriented) solid 2-trees rooted at an edge. This species can be expressed as the following quotient species (see [8, 9, 14, 15] for quotient species related to 2-trees):

$$\mathcal{A}^- = \frac{\mathcal{A}_o^-}{\mathbb{Z}_2} = \frac{Y + YC(\mathcal{B}^2(Y))}{\mathbb{Z}_2}, \tag{48}$$

where $\mathbb{Z}_2 = \{\text{Id}, \tau\}$ is the two-element group consisting of the identity and τ, whose action is to reverse the orientation of the edges. Hence, an unlabelled $\mathcal{A}^-$-structure is an orbit $\{a, \tau \cdot a\}$ under the action of $\mathbb{Z}_2$, where a is any (oriented) unlabelled $\mathcal{A}_o^-$-structure. Roughly speaking, quotient by $\mathbb{Z}_2$ corresponds to forgetting the orientation in the structures. A structure $s = \{a, \tau \cdot a\}$ is said *τ-symmetric* if $a = \tau \cdot a$, that is, the structure s is invariant (or equivalently, left fixed) under the orientation reversing induced by τ.

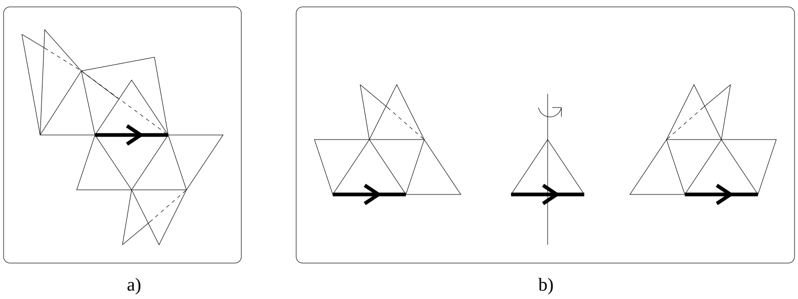

FIGURE 8. A $\mathcal{B}_{\text{Sym}}$-structure.

Let us introduce the auxiliary species $\mathcal{B}_{\text{Sym}}$ of τ-symmetric $\mathcal{B}$-structures; see Figure 8 a) for such a symmetric $\mathcal{B}$-structure, where only the orientation of the planted-edge is indicated. We notice that the orientation of the root-edge can be extended uniquely to the whole structure by first cyclically orienting the triangles incident to the root-edge and then recursively transmitting this orientation to adjacent triangles. Denote by $\widetilde{\mathcal{B}}_{\text{Sym}}(y)$ the ordinary generating series of unlabelled $\mathcal{B}_{\text{Sym}}$-structures. Recall the functional equation satisfied by the species $\mathcal{B}$

$$\mathcal{B} = YL(\mathcal{B}^2).$$

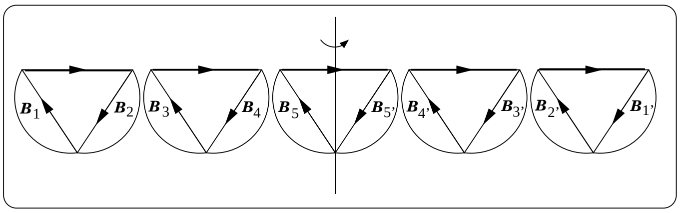

FIGURE 9. A $\mathcal{B}_{\text{Sym}}$-structure, for i odd.

In order to compute $\widetilde{\mathcal{B}}_{\text{Sym}}(y)$, we have to distinguish two cases according to the parity of k, the length of the list of $\mathcal{B}^2$-structures attached to the rooted edge. First consider the case where k is odd (Figures 9 and 8 show examples, where $k = 5$ and $k = 3$, respectively). A $\mathcal{B}$-structure is τ-symmetric if it can be embedded in space in such a way that the action of reversing the orientation of all edges corresponds to flip the whole structure back to itself by reversing the end points of the root edge. When an inversion of the orientation of the rooted edge is applied, the two $\mathcal{B}$-structures glued on the two (non root) sides of the middle triangle (structures $\mathcal{B}_5$ and $\mathcal{B}_{5'}$ in Figure 9) are isomorphically exchanged. The $k-1$ remaining triangles are exchanged pairwise carrying with them each of their attached $\mathcal{B}$-structures as shown in Figure 9, where $\mathcal{B}_{i'} = \tau \cdot \mathcal{B}_i$. This gives a factor of $\mathcal{B}^k(y^2)$. We then have to sum the previous expression over all odd values of k. The case where k is even is very similar except that there is no middle triangle, as shown in Figure 10 and we get the same expression summed over all even values of k. It leads us to

$$\widetilde{\mathcal{B}}_{\text{Sym}}(y) = y \sum_{i \geq 0} \mathcal{B}^i(y^2) = \frac{y}{1 - \mathcal{B}(y^2)}. \tag{49}$$

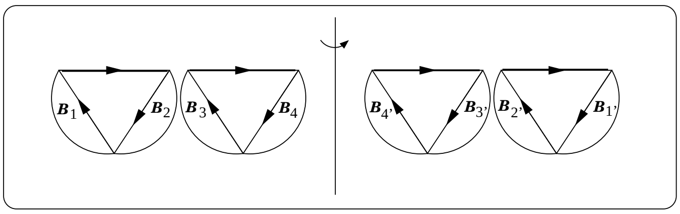

FIGURE 10. A $\mathcal{B}_{\text{Sym}}$-structure, for i even.

From expression (49) and another use of Lagrange inversion, we easily obtain the following result.

Proposition 9. The number $|\widetilde{\mathcal{B}}_{\text{sym}}[m]|$ of τ-symmetric unlabelled oriented $\mathcal{B}$-structures over m triangles is given by

$$|\widetilde{\mathcal{B}}_{\text{sym}}[m]| = \begin{cases} \dfrac{2}{m}\dbinom{3m/2}{m+1}, & \text{if } m \text{ is even,} \\[2ex] \dfrac{1}{m}\dbinom{(3m-1)/2}{m+1} + \dfrac{1}{m}\dbinom{(3m+1)/2}{m+1}, & \text{if } m \text{ is odd.} \end{cases} \tag{50}$$

Proof. To obtain the coefficient $[y^n]\widetilde{\mathcal{B}}_{\mathrm{sym}}(y)$, first write

$$[y^n]\widetilde{\mathcal{B}}_{\mathrm{sym}}(y) = [y^{n-1}]\left(1-\mathcal{B}(y)\right)^{-1} = [y^{\frac{n-1}{2}}]F(\mathcal{B}(y)), \tag{51}$$

where $F(t)=(1-t)^{-1}$. Putting $m=(n-1)/2$ and making use of Lagrange inversion yields to

$$\begin{aligned}[y^m]F(\mathcal{B}(y)) &= \frac{1}{m}[t^{m-1}](1-t)^{-2}(1-t^2)^{-m}\\ &= \frac{1}{m}[t^{m-1}](1+t)^2(1-t^2)^{-2}(1-t^2)^{-m}\\ &= \frac{1}{m}[t^{m-1}](1+2t+t^2)(1-t^2)^{-(m+2)}.\end{aligned}$$

Recalling that $(1-t^2)^{-(m+2)} = \sum_\nu \binom{m+1+\nu}{\nu} t^{2\nu}$, we get in a straightforward way

$$[y^m]F(\mathcal{B}(y)) = \frac{1}{m}\binom{(3m+1)/2}{m+1} + \frac{2}{m}\binom{3m/2}{m+1} + \frac{1}{m}\binom{(3m-1)/2}{m+1}. \tag{52}$$

As $m=(n-1)/2$ is the number of triangles of the solid 2-tree having n edges, we proved the stated formula (50). ■

We can also express $|\widetilde{\mathcal{B}}_{\mathrm{sym}}[m]|$ as follows:

$$|\widetilde{\mathcal{B}}_{\mathrm{sym}}[m]| = \begin{cases} \dfrac{1}{2k+1}\dbinom{3k}{k}, & \text{if } m=2k,\\[2ex] \dfrac{1}{2k+1}\dbinom{3k+1}{k+1}, & \text{if } m=2k+1.\end{cases} \tag{53}$$

It is easy to obtain the last coefficient in terms of the number n of edges replacing m by $(n-1)/2$ in the formula.

Note that the numbers $|\widetilde{\mathcal{B}}_{\mathrm{sym}}[m]|$ also enumerate several classes of symmetric objects (in some sense), in particular, symmetric diagonally convex directed polyominoes, symmetric non-crossing trees, … (see [6, 7]). These numbers are indexed in the on-line Encyclopedia of Integer Sequences [19] as sequence A047749.

We now give an expression for the generating function of unlabelled quotient structures, which will allow us to enumerate various kind of unlabelled solid 2-trees.

Proposition 10. ([4]) Let $F=F_w$ be any (weighted) species and G, a group acting on F. Then the ordinary generating series of the quotient species F/G is given by

$$(F/G)^{\sim}(y) = \frac{1}{|G|}\sum_{g\in G}\sum_{n\geq 0}|\mathrm{Fix}_{\widetilde{F}_n}(g)|_w\, y^n, \tag{54}$$

where $\mathrm{Fix}_{\widetilde{F}_n}(g)$ denotes the set of unlabelled F-structures over n edges left fixed under the action of the element $g\in G$ and $|\mathrm{Fix}_{\widetilde{F}_n}(g)|_w$ represents the total weight of this set. □

Proposition 11. The ordinary generating series of edge-rooted solid 2-trees $\widetilde{\mathcal{A}}^-(y)$ is given by

$$\widetilde{\mathcal{A}}^-(y) = \frac{1}{2}\widetilde{\mathcal{A}}_o^-(y) + \frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{sym}}(y). \tag{55}$$

Proof. Using an unweighted version of Proposition 10, with $F=\mathcal{A}_o^-$ and $G=\mathbb{Z}_2=\{\mathrm{Id},\tau\}$, we obtain

$$\widetilde{\mathcal{A}}^-(y) = \frac{1}{2}\sum_{n\geq 0}|\mathrm{Fix}_{\widetilde{\mathcal{A}}_{o,n}^-}(\mathrm{Id})|y^n + \frac{1}{2}\sum_{n\geq 0}|\mathrm{Fix}_{\widetilde{\mathcal{A}}_{o,n}^-}(\tau)|y^n. \tag{56}$$

We then have to compute

$$\widetilde{\mathcal{A}}_{o,\tau}^-(y) = \sum_{n\geq 0}|\mathrm{Fix}_{\widetilde{\mathcal{A}}_{o,n}^-}(\tau)|y^n,$$

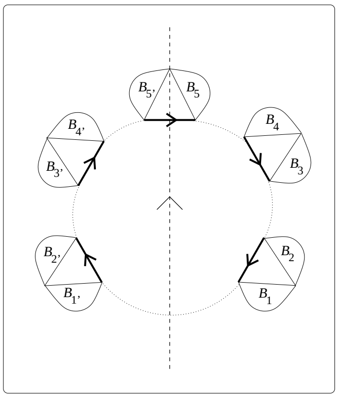

FIGURE 11. A τ-symmetric $\mathcal{A}_o^-$-structure, for k odd.

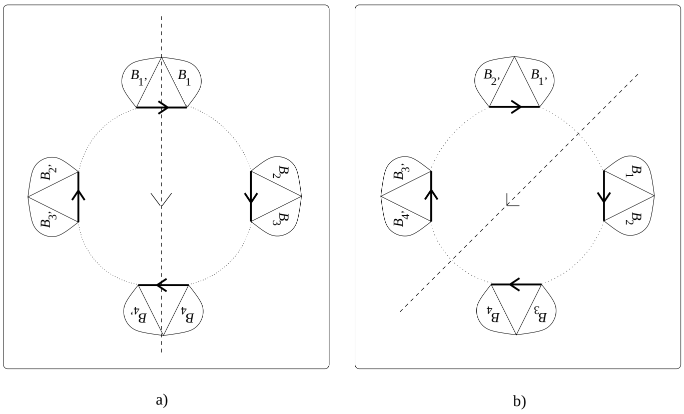

FIGURE 12. $\mathcal{A}_o^-$-structures, for k even with a) edge-edge; b) split; symmetry.

the ordinary generating series of unlabelled $\mathcal{A}_o^-$-structures left fixed under the orientation reversing induced by $\tau \in \mathbb{Z}_2$ (*i.e.*, τ-symmetric). Recall that the species $\mathcal{A}_o^-$ satisfy the relation $\mathcal{A}_o^- = Y + YC(\mathcal{B}^2(Y))$. We treat two cases separately according to the parity of the integer k, the length of the base oriented cycle of triangles around the rooted edge (also called *degree* of the root-edge).

• **Case 1:** The integer k is odd.
We illustrate this case with $k = 5$ and Figure 11; the reader can easily convince himself of the validity of the argument for any odd integer k. In Figure 11, to obtain the spatial representation of the solid 2-tree, the oriented rooted edges have to be identified keeping the cyclic order of triangles given by the plane (counterclockwise, for instance). To leave fixed under orientation reversing such a structure, the planar representation must possess a reflexive axis of symmetry (dashed line in the picture). This axis breaks the base oriented cycle and naturally gives an unlabelled $\mathcal{B}_{\mathrm{sym}}$-structure where the linear order around the planted edge is of lenght k and $\mathcal{B}_i^{'} = \tau \cdot \mathcal{B}_i$, $i = 1, 2, 3, 4, 5$; actually, we recover the situation of Figure 9.
We then get the following expression for $\widetilde{\mathcal{A}}_{o,\tau}^-(y)$, for odd values of the integer k (see equation (49)):

$$y \sum_{k \text{ odd}} \mathcal{B}^k(y^2) = y \sum_{i \geq 0} \mathcal{B}^{2i+1}(y^2). \tag{57}$$

• **Case 2:** The integer k is even (see [15], Proposition 14).
We illustrate the even case with $k = 4$ and Figure 12. As for the odd case, in the planar representations (Figure 12 a) and b)), to be τ-symmetric, an $\mathcal{A}_o^-$-structure must admit a symmetry which can be of two kinds:

(1) *edge-edge symmetry*, that is, an axis passing through two opposite rooted edge; see Figure 12 a);
(2) *split symmetry*, that is, an axis splitting the base oriented cycle of triangles around the root-edge in two equal parts; see Figure 12 b).

Obviously, a structure can have both symmetries.
To make a rigorous analysis of all possible configurations, the reader is referred to the proof of Proposition 14 in [15]. It then suffices to adapt quite directly this proof to the present context, as the reader can check. Therefore, for an $\mathcal{A}_o^-$-structure having a split symmetry, as for the odd case, we can break the base oriented cycle and we get an unlabelled $\mathcal{B}_{\mathrm{sym}}$-structure where the degree of the planted edge is even; see Figure 10. For the split symmetry, we obtain the term

$$\frac{1}{2}y\sum_{k\ \mathrm{even}}\mathcal{B}^k(y^2)=\frac{1}{2}y\sum_{i\geq 1}\mathcal{B}^{2i}(y^2), \tag{58}$$

where the division by 2 stands for the two possible orientations of the axis of symmetry.
For an $\mathcal{A}_o^-$-structure with an edge-edge symmetry, after orienting the axis of symmetry, we can decomposed the structure as follows: a single triangle with a rooted edge and two isomorphic $\mathcal{B}$-structures on its two remaining sides (say $\mathcal{B}_1$ and $\mathcal{B}_1^{'}$, for example), along with a $\mathcal{B}_{\mathrm{sym}}$-structure whose planted edge degree is $k-1$ (3, in Figure 12 b)). Thus, it yields to the following term:

$$\frac{1}{2}y\mathcal{B}(y^2)\sum_{k\ \mathrm{odd}}\mathcal{B}^k(y^2)=\frac{1}{2}y\sum_{i\geq 0}\mathcal{B}^{2i+2}(y^2). \tag{59}$$

Note that the structures possessing both kinds of symmetry are counted half a time in both terms (58) and (59), leading to the correct enumeration of these structures. We also have to take into account the case of a single edge, yielding to the term y.
Collecting terms (57), (58) and (59), we successively deduce

$$\begin{aligned}\widetilde{\mathcal{A}}_{o,\tau}^-(y) &= y\left(\sum_{i\geq 0}\mathcal{B}^{2i+1}(y^2)+\frac{1}{2}\sum_{i\geq 1}\mathcal{B}^{2i}(y^2)+\frac{1}{2}\sum_{i\geq 0}\mathcal{B}^{2i+2}(y^2)+1\right)\\ &= y\left(\sum_{i\geq 0}\mathcal{B}^{2i+1}(y^2)+\sum_{i\geq 1}\mathcal{B}^{2i}(y^2)+1\right)\\ &= y\sum_{i\geq 0}\mathcal{B}^{i}(y^2)\ =\ \widetilde{\mathcal{B}}_{\mathrm{sym}}(y),\end{aligned}$$

which concludes the proof. ■

Hence, it becomes easy to extract the coefficient of y^n in relation (55), and we get the number $|\mathcal{A}^-[n]|$ of edge-pointed solid 2-trees over n edges,

$$|\mathcal{A}^-[n]|=\frac{1}{2}|\widetilde{\mathcal{A}}_o^-[n]|+\frac{1}{2}|\widetilde{\mathcal{B}}_{\mathrm{Sym}}[n]|. \tag{60}$$

We now consider the species $\mathcal{A}^\triangle$ of triangle rooted solid 2-trees. Since $\mathcal{A}^\triangle=\mathcal{A}_o^\triangle/\mathbb{Z}_2$, by virtue of Proposition 10, we have

$$\widetilde{\mathcal{A}}^\triangle(y)=\frac{1}{2}\sum_{n\geq 0}|\mathrm{Fix}_{\widetilde{\mathcal{A}}_{o,n}^\triangle}(\mathrm{Id})|y^n+\frac{1}{2}\sum_{n\geq 0}|\mathrm{Fix}_{\widetilde{\mathcal{A}}_{o,n}^\triangle}(\tau)|y^n, \tag{61}$$

where $|\mathrm{Fix}_{\widetilde{\mathcal{A}}_{o,n}^\triangle}(\tau)|$, the number of τ-symmetric $\mathcal{A}^\triangle$-structures over n edges, has to be determined. As shown in Figure 13, such a structure must have an axis of symmetry which coincides with one of the root-triangle's medians. Since the structure is already determined up to rotation around the root-triangle, the choice among the three possible axes is arbitrary. The base side of the triangle must be a $\mathcal{B}_{\mathrm{Sym}}$-structure while the two other sides must be isomorphic copies of the same $\mathcal{B}$-structure ($\mathcal{B}^{'}=\tau\cdot\mathcal{B}$). Therefore,

$$\widetilde{\mathcal{A}}^\triangle(y)=\frac{1}{2}\widetilde{\mathcal{A}}_o^\triangle(y)+\frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{Sym}}(y)\mathcal{B}(y^2). \tag{62}$$

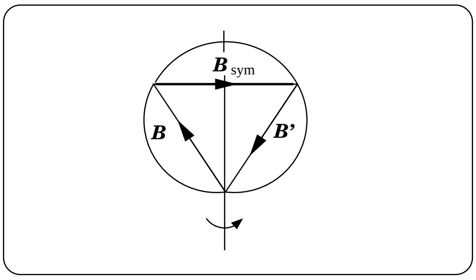

FIGURE 13. A τ-symmetric $\mathcal{A}_o^{\triangle}$-structure.

In a very similar way, since $\mathcal{A}^{\triangle} = \mathcal{A}_o^{\triangle}/\mathbb{Z}_2$, we obtain

$$\widetilde{\mathcal{A}}^{\triangle}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_o^{\triangle}(y) + \frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{Sym}}(y)\mathcal{B}(y^2). \tag{63}$$

Finally, combining equations (55), (62), (63) and the dissymmetry theorem, we get:

Proposition 12. The ordinary generating function of unlabelled solid 2-trees is given by

$$\widetilde{\mathcal{A}}(y) = \frac{1}{2}(\widetilde{\mathcal{A}}_o(y) + \widetilde{\mathcal{B}}_{\mathrm{Sym}}(y)), \tag{64}$$

where $\widetilde{\mathcal{B}}_{\mathrm{Sym}}(y)$ is the ordinary generating series of τ-symmetric $\mathcal{B}$-structures. Consequently, the number $|\widetilde{\mathcal{A}}_t[m]|$ of unoriented solid 2-trees over m triangles is given by

$$|\widetilde{\mathcal{A}}_t[m]| = \frac{1}{2}(|\widetilde{\mathcal{A}}_{o,t}[m]| + |\widetilde{\mathcal{B}}_{\mathrm{sym}}[m]|), \tag{65}$$

where

$$|\widetilde{\mathcal{A}}_{o,t}[m]| = \frac{1}{3m}\sum_{d|m}\phi\left(\frac{m}{d}\right)\binom{3d}{d} + \chi(3|2m+1)\frac{2}{2m+1}\binom{m-1}{\frac{2m-2}{3}} - \frac{2}{3(2m+1)}\binom{3m}{m},$$

and

$$|\widetilde{\mathcal{B}}_{\mathrm{sym}}[m]| = \begin{cases} \dfrac{2}{m}\dbinom{3m/2}{m+1}, & \text{if } m \text{ is even,} \\[2ex] \dfrac{1}{m}\dbinom{(3m-1)/2}{m+1} + \dfrac{1}{m}\dbinom{(3m+1)/2}{m+1}, & \text{if } m \text{ is odd.} \end{cases} \tag{66}$$

□

We note that to express $|\widetilde{\mathcal{A}}_t[m]|$ in terms of the number n of edges, we only have to set $n := 2m+1$ in the previous expressions; see Table 1 for some values of this number $|\widetilde{\mathcal{A}}[n]|$. We note that the sequence $|\widetilde{\mathcal{A}}_t[m]|$ giving the number of solid 2-trees on $2m+1$ edges (m triangles) is listed in [19] under sequence number A082938.

3.2. Enumeration of solid 2-trees according to the edge degree distribution.

We consider again the weight function defined by

$$\begin{array}{rccc} w: & \mathcal{A}[n] & \longrightarrow & \mathbb{Q}[r_1, r_2, \ldots] \\ & s & \mapsto & w(s) \end{array} \tag{67}$$

where $r = (r_1, r_2, r_3, \ldots)$ is an infinite vector of formal variables and n is any positive integer.

• Labelled case

Using the same argument as in the unweighted case, we have

$$|\mathcal{A}[\vec{n}]| = \begin{cases} \frac{1}{2}|\mathcal{A}_o[\vec{n}]|, & \text{if } n > 1, \\ 1, & \text{if } n = 1, \end{cases} \tag{68}$$

where $\vec{n}$ is a valid edge degree distribution, n is the number of edges and $|\mathcal{A}[\vec{n}]| = [r_1^{n_1} r_2^{n_2} \cdots][y^n]\mathcal{A}_w(y)$.

• Unlabelled case

All the structures of this section are weighted in the same way as used in Section 2.2 for the oriented case. Recall that the subscript "$_w$" means that a species is weighted with the function w introduced in Section 2.2.

Proposition 13. The ordinary generating series of the species $\mathcal{A}_w^-$, $\mathcal{A}_w^{\triangle}$ and $\mathcal{A}_w^{\hat{\triangle}}$ are given by

$$\widetilde{\mathcal{A}}_w^-(y) = \frac{1}{2}\widetilde{\mathcal{A}}_{o,w}^-(y) + \frac{1}{2}y\sum_{k\geq 0} r_k \mathcal{B}_{r^2}^k(y^2), \tag{69}$$

$$\widetilde{\mathcal{A}}_w^{\triangle}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_{o,w}^{\triangle}(y) + \frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{sym},r}(y)\mathcal{B}_r(y^2), \tag{70}$$

$$\widetilde{\mathcal{A}}_w^{\hat{\triangle}}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_{o,w}^{\hat{\triangle}}(y) + \frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{sym},r}(y)\mathcal{B}_r(y^2), \tag{71}$$

where

$$\widetilde{\mathcal{B}}_{\mathrm{sym},r}(y) = y\sum_{k\geq 0} r_{k+1}\mathcal{B}_{r^2}^k(y^2), \tag{72}$$

is the ordinary generating series of unlabelled τ-symmetric $\mathcal{B}_r$-structures, that is, unlabelled w-weighted $\mathcal{B}$-structures that are left fixed under orientation reversing.

Proof. We make use of Proposition 10 with the group $G = \mathbb{Z}_2 = \{1, \tau\}$. Using the weighted versions of equations (62) and (63), we easily get

$$\begin{aligned}\widetilde{\mathcal{A}}_w^{\triangle}(y) &= \frac{1}{2}\widetilde{\mathcal{A}}_{o,w}^{\triangle}(y) + \frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{sym},r}(y)\mathcal{B}_r(y^2),\\ \widetilde{\mathcal{A}}_w^{\hat{\triangle}}(y) &= \frac{1}{2}\widetilde{\mathcal{A}}_{o,w}^{\hat{\triangle}}(y) + \frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{sym},r}(y)\mathcal{B}_r(y^2),\end{aligned}$$

where the only unknown is $\widetilde{\mathcal{B}}_{\mathrm{sym},r}(y)$. Before computing it, we first establish an additional condition on the edge degree distribution for an edge-rooted oriented solid 2-tree to be τ-symmetric. Since the root edge must remain fixed and all other edges are exchanged pairwise, the edge degree distribution vector $\vec{n}$ must have all its components even except one odd corresponding to the degree of the rooted edge.

To compute the series $\widetilde{\mathcal{B}}_{\mathrm{sym},r}(y)$, let us consider an unlabelled $\mathcal{B}_{\mathrm{sym},r}$-structure whose planted-edge degree is $k \geq 1$. We recall that a weight of r_{k+1} is attached with the root-edge (see Section 2.2) to take into account a virtual triangle used to build the structure. For the weighted case, the analyis of a $\mathcal{B}_{\mathrm{sym},r}$-structure is the same as for a $\mathcal{B}_{\mathrm{sym}}$-structure (Section 2.2) except the fact that we have to take care about the weight of the structure. Keeping in mind (see Figures 9 and 10) that each of the $\mathcal{B}$-strutures $\mathcal{B}_i$, $1 \leq i \leq k$, must have its reverse $\mathcal{B}$-structure $\mathcal{B}_{i'} = \tau \cdot \mathcal{B}_i$, we obtain the term $\mathcal{B}_{r^2}^k(y^2)$, where, for any structure s, $w^2(s) = (w(s))^2$. The weight is raised to a power of two since all the monomials $r_1^{i_1} r_2^{i_2} \ldots$ occuring in the total weight of each $\mathcal{B}_i$ must appear in the total weight of $\mathcal{B}_{i'}$. Summing over all even and odd values of k, we finally get

$$\widetilde{\mathcal{B}}_{\mathrm{sym},r}(y) = y\sum_{k\geq 0} r_{k+1}\mathcal{B}_{r^2}^k(y^2). \tag{73}$$

For the species $\widetilde{\mathcal{A}}_w^-$, we get

$$\widetilde{\mathcal{A}}_w^-(y) = \frac{1}{2}\sum_{n\geq 0}|\mathrm{Fix}_{\widetilde{\mathcal{A}}_{o,n}^-}(\mathrm{Id})|_w y^n + \frac{1}{2}\sum_{n\geq 0}|\mathrm{Fix}_{\widetilde{\mathcal{A}}_{o,n}^-}(\tau)|_w y^n \tag{74}$$

$$= \frac{1}{2}\widetilde{\mathcal{A}}_o^-(y) + \frac{1}{2}\widetilde{\mathcal{A}}_{o,w,\tau}^-(y). \tag{75}$$

We then have to compute

$$\widetilde{\mathcal{A}}^-_{o,w,\tau}(y) = \sum_{n\geq 0} |\mathrm{Fix}_{\widetilde{\mathcal{A}}^-_{o,n}}(\tau)|_w y^n,$$

the generating series of weighted unlabelled $\mathcal{A}^-_o$-structures invariant under the action of $\tau \in \mathbb{Z}_2$. Following the proof of Proposition 11 taking into account the weight w, we obtain

$$\widetilde{\mathcal{A}}^-_{o,w,\tau}(y) = y \sum_{k\geq 0} r_k \mathcal{B}^k_{r^2}(y^2).$$

Note that this expression is very close to the one of $\widetilde{\mathcal{B}}_{\mathrm{sym},r}(y)$, but the weighting of $\mathcal{A}^-_o$-structures differs from the one of $\mathcal{B}_{\mathrm{sym}}$-structures since, $\mathcal{B}_{\mathrm{sym}}$-structures are built with a virtual triangle around the planted edge. ∎

Making use of the dissymmetry theorem leads to

$$\widetilde{\mathcal{A}}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_{o,w}(y) + \frac{1}{2} y \sum_{k\geq 0} r_k \mathcal{B}^k_{r^2}(y^2). \tag{76}$$

Then, for an edge degree distribution $\vec{n} = (n_1, n_2, \ldots)$ satisfying the additional condition above, we have

$$|\widetilde{\mathcal{A}}^-_{o,w,\tau}[\vec{n}]| = [r_1^{n_1} r_2^{n_2} \cdots][y^n]\widetilde{\mathcal{A}}^-_{o,w,\tau}(y) \tag{77}$$

$$= \frac{2k}{n-1}\binom{\frac{n-1}{2}}{\frac{\vec{n}-\delta_k}{2}}, \tag{78}$$

where k equals the degree of the root-edge. Actually the integer k corresponds to the unique index $i \in \mathbb{N}^*$ such that n_i is odd. We now present the final result of this paper.

Proposition 14. Let $\vec{n}$ be a vector satisfying

$$\sum_i n_i = n \quad \text{and} \quad \sum_i i n_i = 3m.$$

Then, the number $|\widetilde{\mathcal{A}}[\vec{n}]|$ of (non oriented) unlabelled solid 2-trees having $\vec{n}$ as edge degree distribution is given by

$$|\widetilde{\mathcal{A}}[\vec{n}]| = \frac{1}{2}|\widetilde{\mathcal{A}}_o[\vec{n}]| + \frac{1}{2}|\widetilde{\mathcal{A}}^-_{o,w,\tau}[\vec{n}]| \tag{79}$$

where

$$|\widetilde{\mathcal{A}}^-_{o,w,\tau}[\vec{n}]| = \begin{cases} \dfrac{2k}{n-1}\dbinom{\frac{n-1}{2}}{\frac{\vec{n}-\delta_k}{2}}, & \text{if } \vec{n} \text{ has a unique odd component, } n_k, \\ 0, & \text{otherwise,} \end{cases}$$

δ_k being the vector having 1 at the k^{th} component and 0 everywhere else, and

$$|\widetilde{\mathcal{A}}_o[\vec{n}]| = \frac{2}{n-1} \sum_{k\in \mathrm{Supp}(\vec{n})} \sum_{d|\{k,\vec{n}-\delta_k\}} \phi(d)\binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}} + 2\frac{\chi(3|\vec{n})}{n}\binom{n/3}{n_1/3, n_2/3, \ldots} - \frac{2}{n}\binom{n}{n_1, n_2, \ldots}.$$

□

Appendix.

To conclude this paper, we present two tables giving the numbers of unlabelled solid 2-trees oriented and unoriented as well as the number of unlabelled τ-symmetric $\mathcal{B}$-structures. The first table gives these numbers according to the number n of edges for odd values of n from 1 up to 21, and the second, according to edge degree distribution for a few vectors $\vec{n}$. We use the notation $1^{n_1}2^{n_2}\cdots$, where i^{n_i} means n_i edges of degree i.

n	$\lvert\widetilde{\mathcal{A}}_o[n]\rvert$	$\lvert\widetilde{\mathcal{B}}_{\mathrm{sym}}[n]\rvert$	$\lvert\widetilde{\mathcal{A}}[n]\rvert$
1	1	1	1
3	1	1	1
5	1	1	1
7	2	2	2
9	7	3	5
11	19	7	13
13	86	12	49
15	372	30	201
17	1825	55	940
19	9143	143	4643
21	47801	273	24037

TABLE 1. Number of solid 2-trees according to the number of edges.

$\vec{n}$	$\lvert\widetilde{\mathcal{A}}_o[\vec{n}]\rvert$	$\lvert\widetilde{\mathcal{B}}_{\mathrm{sym}}[\vec{n}]\rvert$	$\lvert\widetilde{\mathcal{A}}[\vec{n}]\rvert$
$1^7 2^1 3^1$	2	0	1
$1^8 2^2 3^1$	9	3	6
$1^{12} 2^1 3^1 4^1$	46	0	23
$1^{10} 5^1$	3	1	2
$1^{15} 4^1 5^1$	2	0	1
$1^{16} 3^2 5^1$	17	5	11
$1^{15} 2^2 7^1$	34	0	17

TABLE 2. Number of solid 2-trees according to edge degree distribution.

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E-mail address: bousq2@math.uqam.ca,lamathe@loria.fr

Lacim, Département de Mathématiques, Université du Québec à Montréal.

Loria - INRIA Lorraine