# Counting Permutations by their Rigid Patterns

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**Abstract**

In how many permutations does the *pattern* $\tau$ occur exactly $m$ times? In most cases, the answer is unknown. When we search for *rigid patterns*, on the other hand, we obtain exact formulas for the solution, in all cases considered.



**keywords:** pattern, rigid pattern, permutation, block



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# 1 Introduction

The results of this paper generalize a fun, easy counting problem suggested by Herb Wilf. We begin by introducing and solving this problem. Given a permutation of the set $[n] = \{1, 2, \ldots, n\}$, an *$n$-permutation*, we consider sequences of consecutive integers which appear in consecutive positions. A maximal such sequence is called a *block*. For example, the 8-permutation 12678345 contains three blocks: 12, 678, and 345. Of the six 3-permutations, one contains one block (123), two contain two blocks (312 and 231), and three contain three blocks (132, 213, and 321). Figure 1 shows the 4-permutations grouped by number of blocks.

$m = 1:$ $\boxed{1\,2\,3\,4}$

$m = 2:$ $\boxed{2\,3\,4}\boxed{1}$ $\quad \boxed{4}\boxed{1\,2\,3}$ $\quad \boxed{3\,4}\boxed{1\,2}$

$m = 3:$ $\boxed{1\,2}\boxed{4}\boxed{3}$ $\quad \boxed{3}\boxed{1\,2}\boxed{4}$ $\quad \boxed{4}\boxed{3}\boxed{1\,2}$ $\quad \boxed{2\,3}\boxed{1}\boxed{4}$

$\boxed{4}\boxed{2\,3}\boxed{1}$ $\quad \boxed{1}\boxed{4}\boxed{2\,3}$ $\quad \boxed{3\,4}\boxed{2}\boxed{1}$ $\quad \boxed{1}\boxed{3\,4}\boxed{2}$ $\quad \boxed{2}\boxed{1}\boxed{3\,4}$

$m = 4:$ $\boxed{1}\boxed{3}\boxed{2}\boxed{4}$ $\quad \boxed{2}\boxed{1}\boxed{4}\boxed{3}$ $\quad \boxed{3}\boxed{1}\boxed{4}\boxed{2}$ $\quad \boxed{4}\boxed{1}\boxed{3}\boxed{2}$ $\quad \boxed{2}\boxed{4}\boxed{1}\boxed{3}$

$\boxed{3}\boxed{2}\boxed{1}\boxed{4}$ $\quad \boxed{4}\boxed{2}\boxed{1}\boxed{3}$ $\quad \boxed{1}\boxed{4}\boxed{3}\boxed{2}$ $\quad \boxed{2}\boxed{4}\boxed{3}\boxed{1}$ $\quad \boxed{3}\boxed{2}\boxed{4}\boxed{1}$ $\quad \boxed{4}\boxed{3}\boxed{2}\boxed{1}$

*Figure 1. 4-permutations with $m$ blocks.*

Wilf posed the following question.

$\boxed{\text{How many } n\text{-permutations contain exactly } m \text{ blocks?}}$

To solve this problem, we count, for each choice of $m$ blocks, the number of $n$-permutations which contain exactly those blocks. To choose $m$ blocks, we list the integers from 1 to $n$, and insert $m-1$ "dividers" into the $n-1$ spaces between integers. For example, the insertion of dividers, 12—345—678, yields blocks 12, 345, and 678. Each choice of $m-1$ spaces to place dividers from the set of $n-1$ possible spaces gives a distinct choice of blocks. Thus there are $\binom{n-1}{m-1}$ ways to choose the blocks.

Next we count the number of $n$-permutations which contain a given choice of $m$ blocks. Suppose $n = 8$, $m = 3$, and we are given blocks $\beta_1 = 12$, $\beta_2 = 345$, and $\beta_3 = 678$. There are 3! ways to arrange three blocks, but not every arrangement yields a permutation with exactly the given blocks. For example, the arrangement $\beta_2\beta_3\beta_1$ gives the permutation 34567812. This permutation contains two blocks, 12 and 345678.

Let $F(m)$ denote the number of arrangements of $m$ blocks which give $n$-permutations with exactly those blocks. Then $F(m)$ is the number of $m$-permutations with no two consecutive increasing integers in consecutive positions. For example, $F(3) = 3$ counts the 3-permutations 312, 213, and 321. We see that $F(m)$ is the number of $m$-permutations containing $m$ blocks.

The answer to Wilf's question is given by $\binom{n-1}{m-1}F(m)$. We must determine $F(m)$. To do this, we observe every $n$-permutation contains $m$ blocks for some choice of $m$. Thus

$$n! = \sum_m \binom{n-1}{m-1} F(m).$$

At this point we need the first of two versions of the binomial inversion formula quoted below. We include the second version of the formula for future reference.

$$a_n = \sum_k \binom{n}{k} b_k \quad (n = 0, 1, 2, \ldots) \iff b_n = \sum_k (-1)^{n-k} \binom{n}{k} a_k \quad (n = 0, 1, 2, \ldots) \tag{1}$$

and

$$a_k = \sum_n \binom{n}{k} b_n \quad (k = 0, 1, 2, \ldots) \iff b_k = \sum_n (-1)^{n-k} \binom{n}{k} a_n \quad (k = 0, 1, 2, \ldots). \tag{2}$$

The binomial inversion formula (**??**) tells us

$$F(m) = (m-1)! \sum_k (-1)^{m-k-1} \frac{k+1}{(m-k-1)!}.$$

The number of $n$-permutations which contain $m$ blocks is therefore given by

$$\binom{n-1}{m-1} (m-1)! \sum_k (-1)^{m-k-1} \frac{(k+1)}{(m-k-1)!}. \tag{3}$$

We remark that the sequence $F(2), F(3), F(4), \ldots$ has been well studied. It is sequence A000255 in Sloane's On-Line Encyclopedia of Integer Sequences [**?**], and its exponential generating function is $e^{-x}/(1-x)^2$ (see Kreweras [**?**]).

So far, we have counted $n$-permutations with blocks that look like $i(i+1)(i+2)\cdots$. We now attempt to enumerate $n$-permutations with blocks having a more general form, for example, $i(i-1)(i+1)$ or $i(i-1)(i+2)$. To do so, we first generalize the notion of block.

We now define a *block* to be any sequence of values in a permutation which appear in consecutive positions. The number of values is the *length* of the block. We say a block $\beta = \beta_1\beta_2\ldots\beta_k$ has *type* $\tau = \tau_1\tau_2\ldots\tau_k$ when $\beta_i = \tau_i + c$ for all $i$, where $c$ is some integer constant. In this case, we say $\beta$ is a *$\tau$-block*. For example, when $\tau = 213$, $\tau$-blocks include 213, 324, 435, $\cdots$. When $\tau = 214$, $\tau$-blocks include 214, 325, 436, $\cdots$.

With this new notion of block, the question of interest becomes the following.

*How many $n$-permutations contain exactly $m$ blocks with type $\tau$?*

This question is closely related to a big open problem in the theory of patterns of permutations. A *pattern* $\tau = \tau_1\tau_2\ldots\tau_k$ of *length* $k$ is a fixed $k$-permutation. We say $\tau$ *occurs* in an $n$-permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ when there exist integers $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that $\sigma_{i_s} < \sigma_{i_t}$ if and only if $\tau_s < \tau_t$ for all $1 \leq s < t \leq k$. For example, the pattern $\tau = 213$ occurs exactly five times in the permutation $\sigma = 32415$ as illustrated below.

$$\underline{3}\,\underline{2}\,\underline{4}\,1\,5 \quad \underline{3}\,\underline{2}\,4\,1\,\underline{5} \quad \underline{3}\,2\,4\,\underline{1}\,\underline{5} \quad 3\,\underline{2}\,4\,\underline{1}\,\underline{5} \quad 3\,2\,\underline{4}\,\underline{1}\,\underline{5}$$

*Figure 2. Five occurrences of the pattern* $\tau = 213$ *in the permutation* $\sigma = 32415$.

A permutation $\sigma$ *avoids* a pattern $\tau$ if $\tau$ does not occur in $\sigma$.

Recent results in pattern research establish partial answers to the following question.

$\boxed{\textit{In how many permutations does the pattern } \tau \textit{ occur exactly } m \textit{ times?}}$

When $\tau$ has length 3, and we look for permutations with $m = 0$ occurrences of the pattern $\tau$ (i.e., patterns which avoid $\tau$), we can completely answer this question. Surprisingly, the answer does not depend on the particular pattern chosen. Schmidt and Simion [**?**] have shown that all patterns of length 3 are avoided by the same number of permutations, and this number is a Catalan number.

To obtain results for longer patterns when $m = 0$, we define an equivalence relation on patterns of length $k$ by requiring that the numbers of permutations which avoid equivalent patterns be equal. In this sense, all patterns with length 3 are equivalent. Stankova [**?**] has shown there are three equivalence classes for patterns with length 4. In addition to the results involving equivalence classes, Bona [**?**] found the exact number of $n$-permutations avoiding 1342 and gave an ordinary generating function for them.

When $m > 0$ we also have a few results. Noonan and Zeilberger [**?**] found the number of permutations with exactly $m = 1$ occurrence of the pattern 132. Robertson, Wilf, and Zeilberger [**?**] found the number of $n$-permutations having exactly $p$ occurrences of 123 and $q$ occurrences of 132 in the form of a Maple program which returns the desired generating function.

Another interesting open problem concerns asymptotics for the number of $n$-permutations which avoid a given pattern $\tau$. Let $f(n;\tau)$ denote the number of $n$-permutations which avoid $\tau$. Stanley and Wilf have conjectured that the limit

$$\lim_{n\to\infty} f(n;\tau)^{1/n}$$

exists, is finite, and is nonzero. In all known cases, this limit is an integer.

Further results concern the maximum number of occurrences of $\tau$ in an $n$-permutation, and the permutations which achieve these maximums. For work on problems of this type, see [**?**], [**?**], and [**?**]. For a survey of results on patterns, see [**?**].

Counting occurrences of patterns in permutations is hard because patterns are so flexible. We have very few results for patterns of length greater than 4, and nonzero values of $m$. If we make the patterns more "rigid," then they are easier to count. In particular, we define a *rigid pattern* to be a sequence $\tau = \tau_1\tau_2\ldots\tau_k$ of $k$ distinct positive integers, i.e., a block, and we say $\tau$ *occurs* in an $n$-permutation $\sigma$ when $\sigma$ contains a block with type $\tau$. Note that all patterns are rigid patterns, but not all rigid patterns are patterns.

To illustrate the difference between patterns and rigid patterns, we note the *pattern* $\tau = 213$ occurs exactly five times in the permutation $\sigma = 32415$, as demonstrated above. The *rigid pattern* $\tau = 213$, on the other hand, occurs exactly once as the subsequence 324. Listed below are all 5-permutations in which the rigid pattern $\tau = 213$ occurs exactly one time.

$$\begin{array}{cccccc}
\boxed{2\,1\,3}\,4\,5, & \boxed{2\,1\,3}\,5\,4, & 4\,\boxed{2\,1\,3}\,5, & 5\,\boxed{2\,1\,3}\,4, & 4\,5\,\boxed{2\,1\,3}, & 5\,4\,\boxed{2\,1\,3} \\
\boxed{3\,2\,4}\,1\,5, & \boxed{3\,2\,4}\,5\,1, & 1\,\boxed{3\,2\,4}\,5, & 5\,\boxed{3\,2\,4}\,1, & 1\,5\,\boxed{3\,2\,4}, & 5\,1\,\boxed{3\,2\,4} \\
\boxed{4\,3\,5}\,1\,2, & \boxed{4\,3\,5}\,2\,1, & 1\,\boxed{4\,3\,5}\,2, & 2\,\boxed{4\,3\,5}\,1, & 1\,2\,\boxed{4\,3\,5}, & 2\,1\,\boxed{4\,3\,5}
\end{array}$$

*Figure 2. The rigid pattern $\tau = 213$ occurs exactly once in eighteen distinct 5-permutations.*

In this paper, we consider the question analogous to that above for the more manageable rigid patterns.

*In how many $n$-permutations does the rigid pattern $\tau$ occur exactly $m$ times?*

When we begin to investigate this question we notice an important difference between, for example, the rigid patterns 213 and 214. We observe that a permutation may contain overlapping blocks of type 214, but no permutation contains overlapping blocks of type 213. The 6-permutation 214365, for example, contains 214-blocks 214 and 436, which share the 4.

To account for this distinction, we say a rigid pattern $\tau$ is *nonextendible* if no permutation contains two overlapping blocks both with type $\tau$. Otherwise $\tau$ is *extendible*.

When $\tau$ is a nonextendible rigid pattern satisfying certain conditions, we obtain the answer to our question from Theorem **??**. When the block is extendible, we can compute at least a lower bound, if not the answer itself, using Proposition **??**.

## 2 Nonextendible Blocks

In the previous section, we counted $n$-permutations with exactly $m$ blocks by first selecting $m$ blocks, then counting $n$-permutations with exactly those blocks. As a final step, we used binomial inversion to obtain a formula. To count $n$-permutations with exactly $m$ occurrences of a given nonextendible rigid pattern $\tau$, we follow a similar procedure.

We begin with the case where $\tau$ is a $k$-permutation. We again choose $\tau$-blocks by inserting dividers, but now we do so more carefully. Not just any insertion of dividers will do. Consider the case $\tau = 213$. Here any three consecutive integers determine a $\tau$-block. For example, the integers 3,4,5 determine the $\tau$-block 435. Thus a choice of $m$ $\tau$-blocks corresponds to a choice of $m$ disjoint sets of three consecutive integers each.

In how many ways can we choose $m$ disjoint sets of three consecutive integers each from the set $[n]$? To answer this, we consider an equivalent question. In how many ways can we choose $m$ sets of three consecutive integers each together with $n - 3m$ sets of one integer each from $[n]$ so that all sets are disjoint? Let $n = 10$ and $m = 2$. The insertion of dividers 123—456—7—8—9—10 demonstrates one such choice. We associate this particular insertion of dividers with the composition $3+3+1+1+1+1$ of 10 with $m$ parts size 3 and $n-3m$ parts size 1.

In this way we associate with each choice of $m$ $\tau$-blocks a composition of $n$ with $m$ parts size 3 and $n-3m$ parts size 1. Conversely, for each composition of $n$ with $m$ parts size 3 and $n-3m$ parts size 1, we have a choice of $m$ $\tau$-blocks with which the composition is associated. The number of distinct choices for $m$ $\tau$-blocks is therefore given by the number of distinct compositions of $n$ with $m$ parts size 3 and $n-3m$ parts size 1. There are $\frac{(m+(n-3m))!}{m!(n-3m)!} = \binom{m+(n-3m)}{m}$ such compositions. This observation generalizes naturally to the following.

**Proposition 1** *Let $\tau$ be a nonextendible $k$-permutation. The number of ways to choose $m$ disjoint $\tau$-blocks from the set $[n]$ is given by $\binom{m+(n-km)}{m}$.*

Not all nonextendible rigid patterns are $k$-permutations. The rigid pattern $\tau = 1254$, for example, is nonextendible. To count the ways to choose $m$ disjoint $\tau$-blocks for more general nonextendible $\tau$, we consider the *span* of $\tau$.

The *span* of a rigid pattern $\tau = \tau_1\tau_2\ldots\tau_k$ is the block of consecutive increasing integers from $\min\{\tau_i\}$ to $\max\{\tau_i\}$. The span of 2154 is the block 12345. When disjoint $\tau$-blocks necessarily have disjoint spans, we can choose $\tau$-blocks by choosing their spans. Note that if $\tau$ is a permutation, then disjoint $\tau$-blocks have disjoint spans.

**Lemma 1** *Let $\tau$ be a nonextendible rigid pattern with the property that disjoint $\tau$-blocks have disjoint spans. Suppose the span of $\tau$ has length $l$. The number of ways to choose $m$ disjoint $\tau$-blocks from the set $[n]$ is given by $\binom{m+(n-lm)}{m}$.*

**Proof:** We choose $m$ disjoint $\tau$-blocks from $[n]$ by choosing their disjoint spans. The number of ways to do this is given by the number of ways to choose $m$ sets of $l$ consecutive integers each together with $n-lm$ sets of one integer each from $[n]$. With each such choice we associate a composition of $n$ with $m$ parts size $l$ and $n-lm$ parts size 1. The number of distinct choices for $m$ $\tau$-blocks is therefore given by the number of distinct compositions of $n$ with $m$ parts size $l$ and $n-lm$ parts size 1. There are $\binom{m+(n-lm)}{m}$ such compositions. $\diamond$

The following theorem uses this lemma to enumerate $n$-permutations containing exactly $m$ occurrences of a given nonextendible rigid pattern.

**Theorem 1** *Let $\tau$ be a nonextendible rigid pattern of length $k$ with the property that disjoint $\tau$-blocks have disjoint spans. Suppose the span of $\tau$ has length $l$. Then the number of $n$-permutations in which $\tau$ occurs exactly $m$ times is given by*

$$\sum_{m'}(-1)^{m-m'}\binom{m'}{m}\binom{m'+(n-lm')}{m'}[m'+(n-km')]! \tag{4}$$

**Proof:** We use Lemma **??** to count the number of ways to choose $m$ disjoint $\tau$-blocks from $[n]$. For each such choice we list the $[m+(n-km)]!$ arrangements of the $m$ $\tau$-blocks together with the $n-km$ blocks with length 1 not contained by any $\tau$-block.

For example, suppose $\tau = 1254$, $n = 12$, $m = 2$, and we are given $\tau$-blocks $\tau_1 = 1254$ and $\tau_2 = 67(10)9$. The blocks of length 1 not contained by either of these $\tau$-blocks include $\tau_3 = 3, \tau_4 = 8, \tau_5 = (11)$, and $\tau_6 = (12)$. We list all 6! permutations of the blocks $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5$, and $\tau_6$.

In the end we obtain a list of $\binom{m+(n-lm)}{m}[m + (n - km)]!$ arrangements in total. This list includes all $n$-permutations with at least $m$ $\tau$-blocks.

Consider the case where $\tau = 1254$, $n = 12$, and $m = 1$. We list the $n$-permutation 125467(10)938(11)(12), with $\tau$-blocks 1254 and 67(10)9, once when we arrange the blocks $\tau_1 = 1254, \tau_2 = 3, \tau_3 = 6, \tau_4 = 7, \tau_5 = 8, \tau_6 = 9, \tau_7 = (10), \tau_8 = (11)$, and $\tau_9 = (12)$, and once when $\tau_1 = 67(10)9, \tau_2 = 1, \tau_3 = 2, \tau_4 = 3, \tau_5 = 4, \tau_6 = 5, \tau_7 = 8, \tau_8 = (11)$, and $\tau_9 = (12)$.

Let $F^n_\tau(m)$ denote the number of $n$-permutations in which $\tau$ occurs exactly $m$ times. Then $F^n_\tau(m)$ equals $\binom{m+(n-lm)}{m}[m+(n-km)]!$ take away the number of arrangements in the list in which $\tau$ occurs $m' > m$ times. There are $F^n_\tau(m')$ $n$-permutations in which $\tau$ occurs exactly $m'$ times, each of which appears $\binom{m'}{m}$ times in the list above. Thus

$$F^n_\tau(m) = \binom{m + (n - lm)}{m}[m + (n - km)]! - \sum_{m'>m} \binom{m'}{m} F^n_\tau(m').$$

Now we apply a second form of the binomial inversion formula (**??**) to obtain

$$F^n_\tau(m) = \sum_{m'} (-1)^{m-m'} \binom{m'}{m} \binom{m' + (n - lm')}{m'} [m' + (n - km')]!.$$

$\diamond$

Note that application of this theorem depends on our ability to decide whether or not a given rigid pattern is extendible, and whether or not disjoint $\tau$-blocks have disjoint spans. This is not difficult to do by inspection.

For example, suppose $\tau = 1254$. We first try to extend $\tau$ beginning with the 4. The $\tau$-block starting with 4 is 4587. When we join the two $\tau$-blocks by overlapping the 4's, we obtain 1254587. Since the 5 appears twice in this arrangement, we realize $\tau$ cannot be extended starting with the 4. Clearly it cannot be extended beginning with the 2 or the 5 either, and we easily conclude $\tau$ is nonextendible. It is similarly easy to determine that disjoint $\tau$-blocks have disjoint spans.

For an example of a rigid pattern $\tau$ with the property that disjoint $\tau$-blocks do *not* have disjoint spans, consider $\tau = 152$. In this case, 152 and 376, for example, are disjoint $\tau$-blocks. Their spans, 12345 and 34567, on the other hand, are not disjoint.

When $\tau$ is an actual *pattern*, i.e., a $k$-permutation, we observe (as we would hope) that disjoint $\tau$-blocks always have disjoint spans.

# 3 Extendible Blocks

If $\tau$ is a nonextendible block, then no permutation contains overlapping blocks with type $\tau$, and it easy to count $\tau$-blocks in a given permutation. How do we count blocks when $\tau$

is extendible? Do the overlapping blocks 214 and 436 in the permutation 214365 count as two distinct $\tau$-blocks, or should we count 21436 as one "extended" $\tau$-block? Because we want to generalize the results of the introduction, we choose the latter method of counting occurrences of extendible rigid patterns.

Consider the 8-permutation $\sigma = 12678345$. With the original definition of a block as a maximal sequence of consecutive increasing integers, this permutation contains three blocks, 12, 345, and 678. To include this special case in the more general context in which we now find ourselves, we say $\sigma$ contains three blocks with (extendible) type $\tau = 12$. The block 345 contains overlapping $\tau$-blocks 34 and 45, and the block 678 contains overlapping $\tau$-blocks 67 and 78. Thus to include the motivating problem in the current framework, we count each of the blocks 345 and 678 as one *extended* $\tau$-block, rather than two.

When $\tau$ is an extendible rigid pattern, we count a maximal sequence of overlapping blocks with type $\tau$ as one (extended) $\tau$-block. What do we do when $\tau$-blocks overlap in different ways? Consider the extendible rigid pattern $\tau = 2154$. The $\tau$-blocks $\beta = 4376$ and $\beta' = 5487$ both overlap the block $\tau$. The blocks $\beta$ and $\tau$ have one integer in common, while $\beta'$ and $\tau$ have two integers in common. In fact, any pair of distinct $\tau$-blocks have at most two integers in common. In this case, we say the block 215487, containing both $\tau$ and $\beta'$, is a *proper* $\tau$-block, and $\tau$ has *overlap length* equal to two.

In this section we enumerate $n$-permutations in which a given extendible rigid pattern occurs exactly $m$ times. We say an extendible rigid pattern $\tau$ *occurs* exactly $m$ times in a give $n$-permutation $\sigma$ when $\sigma$ contains exactly $m$ proper blocks of type $\tau$. To do this, we must know the possible lengths of $\tau$-blocks and their spans. When $\tau = 2154$, for example, blocks of type $\tau$ include 2154, 215487, 215487(11)(10), $\cdots$. In this case, $\tau$-blocks have lengths 4, 6, 8, and so on.

**Lemma 2** *Suppose $\tau = \tau_1\tau_2\ldots\tau_k$ is an extendible rigid pattern with span length $l$. Let $p$ denote the overlap length of $\tau$, and set $q = \tau_{k-p+1} - \tau_1$. A proper $\tau$-block in $[n]$ has length $k + r(k-p)$ and its span has length $l + rq$, where $0 \le r \le \lfloor \frac{n-l}{q} \rfloor$.*

**Proof:** Let $\beta$ be a proper $\tau$-block in $[n]$ with maximal length. The length of $\beta$ is $k+r(k-p)$, where $r$ is the greatest nonnegative integer for which the span of $\beta$ has length at most $n$. The span of $\beta$ has length $l + rq$. Thus $l + rq \le n$, which means $0 \le r \le \lfloor \frac{n-l}{q} \rfloor$. $\diamond$

When $\tau = 2154$, for example, we have $l = 5$, $p = 2$, and $q = 3$. In this case, the lemma tells us $\tau$-blocks have lengths $4 + 2r$, where $0 \le r \le \lfloor \frac{n-5}{3} \rfloor$. In particular, when $n = 12$, $\tau$-blocks in [12] have lengths 4, 6 and 8. Examples of such $\tau$-blocks include 2154, 215487, and 2154(11)(10).

When $\tau$ is a nonextendible rigid pattern, each choice of $m$ $\tau$-blocks in $[n]$ is associated with a composition of $n$ with $m$ parts size $l$ and $n - ml$ parts size 1. When $\tau$ is an extendible rigid pattern, on the other hand, the compositions associated with the various choices of $m$ $\tau$-blocks have parts of several sizes. As in the previous section, we consider only those $\tau$-blocks for which any two disjoint $\tau$-blocks have disjoint spans, and we choose disjoint $\tau$-blocks by choosing their disjoint spans.

Lemma **??** tells us that blocks in [12] with type $\tau = 2154$ have spans with lengths 5, 8, and 11. To choose $m$ disjoint spans in [12], we choose $m$ disjoint sets of consecutive integers in [12], where each set consists of either 5, 8, or 11 integers. We associate each choice of spans with lengths $s_1, s_2, \ldots, s_m$ with a composition of $[n]$ with parts sizes $s_1, s_2, \ldots, s_m$ and 1. There are $n - \sum s_i$ parts size 1. Let $C_\tau^n(m)$ denote the set of all such compositions.

For example, let $n = 12$ and $\tau = 2154$. Then $C_\tau^n(m)$ consists of compositions with exactly $m$ parts in the set $\{5, 8, 11\}$, and remaining parts 1. We list compositions in $C_\tau^n(m)$ below.

$m = 0$: 1+1+1+1+1+1+1+1+1+1+1+1
$m = 2$: 5+5+1+1, 5+1+5+1, 5+1+1+5, 1+5+1+5, 1+1+5+5
8+1+1+1+1, 1+8+1+1+1, 1+1+8+1+1, 1+1+1+8+1, 1+1+1+1+8
1+11, 11+1

*Figure 3. Compositions in $C_{2154}^{12}(m)$ for $m = 0, 2$.*

We see that the number of ways to choose $m$ $\tau$-blocks in $[n]$ is given by the cardinality of the set $C_\tau^n(m)$. To determine this, we choose $m$ (not necessarily distinct) span sizes $s_1, s_2, \ldots, s_m$ from the set $\{l + rq : 0 \le r \le \lfloor \frac{n-l}{q} \rfloor\}$ of possible span sizes (see Lemma **??**). For each choice of span sizes $s_1, s_2, \ldots, s_m$, we enumerate compositions of $n$ with parts $s_1, s_2, \ldots, s_m$, and $n - \sum s_i$ parts 1. Suppose we have chosen $s$ distinct span sizes. Let $t_1, t_2, \ldots, t_s$ denote the multiplicities of the $s$ distinct span sizes (so $\sum t_i = m$). Then the number of ordered compositions with parts $s_1, s_2, \ldots, s_m$, and $n - \sum s_i$ parts 1 is given by

$$\frac{n}{\prod t_i!} = \binom{n}{t_1, t_2, \ldots, t_s}.$$

Each $\sigma$ in $C_\tau^n(m)$ is associated with a choice of $m$ disjoint $\tau$-blocks in $[n]$. Let $s_n(\sigma)$ denote the number of integers in $[n]$ not contained by any of these $\tau$-blocks. For example, consider $\sigma = 5 + 5 + 1 + 1$ in $C_{2154}^{12}(2)$. The composition $\sigma$ is associated with the 2154-blocks 2154 and 76(10)9. In this case, $s_n(\sigma) = 4$ counts the integers 3, 8, 11, and 12 in [12].

We are now ready to count $n$-permutations with exactly $m$ occurrences of an extendible rigid pattern.

**Proposition 2** *Suppose $\tau$ is an extendible rigid pattern with the propertly that disjoint $\tau$-blocks have disjoint spans. Let $F_\tau^n(m)$ denote the number of $n$-permutations in which $\tau$ occurs exactly $m$ times. Then*

$$F_\tau^n(m) = \sum_{\sigma \in C_\tau^n(m)} [m + s_n(\sigma)]! - \sum_{m' > m} \binom{m'}{m} F_\tau^n(m') \tag{5}$$

**Proof:** Above we showed that each choice of $m$ disjoint $\tau$-blocks in $[n]$ is associated with a composition $\sigma$ in $C_\tau^n(m)$. For each choice of $\tau$-blocks we list the $[m + s_n(\sigma)]!$ arrangements of the $m$ $\tau$-blocks together with the $s_n(\sigma)$ integers not contained by any $\tau$-block. We obtain a list of

$$\sum_{\sigma \in C_\tau^n(m)} [m + s_n(\sigma)]!$$

arrangements in total. This list includes all $n$-permutations with at least $m$ $\tau$-blocks.

Now $F_\tau^n(m)$ equals the number of permutations in the list above minus the number of arrangements in the list which contain $m' > m$ $\tau$-blocks. There are $F_\tau^n(m')$ $n$-permutations which contain exactly $m'$ $\tau$-blocks, each of which appears $\binom{m'}{m}$ times in the list above. Thus equation (**??**) holds. $\diamond$

For example, when $\tau = 214$ and $n = 8$, $F_\tau^n(2) = 24$ counts the 4! 8-permutations of the blocks $\tau_1 = 214$, $\tau_2 = 658$, $\tau_3 = 3$, and $\tau_4 = 7$. Thus formula (**??**) for $m = 1$ becomes

$$F_\tau^n(1) = \sum_{\sigma \in C_\tau^n(1)} [1 + s(\sigma)]! - 24.$$

The set $C_\tau^n(1)$ contains ordered compostions $\sigma_1 = 8$, $\sigma_2 = 6+1+1$, $\sigma_3 = 1+6+1$, $\sigma_4 = 1+1+6$, $\sigma_5 = 4+1+1+1+1$, $\sigma_6 = 1+4+1+1+1$, $\sigma_7 = 1+1+4+1+1$, $\sigma_8 = 1+1+1+4+1$, and $\sigma_9 = 1+1+1+1+4$. Now $s(\sigma_1) = 1$, $s(\sigma_2) = s(\sigma_3) = 3$, and $s(\sigma_5) = s(\sigma_6) = s(\sigma_7) = s(\sigma_8) = s(\sigma_9) = 5$. Thus $F_\tau^n(1) = ([1+1]! + 2[1+4]! + 5[1+5]!) - 24 = 3,626$. Finally $F_\tau^n(0) = 8! - (3626 + 24) = 40,320$.

# 4 Conclusions

When we search for rigid patterns, we restrict the flexibility of (ordinary) patterns in two ways: 1) we require that values occur in consecutive positions, and 2) we insist that values in a $\tau$-block $\beta$ differ by exactly the difference between corresponding values in the rigid pattern, i.e., $\beta_i - \beta_j = \tau_i - \tau_j$. To bound the number of $n$-permutations in which an actual *pattern* occurs exactly $m$ times, we would ideally eliminate these restrictions. With our approach, we're stuck with the first restriction, but we can at least reduce the second.

For example, consider enumerating $n$-permutations with exactly $m$ occurrences of the *pattern* 213 in consecutive positions. To do this, we count $n$-permutations with exactly $m$ occurrences of the *rigid* patterns 213, 214, 215, ..., 314, 315, ..., 324, 325, and so on. When the rigid pattern is nonextendible, and satisfies the span condition, we have the count from Theorem **??**. When the rigid pattern is extendible, we can compute at least a lower bound using Proposition **??**.

# 5 Open Problems

The alternating sums and binomial coefficients in (**??**) and (**??**) suggest a search for asymptotics.

> *For $m$ a fixed constant or function of $n$, can we find asymptotics for* (**??**) *and* (**??**) *as $n \to \infty$? What is the probability that a rigid pattern $\tau$ occurs exactly $m$ times in an $n$-permutation?*

Recall that when $\tau$ is an extendible rigid pattern, we count only those permutations in which every pair of overlapping $\tau$-blocks overlap properly.

*Can we extend the enumeration to include cases when blocks with a given rigid pattern do not overlap properly?*

Given the connection with the theory of patterns, we would like to count overlapping blocks of extendible type individually.

*If we count every block with a given type individually, regardless of overlapping, can we enumerate $n$-permutations which contain exactly $m$ blocks with this type?*

Finally, we would like to eliminate from our results the condition that disjoint blocks with the same type have disjoint spans.

*Can we find results for a rigid pattern $\tau$ with the property that disjoint $\tau$-blocks have nondisjoint spans?*

# 6 Acknowledgments

I am grateful to Herb Wilf for stimulating discussions which initiated this study of rigid patterns.

# 7 Figures

I include all figures again here at the end as requested in the instructions for submission.

$m = 1:$ $\boxed{1\ 2\ 3\ 4}$

$m = 2:$ $\boxed{2\ 3\ 4}\boxed{1}$ $\boxed{4}\boxed{1\ 2\ 3}$ $\boxed{3\ 4}\boxed{1\ 2}$

$m = 3:$ $\boxed{1\ 2}\boxed{4}\boxed{3}$ $\boxed{3}\boxed{1\ 2}\boxed{4}$ $\boxed{4}\boxed{3}\boxed{1\ 2}$ $\boxed{2\ 3}\boxed{1}\boxed{4}$

$\boxed{4}\boxed{2\ 3}\boxed{1}$ $\boxed{1}\boxed{4}\boxed{2\ 3}$ $\boxed{3\ 4}\boxed{2}\boxed{1}$ $\boxed{1}\boxed{3\ 4}\boxed{2}$ $\boxed{2}\boxed{1}\boxed{3\ 4}$

$m = 4:$ $\boxed{1}\boxed{3}\boxed{2}\boxed{4}$ $\boxed{2}\boxed{1}\boxed{4}\boxed{3}$ $\boxed{3}\boxed{1}\boxed{4}\boxed{2}$ $\boxed{4}\boxed{1}\boxed{3}\boxed{2}$ $\boxed{2}\boxed{4}\boxed{1}\boxed{3}$

$\boxed{3}\boxed{2}\boxed{1}\boxed{4}$ $\boxed{4}\boxed{2}\boxed{1}\boxed{3}$ $\boxed{1}\boxed{4}\boxed{3}\boxed{2}$ $\boxed{2}\boxed{4}\boxed{3}\boxed{1}$ $\boxed{3}\boxed{2}\boxed{4}\boxed{1}$ $\boxed{4}\boxed{3}\boxed{2}\boxed{1}$

*Figure 1. 4-permutations with $m$ blocks.*

$m = 0$: 1+1+1+1+1+1+1+1+1+1+1+1
$m = 2$: 5+5+1+1, 5+1+5+1, 5+1+1+5, 1+5+1+5, 1+1+5+5
8+1+1+1+1, 1+8+1+1+1, 1+1+8+1+1, 1+1+1+8+1, 1+1+1+1+8
1+11, 11+1

*Figure 2. Ordered compositions in* $C^{12}_{2154}(m)$ *for* $m = 0, 2$.

$$\underline{3}\,\underline{2}\,\underline{4}\,1\,5 \quad \underline{3}\,\underline{2}\,4\,1\,\underline{5} \quad \underline{3}\,2\,4\,\underline{1}\,\underline{5} \quad 3\,\underline{2}\,4\,\underline{1}\,\underline{5} \quad 3\,2\,\underline{4}\,\underline{1}\,\underline{5}$$

*Figure 3. Five occurrences of the pattern* $\tau = 213$ *in the permutation* $\sigma = 32415$.

$$\begin{array}{llllll}
\boxed{2\,1\,3}\,4\,5, & \boxed{2\,1\,3}\,5\,4, & 4\,\boxed{2\,1\,3}\,5, & 5\,\boxed{2\,1\,3}\,4, & 4\,5\,\boxed{2\,1\,3}, & 5\,4\,\boxed{2\,1\,3} \\
\boxed{3\,2\,4}\,1\,5, & \boxed{3\,2\,4}\,5\,1, & 1\,\boxed{3\,2\,4}\,5, & 5\,\boxed{3\,2\,4}\,1, & 1\,5\,\boxed{3\,2\,4}, & 5\,1\,\boxed{3\,2\,4} \\
\boxed{4\,3\,5}\,1\,2, & \boxed{4\,3\,5}\,2\,1, & 1\,\boxed{4\,3\,5}\,2, & 2\,\boxed{4\,3\,5}\,1, & 1\,2\,\boxed{4\,3\,5}, & 2\,1\,\boxed{4\,3\,5}
\end{array}$$

*Figure 4. The rigid pattern* $\tau = 213$ *occurs exactly once in eighteen distinct* 5*-permutations.*

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