RESEARCH STATEMENT

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1. Introduction

Since the 1960's, relationships between algebraic K-theory and number theory have been investigated. For number fields F and their rings of integers $\mathcal{O}_F$, the K-groups $K_0(\mathcal{O}_F)$ and $K_1(\mathcal{O}_F)$ are related to classical objects in number theory. From [25] we have

$$K_0(\mathcal{O}_F) \cong \mathbb{Z} \times C(F)$$

where $C(F)$ is the ideal class group of F, and

$$K_1(\mathcal{O}_F) \cong \mathcal{O}_F^*,$$

the group of units of $\mathcal{O}_F$.

What can we say in general about $K_2(\mathcal{O}_F)$? For a ring R with unity, Milnor [25] defined $K_2(R)$ as the kernel of the natural surjection $St(R) \to E(R)$ where $St(R)$ is the Steinberg group of R and $E(R)$ is the direct limit of the group generated by elementary matrices. In particular, $K_2(R)$ is the center of $St(R)$, hence abelian. For a field F the group $K_2(F)$ has been computed by Matsumoto [24] as the universal symbol group:

$$K_2(F) = F^* \otimes_{\mathbb{Z}} F^* \diagup < u \otimes (1-u) : u \neq 1 >.$$

The kernel of the surjective homomorphism

$$K_2(F) \to \bigoplus_{\mathfrak{p}} \left(\mathcal{O}_F/\mathfrak{p}\right)^*,$$

given by the "tame symbols" at all finite primes $\mathfrak{p}$ of F, is called the **tame kernel** of F and is known to be finite [13] and isomorphic to $K_2(\mathcal{O}_F)$ [40]. For this reason, $K_2(\mathcal{O}_F)$ is commonly referred to as the **tame kernel** of F. In 1970, J. Birch [4] and J. Tate [44] conjectured for totally real number fields F that the order of $K_2(\mathcal{O}_F)$ is related to the value of the Dedekind zeta-function of F at -1, i.e

$$\# K_2(\mathcal{O}_F) = |w_2(F) \cdot \zeta_F(-1)|$$

where $w_2(F)$ is a readily computable term (page 26 in [45]). The Birch-Tate conjecture is a special case of the Lichtenbaum conjecture [22] which attempts to generalize Dirichlet's class number formula. The Birch-Tate conjecture was confirmed up to powers of 2 by Wiles [49].

Determining the structure of $K_2(\mathcal{O}_F)$ remains a difficult and intriguing problem. Much research (e.g. [5], [7], [9], [11], [19], [20], [21], [36], [37], [38], [39], [46], [47], [48]) has focused on the 2-Sylow subgroup of $K_2(\mathcal{O}_F)$. We say the 2^j**-rank**, $j \geq 1$, of $K_2(\mathcal{O}_F)$ is the number of cyclic factors of

$K_2(\mathcal{O}_F)$ of order divisible by 2^j. A formula of Tate [43] computes the 2-rank of the tame kernel. If F is a quadratic number field, Browkin and Schinzel [6] simplified the 2-rank formula. What about the 4-rank of $K_2(\mathcal{O}_F)$?

2. RESULTS

In [36], [37], and [38], Qin determined the 4-rank of the tame kernel for quadratic number fields F in terms of indefinite quadratic forms. Hurrelbrink and Kolster [18] generalized Qin's approach and obtained 4-rank results by computing $\mathbb{F}_2$-ranks of certain matrices of local Hilbert symbols. This approach is an effective technique and has led to connections between densities of certain sets of primes and 4-rank values. In [33], the author considered the 4-rank of $K_2(\mathcal{O})$ for the fields $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{2pl})$, $\mathbb{Q}(\sqrt{-pl})$, $\mathbb{Q}(\sqrt{-2pl})$ for primes $p \equiv 7 \bmod 8$, $l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$. In [10], it was shown that for the fields $E = \mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{2pl})$ and $F = \mathbb{Q}(\sqrt{-pl})$, $\mathbb{Q}(\sqrt{-2pl})$,

$$\text{4-rank } K_2(\mathcal{O}_E) = 1 \text{ or } 2,$$
$$\text{4-rank } K_2(\mathcal{O}_F) = 0 \text{ or } 1.$$

The idea in [33] was to fix a prime $p \equiv 7 \bmod 8$ and consider the set

$$\Omega = \{l \text{ rational prime} : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = 1\}.$$

In [33], we proved the following:

Theorem 2.1. *For the fields $\mathbb{Q}(\sqrt{pl})$ and $\mathbb{Q}(\sqrt{2pl})$, 4-rank 1 and 2 each appear with natural density $\frac{1}{2}$ in Ω. For the fields $\mathbb{Q}(\sqrt{-pl})$ and $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density $\frac{1}{2}$ in Ω.*

In [26], we extended the results in [33] by providing a complete densitiy picture for the 4-ranks of tame kernels of the fields $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{-pl})$ for primes p, l. One can show that 0, 1, or 2 are the possible 4-rank values for $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})})$ and $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})})$. Now, for squarefree, odd integers d, consider the sets

$$X = \{d : d = pl\}$$

and

$$Y = \{d : d = -pl\}$$

for distinct primes p and l. As a consequence of Theorems 1.2 and 1.3 in [26], we obtain

Corollary 2.2. *For the fields $\mathbb{Q}(\sqrt{pl})$, 4-rank 0, 1, and 2 appear with natural density $\frac{13}{64}$, $\frac{97}{128}$, $\frac{5}{128}$ respectively in X. For the fields $\mathbb{Q}(\sqrt{-pl})$, 4-rank 0, 1, and 2 appear with natural density $\frac{37}{64}$, $\frac{13}{32}$, and $\frac{1}{64}$ respectively in Y.*

The rough idea behind Theorem 2.1 and Corollary 2.2 is to use the matrices of local Hilbert symbols to get a correspondence bewteen 4-rank values and characterizations of the primes p and l by positive definite binary quadratic forms. This characterization then determines the splitting of p and l in a certain normal extension of $\mathbb{Q}$. Associating Artin symbols to p and l, we then use the Cébotarev Density theorem.

3. Further Research

3.1. **4-rank densities.** These matrices of local Hilbert symbols are analogous to Rédei matrices [41] which were used in the 1930's to study the structure of ideal class groups. In [34], this analogy is discussed along with density results of Gerth [15]. In the appendix of [34], we give a product formula for a local Hilbert symbol. Do Gerth's methods [15] coupled with this product formula yield, for any quadratic number field, asymptotic formulas for 4-rank densities of tame kernels?

3.2. **Higher 2-power ranks.** Is it possible to classify higher 2-power ranks of tame kernels of quadratic number fields in terms of positive definite binary quadratic forms? Do density results exist for higher 2-power ranks? Little is known about 8-ranks of tame kernels. Recent results in [18], [39], and [48] still need to be studied in order to provide a more unified approach.

3.3. **Question of Erdös.** During a conference in honor of D.H Lemher [16], Ron Graham posed the following question of Erdös: Are there infinitely many n such that the middle binomial coefficient $\binom{2n}{n}$ is relatively prime to 105? Lucas knew [23] that for a prime p, $(\binom{2n}{n}, p) = 1$ if and only if every coefficient in the base p expansion of n is $< \frac{p}{2}$. This implies that there are infinitely many n such that $(\binom{2n}{n}, p) = 1$ for a given prime p. Erdös, Graham, Ruzsa, and Straus [12] proved that for any two primes p and q, there exist infinitely many n for which $(\binom{2n}{n}, pq) = 1$. By Lucas' theorem, Erdös' question can be rephrased: Are there infinitely many n that have the digits 0, 1 or 0, 1, 2 or 0, 1, 2, 3 when written in bases 3, 5, or 7 respectively? A list of known n's is given by sequence $\#A030979$ [42].

3.4. **t-cores.** A **partition** of a positive integer n is a non-increasing sequence of positive integers whose sum is n. The number of such partitions is denoted by $p(n)$. If $\Lambda = \lambda_1 \geq \lambda_2 \geq \ldots \lambda_s$ is a partition of n, then the **Ferrers-Young diagram** of Λ is the s-row collection of nodes:

$$\begin{array}{llllll}
\bullet & \bullet & \ldots & \bullet & \bullet & \lambda_1 \text{ nodes} \\
\bullet & \bullet & \ldots & \bullet & & \lambda_2 \text{ nodes} \\
\vdots & & & & & \\
\bullet & \ldots & \bullet & & & \lambda_s \text{ nodes}
\end{array}$$

Label the nodes as if it were a martix. Let $\lambda_j{}'$ denote the number of nodes in column j. Define the **hook number** $H(i,j)$ of the (i,j) node to be $H(i,j) := \lambda_i + \lambda_j{}' - j - i + 1$. If t is a positive integer, then a partition of n is called a **t-core** of n if none of the hook numbers of its associated Ferrers-Young diagram are multiples of t. Let $C_t(n)$ denote the number of t-core partitions of n. In [32], Ono and Sze made the following remarkable discovery: If $8n + 5$ is square-free, then $C_4(n) = \frac{1}{2}h(-32n - 20)$ where $h(N)$ is the order of the class group of discriminant N binary quadratic forms. There are two proofs of this theorem. One proof uses the generating function for $C_4(n)$ [14] and properties of class numbers. The second proof relies on an explicit map from the set of 4-core partitions of n to the class group of binary quadratic forms of discriminant $-32n - 20$. Does such an explicit map exist between other t-cores and class numbers? Between t-cores and orders of K-groups?

3.5. **Partition congruences.** There has been recent exciting work [1], [2], [31] on congruence properties of the partition function $p(n)$. There are still many interesting open questions concerning the distribution of $p(n)$ modulo integers M, see [3] or [8]. The "folklore conjecture" [35] states that the values of $p(n)$ are distributed evenly modulo 2. Of the first 10000 values of $p(n)$, 4996 are even and 5004 are odd. The pattern seems to continue with 2 replaced by 3. Namely, the values of $p(n)$ seem to be evenly distributed modulo 3. Currently, there is no known explanation for this behavoir. In fact it is not known whether there are infinitely many n for which $p(n) \equiv 0 \bmod 3$.

3.6. **Sign Ambiguities.** Gauss, Jacobi, Stern, E. Lehmer, Whiteman, and others have obtained congruences for binomial coefficients in terms of parameters coming from representations of primes by quadratic forms. In [17], many other beautiful binomial coefficient congruences are proved. In certain cases, the key step is the resolution of a sign ambiguity. These sign ambiguities are counterexamples to Hasse's conjecture that all multiplicative relations between Gauss sums follow from the Davenport-Hasse product formula and the norm relation for Gauss sums. Very few ([28], [29], [30], [50]) sign ambiguities have been given. Recently, Brian Murray [27] has proved a remarkable product formula which yields an infinite class of new sign ambiguities. Can these new resolutions of sign ambiguities be used to obtain new congruences for binomial coefficients?

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