RICE UNIVERSITY

On Eliminating Square Paths in a Square Lattice

by

Nikki L. Williams

A Thesis Submitted
in Partial Fulfillment of the
Requirements for the Degree

Master of Arts

Approved, Thesis Committee:

Nathaniel Dean, Chairman
Associate Professor of Computational and Applied Mathematics

Richard A. Stong
Professor of Mathematics

Richard A. Tapia
Noah Harding Professor of Computational and Applied Mathematics

---

Yin Zhang
Associate Professor of Computational and Applied Mathematics

Houston, Texas

April, 2000

Abstract

On Eliminating Square Paths in a Square Lattice

by

Nikki L. Williams

Removing the minimum number of vertices or points from a square lattice such that no square path exists is known as the square path problem. Finding this number as the size of the lattice increases is not so trivial. Results provided by Erdös-Pósa and Bienstock-Dean provides an upper bound for eliminating all cycles from a planar graph but sheds little light on the case of the square lattice. This paper provides several values for the minimum number of vertices needed to be removed such that no square path exists.

Acknowledgments

I would like to thank the consistent support and advice of many people.

First and foremost I have to thank God for his grace and allowing me to make it thus far. He deserves all the honor and the praise.

I am very grateful for my advisor Dr. Nathaniel Dean who introduced me to this problem. Thanks Nate for hanging in there with me. You are indeed "The Best Advisor in the Whole World." I also want to sincerely thank Dr. Richard Tapia for his constant support, advice, time, and teaching me how to dance. I am extremely grateful to Dr. Yin Zhang and Dr. Richard Stong for much guidance during this process.

Much thanks to the National Defense Science and Engineering Graduate Fellowship for supporting me. Also much gratitude to the Andrew W. Mellon Foundation and the SSRC/Mellon Graduate Fellows.

Thanks to Dr. Robert Bixby, Dr. Cassandra McZeal, Dr. Jennifer Rich, Tim Redl, Sripriya Venkataraman, and Melisa Ramos for all that you have done.

Also, I want to acknowledge Dr. Rhonda Hughes and Dr. Sylvia Bozeman for constantly inspiring me to reach my goals. You both are true role models. Thanks Dr. Pamela J. Williams for taking me in and showing me the way.

I also want to thank the following: Bible Book Club (BBC) members; Lilly Grove Missionary Baptist Church and Rev. Terry K. Anderson for your prayers and encouragement; and Silver Hill Baptist Church.

Sincere thanks to the following individuals who have had a major impact in my life academically and spiritually: Nikeya C. Harper (Rin-Tin-Tin) for being my soror, my sister, and my true friend; Donald Williams for your advice; Ronald Session for your prayers long talks, and testimonies; Dragon for believing me when I really needed it most; Illya Hicks for constantly pushing me; Boo; and Rayzov Sonlight for being virtually who you are.

Lastly, I must thank my family: Thanks to my mommy, Mary R. Williams, for her constant support, advice, and wisdom. You have always believed that I could reach the stars and never once failed to tell me. Thanks mom! Thanks to my dad, Jimmie L. Williams, for his continuous support and teaching me perseverance. Thanks to my sister, LaTanya C. Williams, for everything! Words just can not explain! Thanks to my aunt, Marjorie Rich, for all that you have done for me over the years. Your kind works never go unappreciated. Thanks to the rest of my family and friends for all that you have done for me.

Contents

Illustrations

Tables

Chapter 1

Introduction

Consider an $n \times n$ square grid (also called a lattice) with vertices colored either black or white. A *path* is a chain of edges such that the end vertex of one edge is the beginning vertex of the next edge and no vertices are repeated, except possibly the beginning is the end. A *square path* is a closed path in the shape of a square with sides parallel to the edges of the lattice. Define $M(n)$ to be the minimum number of black points needed for an $n \times n$ square lattice so that every square path has at least one black point. We seek to find $M(n)$ for any given n. This is known as the Square Path Problem [5]. For example, $M(2) = 1$. For the single point or 1×1 lattice we define $M(1) = 0$. Even though these examples are obvious, finding $M(n)$ is not so trivial as n increases.

According to the 1988 editors of *Mathematics Magazine* [7, 4] the Square Path Problem (SPP), which is one of three problems posed by Morris [5], has not been solved. Several web and library searches using key words, such as Hamiltonian square path, square cycle, square path, square circuit, square lattice, lattice packing, path packing, and packing, were done in order to obtain information on the problem. Some of these searches returned nothing while others returned articles that were not related to the SPP.

However, the literature search did reveal one related problem. If we require the black points in the SPP to cover not only square paths (or circuits) but also circuits of any shape, then this new value is referred to as $\tau(n)$. We say a graph is *planar* if it can be drawn in the plane such that there are no edge crossings. Thus, the grid in the SPP is planar. This related problem seeks to find the minimum number of vertices to be removed from a planar graph such that no circuit exists. Dean refers to this minimum number as $\tau(n)$. In relation to the Square Path problem discussed in this work, $\tau(n)$ provides an upper bound for $M(n)$. Several upper bounds for n up to 14 are included in the Results chapter.

Bienstock-Dean [1] consider covering points of a planar graph with a minimum number of faces. The Erdös-Pósa theorem [3] on independent circuits in graphs can be applied when graphs with a specific embedding are considered. Erdös-Pósa define a family of cycles in a graph *independent* if they are pairwise vertex-disjoint.

The SPP can be transformed into a node covering problem in a bipartite graph. Let A,B be a bipartition of our graph. Then the set A contains a node for every vertex in the lattice, and the set B contains a node for every square path. An edge joins a vertex in A to a vertex in B if a node in the set A is a black point. This problem can be stated as follows: Find the minimum cardinality set S of nodes in the set A such that every node in the set B is adjacent to a member of S.

Chapter 2

Square and Non-square Lattices

2.1 Bounds for M(n)

Erickson [2] showed that

$$\lim_{n\to\infty} \frac{M(n)}{n^2}$$

exists and that

$$\frac{M(n)}{(n-1)^2} \geq \frac{2}{7}.$$

He also replicated a pattern to show that $\leq 2/7$ of the points of the $n \times n$ lattice need to be black. Thus,

$$\frac{2}{7}(n-1)^2 \leq M(n) \leq \frac{2}{7}n^2.$$

Proof of lowerbound (Erickson): Let B be a black point in the lattice, and suppose S is a 2×2 square path that passes through B. We will assign B a ”credit” of 1/k if S passes through exactly k black points. Let T(B) be the sum of all the credits assigned to B as S varies over all 2×2 squares that pass through B.

Note that the sum of T(B) as B varies over all black points in the square array is $(n-1)^2$ since each of the 2×2 arrays contributes 1 to the total.

It is clear that the sum of $T(B) \leq 1$ if B is a corner point, and $T(B) \leq 2$ if B is on the outer edge. Suppose that B is a point in the interior of the lattice. It lies on exactly four 2×2 square paths, and there must be at least one black point on the 3×3 square path surrounding B. Thus, for such a B, $T(B) \leq 7/2$.

Thus, in all cases, $T(B) \leq 7/2$, and so $(7/2)M(n) \geq (n-1)^2$, or equivalently, $M(n) \geq 2(n-1)^2/7$. □

Hence,

$$\lim_{n \to \infty} \frac{M(n)}{n^2} = \frac{2}{7}.$$

2.2 Trivial values for M(n)

In order to find a general formula for $M(n)$, values for n small were easily computed. The following results were used to obtain more information about $M(n)$.

Theorem 2.1 $M(2) = 1$.

Theorem 2.2 $M(3) = 2$. Moreover, if one black point is a corner point, then the other is the center.

Figure 2.1 Two configurations to show $M(3) = 2$.

Note that Figure 2.1 shows that $M(3) = 2$ does not have a unique solution. Thus, there might be several optimal configurations.

Theorem 2.3 $M(4) = 4$.

Proof Since a 4×4 lattice contains four distinct 2×2 lattices and $M(2) = 1$, then $M(4) \geq 4$. Choosing the four points indicated in Figure 2.2 eliminates all square paths, and so $M(4) = 4$. □

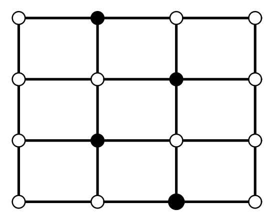


Figure 2.2 $M(4) = 4$

2.3 Non-square Lattices

Proving $M(n)$ for a specific n can be difficult as n increases. One possible method involves considering non-square lattices which partition the lattice into several regions. This technique allowed $M(n)$ to be determined for larger values of n.

Suppose we are given an $a \times b$ lattice or rectangle. Then we let $M(a, b)$ denote the minimum number of points to be removed from an $a \times b$ size rectangle such that no square path exists. Note that $M(a, b) = M(b, a)$. The following results are trivial.

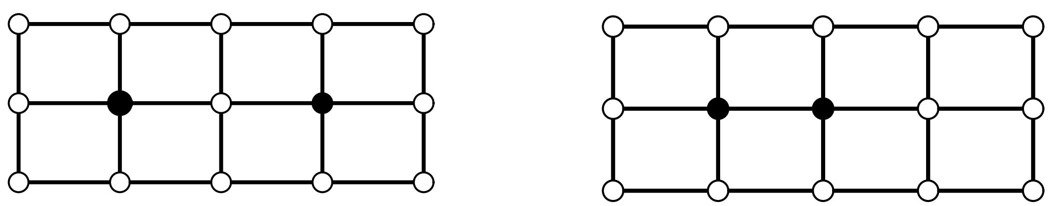
Lemma 2.1 $M(2, 3) = 1$.

Lemma 2.2 $M(3, 4) = 2$, and the solution is unique.

The following results are used in the next section to prove cases for square lattices.

Lemma 2.3 $M(3,5) = 3$.

Proof Since a 3×3 lattice is contained in a 3×5 rectangle, then $M(3,5) \geq 2$. Suppose $M(3,5) = 2$. Since we want to remove a minimum number of vertices, then we want to choose points that eliminate as many square paths as possible. Choosing the points in Figure 2.3(a) covers the eight distinct 2×2 square paths. However, a 3×3 square path exists containing the center point. If we were to choose the two center points indicated in Figure 2.3(b), then there is a 2×2 square path not covered. Thus, at least one more black point is needed. Hence, $M(3,5) \geq 3$. In fact, the configurations in Figure 2.4 show that $M(3,5) \leq 3$. □



(a) $M(3,5) \geq 2$

(b) $M(3,5) \geq 2$

Figure 2.3 Two subfigures

Lemma 2.4 $M(4,5) = 4$, and the solution is unique.

Proof Since a 5×4 contains a 4×4 square lattice and $M(4) = 4$, then $M(4,5) \geq 4$. In fact, Figure 2.5 shows that $M(4,5) = 4$ by choosing the four corners of the inner 2×3 rectangle.

Figure 2.4 Two configurations showing that $M(3,5) \le 3$

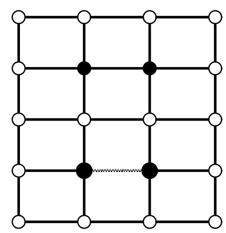


Figure 2.5 $M(4,5) = 4$

It's easy to see that there are only 4 solutions for $M(2,5) = 2$. When we partition the 4×5 lattice into two 2×5 lattices, the only solutions that avoid a square path are combined as shown in Figure 2.5. Thus, the solution is unique. □

Lemma 2.5 $M(3,7) = 4$.

Proof We can divide the 3×7 rectangle into a 3×3 lattice and a 3×4 rectangle. Since $M(3,4) = 2$ and $M(3) = 2$, then $M(3,7) \ge 4$. But, Figure 2.6 shows that $M(3,7) \le 4$. □

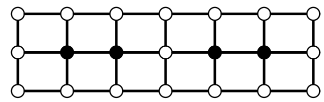


Figure 2.6 $M(3,7)=4$

Lemma 2.6 $M(4,7)=6$.

Proof Since we can divide the 4×7 rectangle into a 4×4 lattice and a 3×4 rectangle, then $M(4,7)\geq 6$. But, Figure 2.7 shows that $M(4,7)\leq 6$. □

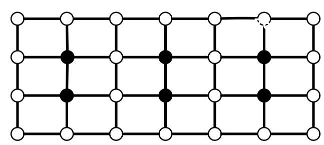


Figure 2.7 $M(4,7)=6$

Lemma 2.7 $M(5,6)=7$.

Proof The 5×6 lattice can be partitioned into two 3×3 lattices and three 2×2 lattices which are all pairwise disjoint. Hence, $M(5,6)\geq 1+1+1+2+2=7$. Figure 2.8 shows that $M(5,6)\leq 7$. □

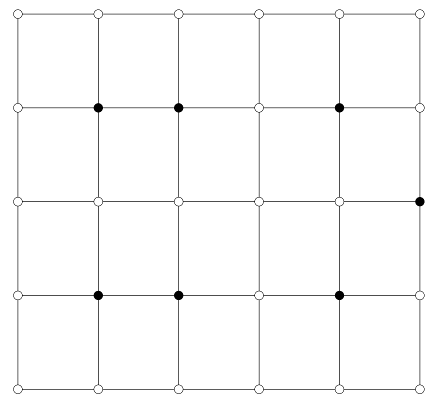


Figure 2.8 $M(5,6) = 7$

2.4 Square Lattices

Recall that the original problem seeks to remove the minimum number of vertices on a square lattice such that no square path exists. Our approach will be for us use the results from the previous section to prove values of $M(n)$. We begin with $n = 5$.

Theorem 2.4 $M(5) = 6$.

Proof

Since the 5×5 lattice contains a 4×4 lattice, then $M(5) \geq 4$. Suppose all square paths can be covered by five black points. Divide the 5×5 lattice into four 2×3 rectangles as in Figure 2.9, and label the regions I, II, III, and IV such that the middle point is not included in any or the 2×3 rectangles.

Case 1: Center is a black point.

If the center is a black point, then there is one point from each region. We choose the inner middle point so that all the square paths in that region are covered. Consider

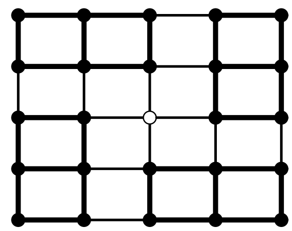


Figure 2.9 Divide into 2×3 rectangles and center

region IV. See Figure 2.10. If we choose the point 1 or 2 instead of a, then we have the 2×2 square path that contains a and the cornerpoint 5 of the 5×5 lattice. If we choose 3 or 5 instead of a, then we have a 2×2 square path between regions I and IV. If we choose point 4, then we have the 2×2 lattice in IV. But, notice that the 5×5 square path is not covered, a contradiction.

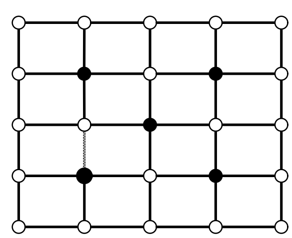


Figure 2.10 Center selected

Case 2: Center is not a black point.

If the center is not black, then there is a region that contains two black points, say region I. Using the same strategy from Case 1 we can choose the points in regions II, III, and IV. See Figure 2.11. Regardless of how the two points are selected in region I, there are two 3×3 square paths that contain the center in regions II and III, a contradiction.

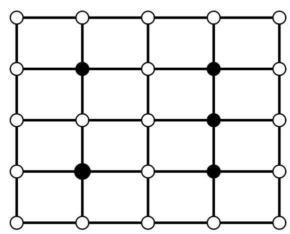


Figure 2.11 center point not selected

Thus, $M(5) > 5$. In fact, Figure 2.12 shows that $M(5) \leq 6$.

□

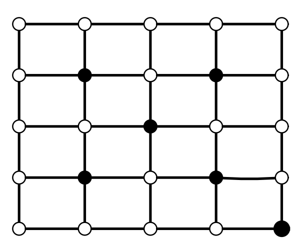


Figure 2.12 $M(5) = 6$

Theorem 2.5 $M(6) = 9$

Proof Since the 6×6 lattice contains 9 disjoint 2×2 lattices, then $M(6) \geq 9$. In fact, Figure 2.13 shows that $M(6) \leq 9$. □

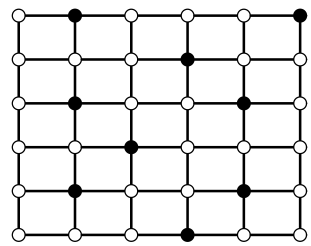


Figure 2.13 $M(6) = 9$

Theorem 2.6 $M(7) = 12$.

Proof Divide the 7×7 lattice into four 3×4 rectangles such that the center point of the lattice is not included in any 3×4 lattice. Recall that $M(3,4) = M(4,3) = 2$, and the solution is unique.

Suppose our 7×7 lattice has exactly 11 black points such that no square path exists.

Case 1 Center point is a black point.

If the center point is a black point, then the remaining 10 points are in each of the 3×4 lattices.

Case 1.1 One 3×4 lattice has 4 black points.

WLOG assume that II has 4 black points. The remaining six black points are in the three 3×4 lattices each containing two black points. Then there exists a 2×2 square path between III and IV regardless of how the four black points are arranged in II, a contradiction. See Figure 2.14.

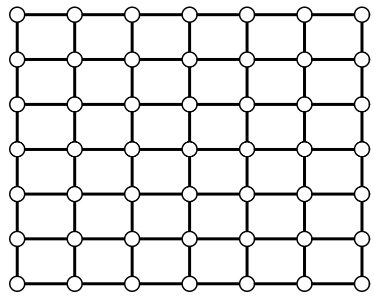


Figure 2.14 11 black points for $M(7)$

Case 1.2 Two 3×4 lattices have 3 black points.

Case 1.2.a

If the two 3×4 lattices that contain 3 black points are I and II (or any two 3×4 lattices that are not horizontal), then a 2×2 square path exists regardless of how the 3 black points in each of these two 3×4 lattices are arranged, a contradiction. See Figure 2.15.

Case 1.2.b

Suppose the two 3×4 lattices that contain 3 black points are in II and IV. WLOG consider region II. The black points in III and I are fixed according to Lemma 2.2. See Figure 2.16.

Notice that the 2×2 lattices labeled 1 and 5 and the 3×3 lattice labeled 8 are disjoint, and so at least $1+1+2 = 4$ points are required to cover them, a contradiction.

Case 2 Center point is not a black point.

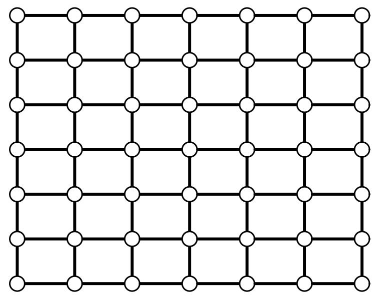


Figure 2.15 11 black points for $M(7)$

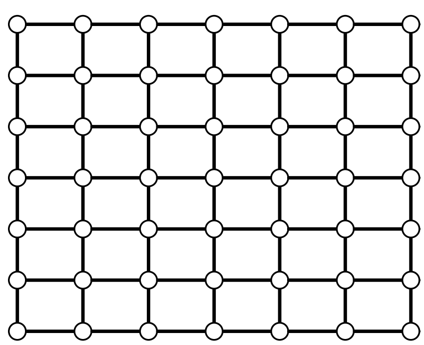


Figure 2.16 11 black points for $M(7)$

Case 2.1 One 3×4 contains 5 black points.

The result is similar to Case 1.1 which contains a square path, a contradiction. See Figure 2.17.

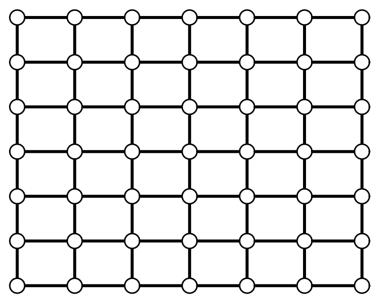


Figure 2.17 11 black points for $M(7)$

Case 2.2 One 3×4 contains 4 black points and another has 3 black points.

Case 2.2.a

If the lattices are not diagonal, for example quadrants II, III, then WLOG assume III has 4 black points and II has 3 black points. Then regardless of how these points in II and III are chosen, a square path exists between I and IV, a contradiction. See Figure 2.18.

Case 2.2.b

If the lattices are diagonal, for example regions II, IV, then WLOG assume II has 4 black points and IV has 3 black points. We note as in Case 1.2.b, that we must eliminate the 2×2 lattices labeled 4 and 5 and the 3×3 lattice labeled 8 with only 3 points. This is impossible, because they are disjoint. See Figure 2.19.

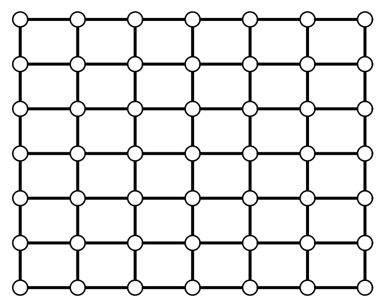


Figure 2.18 11 black points for $M(7)$

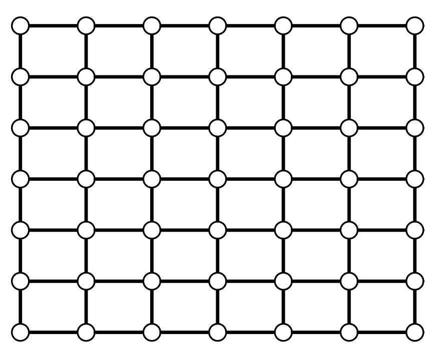


Figure 2.19 11 black points for $M(7)$

<u>Case 2.3</u> Three 3×4 rectangles contain 3 black points.

Assume WLOG that I,II, and III contain 3 black points and IV contains 2 black points. Since the center is not included and IV contains 2 fixed points, then we can extend III to a 4×4 lattice. We know a 4×4 lattice requires at least 4 black points. But, III is only allowed 3 black points. Thus, a square path exists, a contradiction. See Figure 2.20.

Thus, $M(7) \geq 12$. But, we can in fact show that $M(7) \leq 12$. See Figure 2.21. Thus, $M(7) = 12$. □

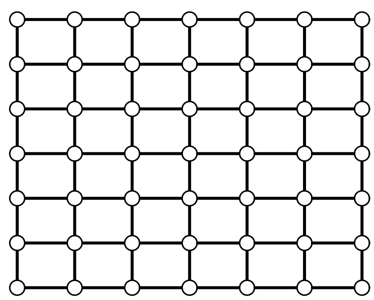


Figure 2.20 11 black points for $M(7)$

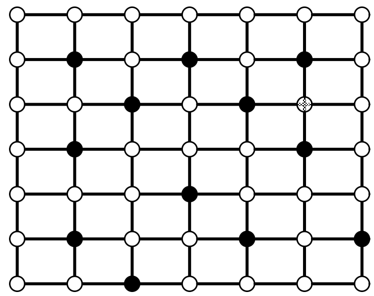


Figure 2.21 $M(7) = 12$

Theorem 2.7 $M(8) = 16$

Proof Since the 8×8 lattice contains 16 disjoint 2×2 lattices, then $M(8) \geq 16$. In fact, Figure 2.22 shows that $M(8) \leq 16$. □

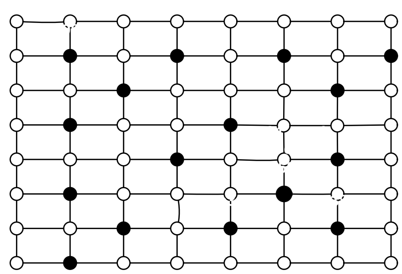


Figure 2.22 $M(8) = 16$

Theorem 2.8 $M(9) = 20$

Proof Suppose 19 black points is enough to cover a 9×9 lattice. We can divide the lattice into the center and four regions of size 4×5. Since $M(4,5) = 4$, then each of these regions contains at least four black points. Table 2.1 lists the five possibilities:

Table 2.1 Possibilities for 9×9 lattice

case	center	I	II	III	IV
1.1	1	4	4	4	6
1.2a	1	4	5	4	5
1.2b	1	5	5	4	4
2.1	0	4	4	4	7
2.2a	0	4	4	5	6
2.2b	0	4	5	4	6
2.3	0	4	5	5	5

<u>Case 1</u> Center point is a black point.

<u>Case 1.1</u> Assume region IV contains six black points. Recall that the configuration for $M(4,5)$ is unique from Lemma 2.4. Regardless of how the six nodes are placed, we will have a 2×2 square path between regions that contain only four black points as indicated by Figure 2.23.

<u>Case 1.2</u> Have 5,5,4,4 black points in the regions

<u>Case 1.2.a</u> The regions that contain five black points are on diagonal, say II and IV. Since the configuration of black points in regions I and III are fixed by Lemma 2.4 to contain no points from their perimeters and the four 2×2 lattices indicated are disjoint, at least four black points of IV are needed to eliminate these square paths. This leaves only one black point to eliminate all square paths in the remaining 4×3 of IV which is impossible. See Figure 2.24.

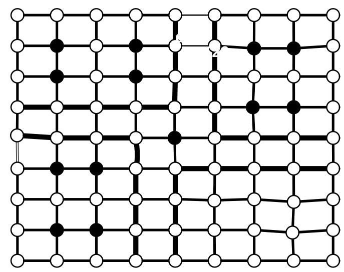


Figure 2.23 Case 1.1 of Proof of Theorem 2.8

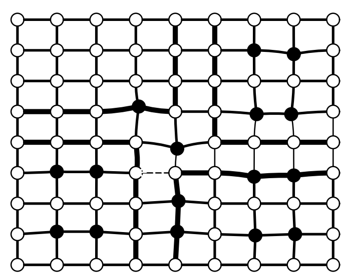


Figure 2.24 Case 1.2.a of Proof of Theorem 2.8

Case 1.2.b The regions that contain five black points are not on diagonal. This case breaks down like Case 1.1 since will have a 2×2 square path between regions that only contain four black points.

Case 2 Center is not a black point

Case 2.1 This case is also similar to Case 1.1.

Case 2.2 We have 4,4,5,6 black points in the regions.

Case 2.2.a The regions that contain four black points are not diagonal
This is also similar to Case 1.1.

Case 2.2.b The regions that contain four black points are diagonal, say I and III.

Since the center is not black and the selection of black points in I and III is fixed (Lemma 2.4), region IV can be extended to a 5×6 lattice without adding more black points, i.e., $M(5,6) \leq 6$ contradicting Lemma 2.7. See Figure 2.25.

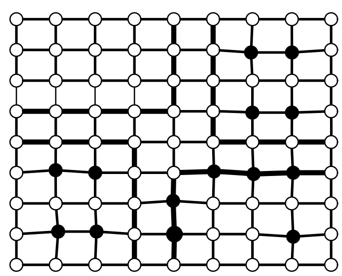


Figure 2.25 Case 2.2.b of Proof of Theorem 2.8

Case 2.3 We have 4,5,5,5 black points in the regions.
As in Case 2.2.b, region IV can be extended to a 5×5 without adding more black points. Hence, $M(5) \leq 5$, contradicting Theorem 2.4. See Figure 2.26.

Thus, $M(9) \geq 20$. In fact, Figure 2.27 gives an optimal configuration that uses exactly 20 black points. Thus, $M(9) = 20$. □

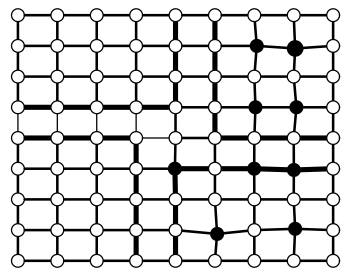


Figure 2.26 Case 2.3 of Proof of Theorem 2.8

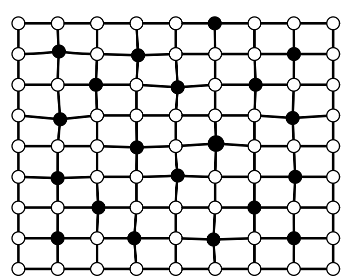


Figure 2.27 $M(9) = 20$ of Proof of Theorem 2.8

Chapter 3

Binary Integer Programming Formulation

The SPP can be modeled as a {0,1}-integer programming problem or a binary integer programming (BIP) problem. In the formulation we assign a variable to every point in the square lattice.

Let L be an n× n lattice. For each point $x_i \in$ L define

$$x_i = \begin{cases} 1 & \text{if } i \text{ is a black point,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

where $i = 1, ..., n^2$

The variables are indexed over the total number of vertices in the lattice. The points are assigned the value 1 if they are black and 0 if not black. We want to minimize the total number of black points in the square lattice subject to the constraint that every square path contains a black point. Thus, the SPP can be formulated as follows:

$$\begin{array}{ll} \text{minimize} & \sum_i x_i \\ \text{subject to} & \sum_{i \in S} x_i \geq 1, for\ each\ 2 \times 2\ square\ S \\ & \sum_{i \in S} x_i \geq 1, for\ each\ 3 \times 3\ square\ S \\ & \quad \vdots \\ & \sum_{i \in S} x_i \geq 1, for\ each\ n \times n\ square\ S \\ & x_i = \{0, 1\} \end{array} \tag{3.2}$$

These constraints require at least one black point on every square path, and the last constraint forces the variables to be binary. CPLEX version 6.0.1 was employed to solve the BIP formulation and obtain values for $M(n)$. The following result describes the growth of the formulation which indicates how efficiently values for $M(n)$ were obtained from the BIP formulation.

Theorem 3.1 As n increases, the number of inequalities in the BIP formulation grows cubically.

Proof There are
$(n-1)^2$ 2×2 equations, for $n \geq 2$
$(n-2)^2$ 3×3 equations, for $n \geq 3$
$(n-3)^2$ 4×4 equations, for $n \geq 4$
$\vdots$
1 $n \times n$ equation generated
and taking the sum gives

$$(n-1)^2 + (n-2)^2 + \ldots + 1^2 = \sum_{i=1}^{n-1} i^2 = \frac{n(n-1)(2n-1)}{6}$$

$\Rightarrow O(n^3)$ growth. □

Since the rapid growth prevented an efficient solution, there was a need to add more constraints to the BIP so that CPLEX considers a minimum feasible region. This led to the following results.

Theorem 3.2 There exists an optimal solution that contains no corner point.

Proof Suppose we have an optimal solution with one corner point. Then removal of this corner point yields a square path. We can replace this point by an adjacent

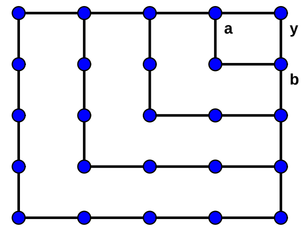


Figure 3.1 no corner point

point, say either a or b, and still obtain an optimal solution, a contradiction. See Figure 3.1.

□

This result can be formulated as $x_1 + x_n + x_{n^2-n+1} + x_{n^2} = 0$ which is equivalent to saying no corners are black.

Theorem 3.3 There exists an optimal solution that contains no boundary contains two consecutive black points.

Proof Suppose we have two consecutive black points on a boundary. We can force one of these to be white by making its perpendicular neighbor black and still cover the same square paths. Figure 3.2 becomes Figure 3.3. □

3.1 Unproven Claims for $M(14)$

The following conjectures were used to add more constraints to the BIP for the 14×14 case.

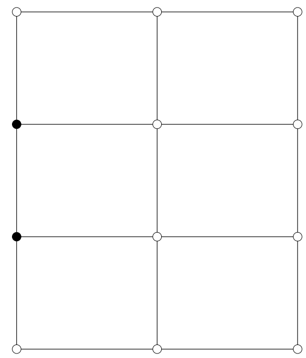


Figure 3.2 2 consecutive black points on boundary

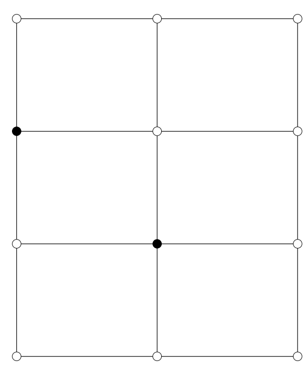


Figure 3.3 No consecutive black points on boundary

Conjecture 1 There does not exist an optimal solution that contains a black point on each of the four distinct boundaries.

Since the 14×14 square must be covered and if Conjecture 1 holds, then the optimal configuration for the 14×14 case has either 1, 2, or 3 points on the boundary.

Conjecture 2 There exists an optimal solution that contains no black points on one of the boundaries.

Argument for Conjecture 2 Suppose there exists an optimal solution with 3 boundaries containing exactly one black point. Consider the 12×12 region which requires 38 black points. WLOG, let 3 of these black points share the boundary with the 3 black points from the 14×14 lattice such that there is no overlap in squares covered thus allowing a possible optimal solution. See Figure 3.4. Then we need at least 53 black points since the 12×12 region requires 38, the 3×5 region and the 3×7 region requires 3 and 4 black points respectively. Also, the seven 2×3 regions require 1 each and the two 2×2 regions may be in a 2×3 form so only require at least one black point. But, the configuration in Figure 3.5 requires 52 black points which is one less black point than the $38 + 3 + 4 + 7 + 1 = 53$, a contradiction.
End of Argument

If Conjecture 2 holds, then it can be formulated for the BIP as $x_1 + x_2 + \ldots + x_n = 0$, i.e., first row of the lattice has no black points.

Remark Since the lattice can be rotated 90,180, or 270 degrees any boundary may be considered the first row of the lattice.

Conjecture 3 There exists an optimal solution such that two nonadjacent boundaries have no black points.

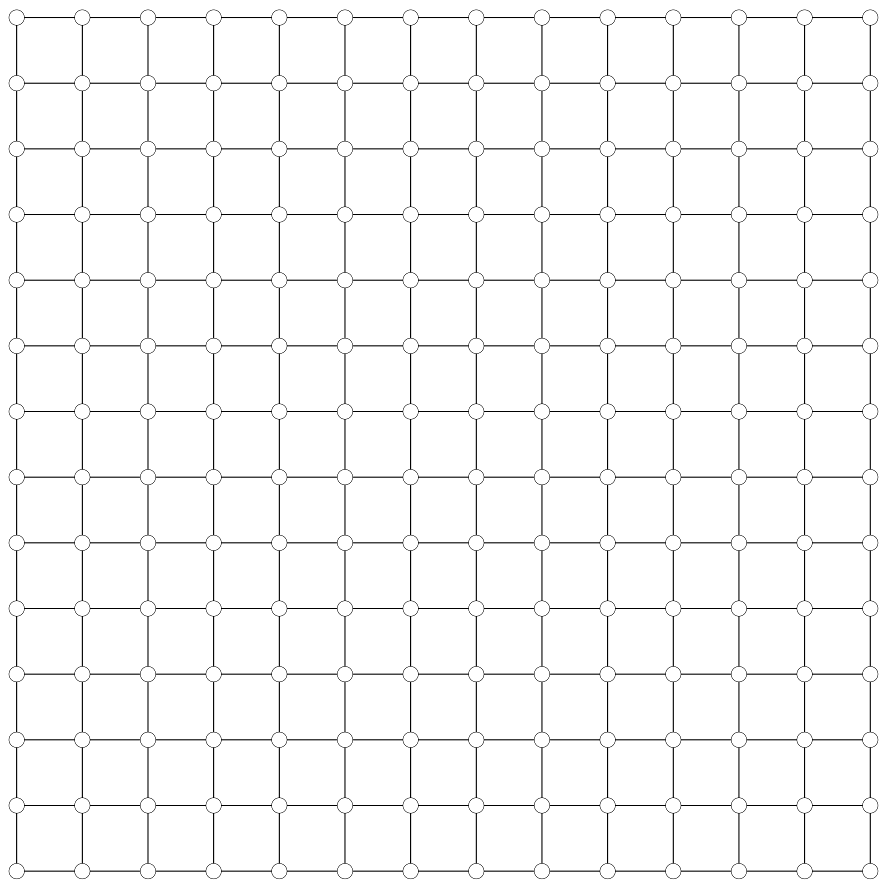


Figure 3.4 3 black points on the boundary

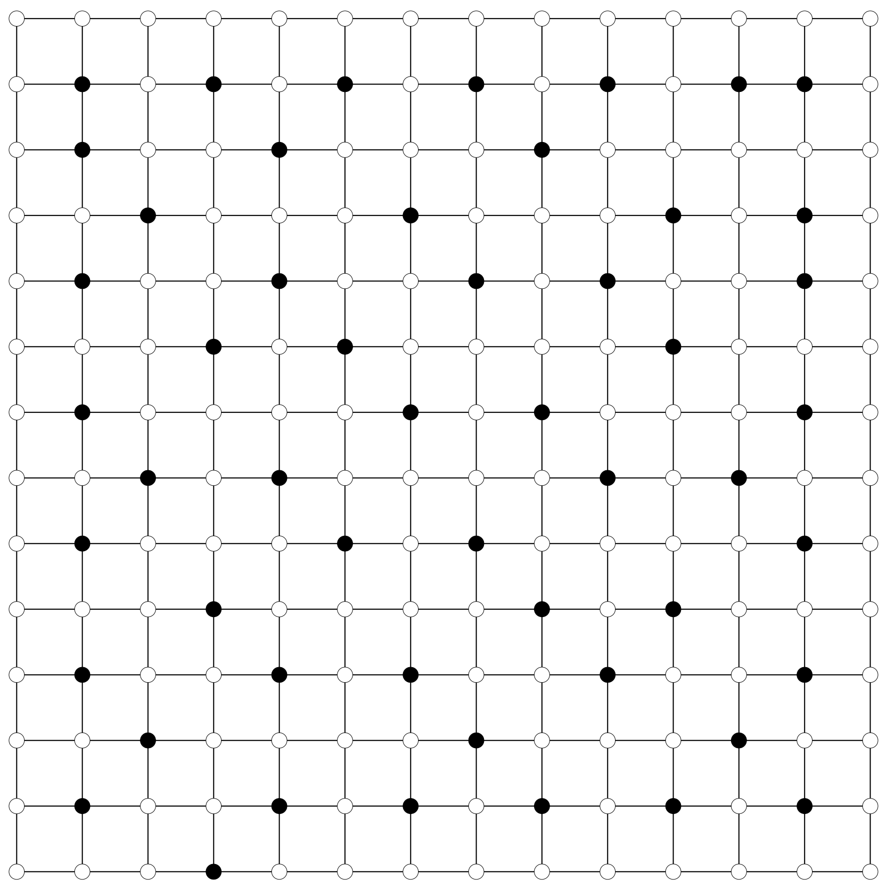


Figure 3.5 $M(14) = 52$

Argument for Conjecture 3 Moving the black point in row 2, column 2 in Figure 3.5 up to the top boundary gives us a configuration that requires 52 black points. Moving this black point up still covers the same square paths. This solution has two black points on the boundary that are on nonadjacent boundaries.

End of Argument

If Conjectures 2 and 3 hold, then clearly Conjecture 4 follows.

Conjecture 4 There exists an optimal solution such that three of the four boundaries do not contain a black point.

The following conjecture holds if all of the above are true.

Conjecture 5 There exists an optimal solution such that the points in positions $n+2$, $2n-1$, n^2-n-1 are black points.

Argument for Conjecture 5 Since there exist an optimal solution such that three of the four boundaries do not contain any black points, then it follows that the points labeled a and b must be black since the boundaries adjacent to the corner nodes do not contain a black point. See Figure 3.6. This forces either c or e and d or f to be black points. Since there is an optimal solution that contains a black point on the outer $n \times n$ boundary, then we can force say e to be a black point. Thus, this requires d to be a black.

End of Argument

The new constraints contributed significantly in the improvement for the time to obtain $M(n)$. For n=12, the original BIP in (wherever it is located) required 3111.67 seconds while the new BIP took only 1157.18 seconds *. With the additional constraints the final BIP required only 182.53 seconds. These computational

*All computations were obtained from a Sun Sparc Ultra 30 Model with 256M memory.

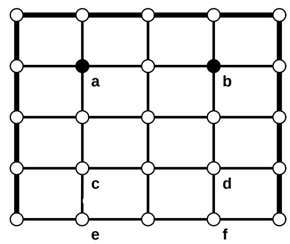


Figure 3.6 No black points on one boundary.

results quantify how the final mathematical formulation is significantly better than the original. All computed values for $M(n)$ can be found in the Results chapter.

Chapter 4

A Similar Problem

One related problem to the SPP is the problem that seeks to find the minimum number of vertices to remove from a planar graph G such that no circuits exist. Dean refers to this minimum value as $\tau(G)$ in general and $\tau(n)$ for the SPP. The graphs are planar in both of these problems. In relation to to the Square Path problem discussed in this work, $\tau(n)$ provides an upper bound for $M(n)$. Bienstock-Dean consider covering points of a planar graph with a minimum number of faces. The Erdös-Pósa theorem on independent circuits in graphs can be applied when we consider graphs with a specific embedding.

Several upper bounds for $\tau(n)$ for n up to 14 are included in the Results chapter. The following is a conjecture about the bounds of $\tau(n)$.

Conjecture 1

$$\frac{1}{3}(n-1)^2 \leq \tau(n) \leq \frac{1}{3}n^2$$

The following figures show an example of $\tau(n)$ where n=4. Figure 4.2 illustrates the new lattice after the black points indicated in Figure 4.1 have been removed.

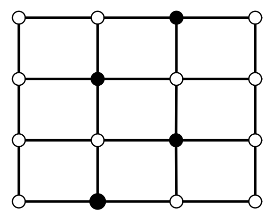


Figure 4.1 $\tau(4) = 4$

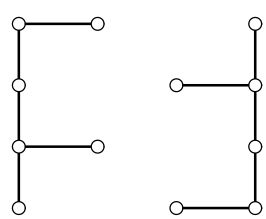


Figure 4.2 No circuits in $\tau(4) = 4$

Chapter 5

Results

Table 5.1 indicates proven results for the non-square lattice.

Table 5.1 Results for $M(a, b)$

$M(2,3) = M(3,2) = 1$
$M(3,4) = M(4,3) = 2$
$M(3,5) = M(5,3) = 3$
$M(5,4) = M(4,5) = 4$
$M(3,7) = M(7,3) = 4$
$M(4,7) = M(7,4) = 6$
$M(5,6) = M(6,5) = 7$

Table 5.2 indicates proven and computed values for $M(n)$ and $\tau(n)$. Note that an underlined value provides an upper bound and a "*" indicates values computed via CPLEX and not proven theoretically. Also note that "**" indicates that constraints that were not proven theoretically were added to the BIP to obtain the indicated solution.

Table 5.2 Results for both $M(n)$ and $\tau(n)$

n	2	3	4	5	6	7	8	9	10	11	12	13	14
$M(n)$	1	2	4	6	9	12	16	20	26*	31*	38*	44*	52**
$\tau(n)$	1	2	4	6	$\underline{10}$	$\underline{13}$	$\underline{19}$	$\underline{24}$	$\underline{32}$	$\underline{38}$	$\underline{47}$	$\underline{56}$	$\underline{64}$

5.1 Computational Results

The table below will include the amount of time it took to solve the BIP with the original formulation and also with the new formulation which includes the new constraints mentioned in chapter 3. The computational results were obtained from a Sun Sparc Ultra 30 Model with 256M memory using CPLEX version 6.0.1 with the exception of the new results for $n = 12, 13, 14$. These results were run on four processors and used an unpublished version of CPLEX and are indicated by * in the table. We might expect the case for $n = 15$ to take at least 2 days if run on multiple processors. Otherwise, it may take about six days since the lack of memory forced us to abort the run after 175057.52 sec which is roughly 48.63 hours. The upper and lower bounds for this problem at the time of abortion was 62 and 58, respectively. Storing only necessary information in CPLEX should eventually lead us to the solution for $n = 15$.

Table 5.3 Computational Results

n	$M(n)$	Old BIP (sec)	New BIP (sec)
3	2	0.00	0.0
4	4	0.02	0.0
5	6	0.02	0.0
6	9	0.02	0.0
7	12	0.13	0.7
8	16	0.15	0.11
9	20	3.78	2.93
10	26	40.92	17.45
11	31	568.42	313.87
12	38	3111.67	182.53*
13	44	--	19506.30*
14	52	--	26206.80**

We want to stress the importance of having more constraints to define the feasible region of the BIP. As mentioned earlier, adding the new constraints decreases the running time. Finding more constraints should direct us to solutions for larger n.

5.2 Configurations for Larger n

The following figures show configurations obtained by CPLEX.

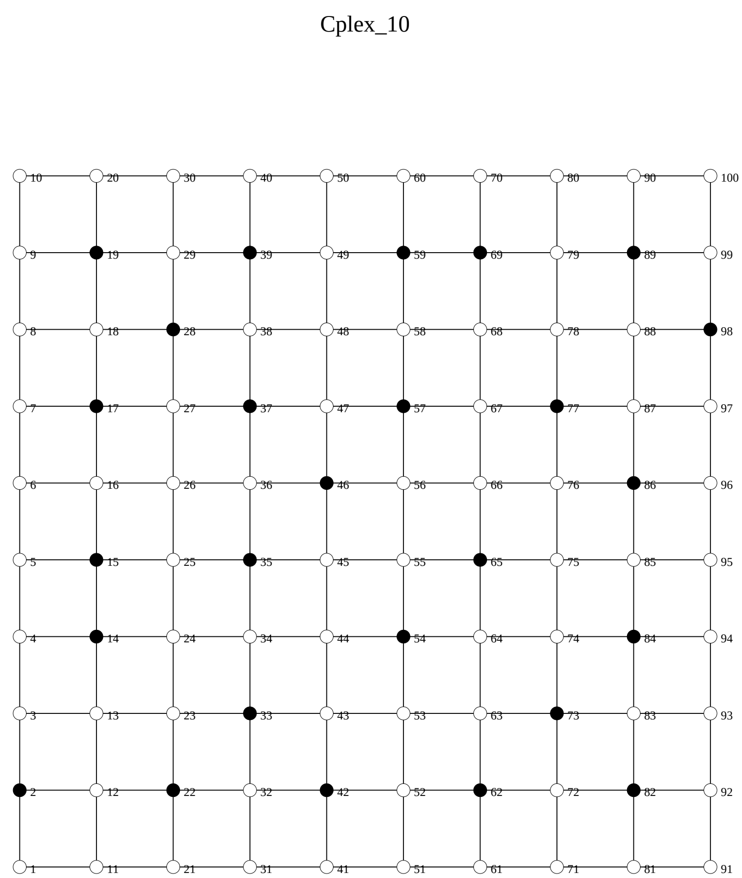


Figure 5.1 $M(10)$

Cplex_11

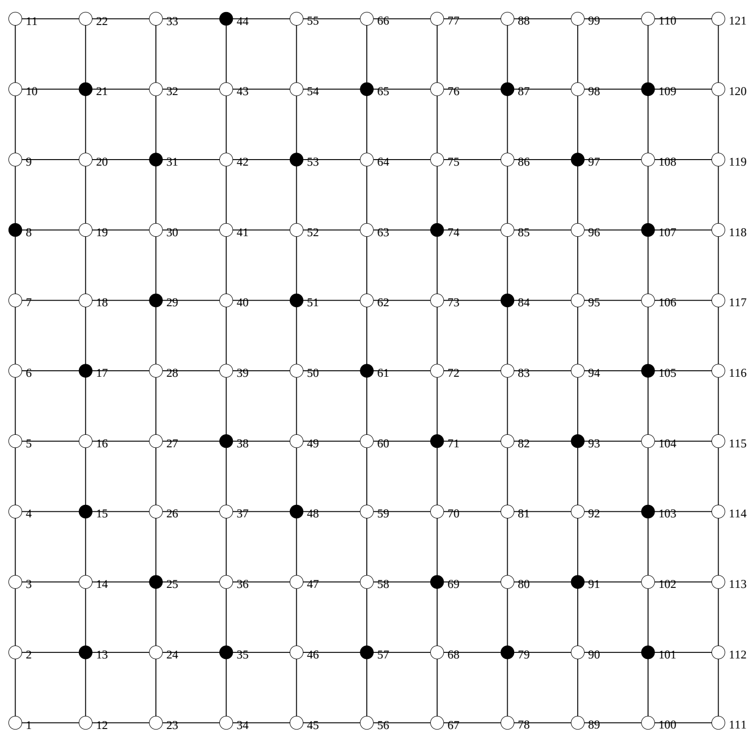


Figure 5.2 $M(11)$

Cplex_12

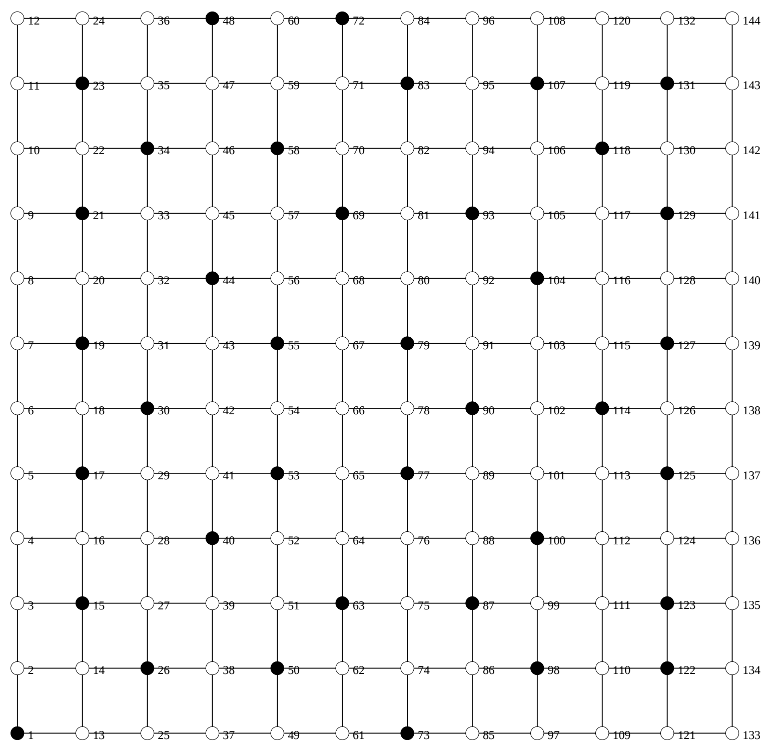


Figure 5.3 $M(12)$

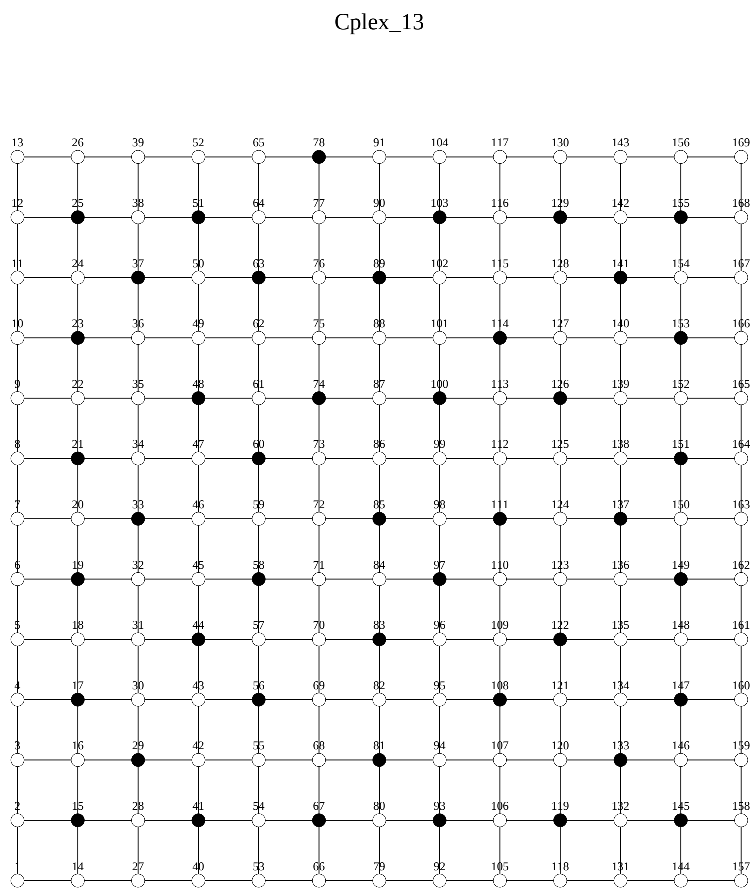


Figure 5.4 $M(13)$

5.3 Closed Form Attempt

In an attempt to obtain insight for a closed form expression or formula for $M(n)$, the computed values from the CPLEX solution of the BIP formulation were fed into Sloane's On-Line Encyclopedia of Integer Sequences [6], but no formula was found. However, there exists a formula for even values of n due to Kimberling [6]. Kimberling describes this sequence as the index of 5^n within the sequence of numbers of the form 2^i5^j. For example, the first nine terms of this sequence are $\underline{1}$, 2, 4, $\underline{5}$, 8, 10, 16, 20, $\underline{25}$ and the underlined terms are the first, fourth and ninth terms of the sequence. These indices are indeed the values of $M(n)$ for $n \geq 2$ with n even. Even though this provides more information for a formula for the SPP, a general formula for any n is still desired.

5.4 Future Work

Adding more constraints to the BIP as well as taking advantage of symmetry should aid in providing a a formula for $M(n)$ efficiently. Also passing known bounds to CPLEX for $M(n)$ and using tricks in CPLEX should decrease the running time compared to the time for n=14. Proving more values for $M(n)$ should eventually help in obtaining a general formula for $M(n)$. This work can be investigated further.

Bibliography

[1] D. Bienstock and N. Dean, On Obstructions to Small Face Covers in Planar Graphs, *Journal of Combinatorial Theory, Series B* **55** (1992), 163-189.

[2] D. Erickson, "Solution 1296", *Mathematics Magazine*, **62** (1988), 142.

[3] P. Erdös and L. Pósa, On Independent Circuits Contained in a Graph, *Canadian Journal of Mathematics* **17** (1965), 347-352.

[4] L. Larson, personal communication, 1999.

[5] H.C. Morris, "Proposal 1296", *Mathematics Magazine*, **61** (1988), 11 5.

[6] `http://akpublic.research.att.com/~njas/sequences/index.html`

[7] P. Zorn, personal communication, 1999.