Convergents of Folded Continued Fractions

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Abstract

Let $\varepsilon_i = \pm 1$ for $i \geq 1$. We discuss the convergents of the continued fractions of two formal Laurent series in $\mathbb{Q}((X^{-1}))$: $g_\varepsilon(X) = \sum_{i \geq 0} \varepsilon_i X^{-2^i}$ and $h_\varepsilon(X) = Xg(X)$. In particular, we show that the denominators of the convergents to both series have the remarkable property that their coefficients all lie in $\{0, 1, -1\}$.

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1 Introduction

Let $t(X) = \sum_{k \geq -n} a_k X^{-k}$ be a formal Laurent series in $\mathbb{Q}((X^{-1}))$. Then it is well known that $t(X)$ may be expanded uniquely as a continued fraction

$$[a_0, a_1, a_2, \ldots]$$

where $a_i \in \mathbb{Q}[X]$ for $i \geq 0$ and $\deg a_i > 0$ for $i > 0$. (This fact was apparently first proved by E. Artin in his thesis [2, §12]; for alternative treatments, see [11, 3, 14].) The a_i are called the *partial quotients*. These expansions share many of the properties of ordinary continued fractions for real numbers; for example, if we set $p_{-2} = 0$, $p_{-1} = 1$, $q_{-2} = 1$, $q_{-1} = 0$, and $p_n = a_n p_{n-1} + q_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 0$, then

$$[a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n}. \tag{1}$$

We call p_n/q_n the nth *convergent* to the continued fraction for $t(X)$. The following three identities will prove useful [7, 16]:

$$[a_n, a_{n-1}, \ldots, a_1] = \frac{q_n}{q_{n-1}}; \tag{2}$$

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}; \tag{3}$$

$$\frac{p_n}{q_n} = a_0 + \sum_{0 \leq i < n} \frac{(-1)^i}{q_i q_{i+1}}. \tag{4}$$

Let $\varepsilon_0 = 1$ and $\varepsilon_i = \pm 1$ for $i \geq 1$. Define

$$g_\varepsilon(X) = \sum_{i \geq 0} \varepsilon_i X^{-2^i}; \tag{5}$$

$$h_\varepsilon(X) = X \sum_{i \geq 0} \varepsilon_i X^{-2^i}. \tag{6}$$

If the choice of signs is arbitrary, or is clear from the context, we will often omit the subscript and simply write $g(X)$ and $h(X)$. Previous papers have discussed the continued fractions for $g_\varepsilon(X)$ and $h_\varepsilon(X)$; see, for example, [19, 8, 10, 12, 20, 4, 21, 15]. It is known that, no matter what the choice of signs is, the continued fraction for $g_\varepsilon(X)$ has partial quotients in the set $\{0, X - 1, X + 1, X, X + 2, X - 2\}$, and the continued fraction for $h_\varepsilon(X)$ has partial quotients in the set $\{1, X, -X\}$; furthermore, explicit formulas are known for these partial quotients in terms of the choice of signs made. These explicit formulas use a description arising from iterated paperfolding [5].

In this paper, we examine the convergents to the continued fractions for $g(X)$ and $h(X)$. In particular, we prove the surprising fact that the denominators of the convergents are polynomials in $\mathbb{Z}[X]$ with coefficients $0, 1, -1$.

Here is a brief outline of this paper. In Section 2, we recall the descriptions of the continued fractions for $g(X)$ and $h(X)$ in terms of the folding and perturbed symmetry maps. In Section 3, we recall some basic facts about continuants and prove some technical

lemmas. In Section 4, we prove some theorems about the constant terms of the convergents to $g(X)$ and $h(X)$. In Section 5 we give a recursion relation for computing the denominators of the convergents and characterize their support. In Section 6, we show how to describe the convergents for $h(X)$ in terms of those for $g(X)$. In Section 7, we obtain further results using some simple Diophantine approximation properties.

2   Paperfolding and Continued Fractions

Let w and x be finite sequences over an additive group G, and define the *folding map* $f_x(w) = w \cdot x \cdot (-w^R)$, where "$\cdot$" represents concatenation, $-w$ represents the same sequence as w, but with all signs reversed, and w^R represents the sequence w taken in reverse order.

Let G be a group written additively, and let $x_i \in G$ for $i \geq 1$.

$$\mathcal{F}(x_1, x_2, \ldots, x_k) = f_{x_k}(f_{x_{k-1}}(\cdots(f_{x_2}(f_{x_1}(\lambda)))\cdots)),$$

where λ represents the empty sequence. Since $\mathcal{F}(x_1, x_2, \ldots, x_k)$ is a prefix of the sequence $\mathcal{F}(x_1, x_2, \ldots, x_k, x_{k+1})$, the limiting sequence $\mathcal{F}(x_1, x_2, \ldots)$ is well-defined. For example, $\mathcal{F}(1, 2, 3, 4, \ldots)$ denotes the infinite sequence

$$(1, 2, -1, 3, 1, -2, -1, 4, 1, \ldots).$$

We now have the following

Theorem 1 *If* $(a_1, a_2, a_3, \ldots) = \mathcal{F}(x_1, x_2, x_3, \ldots)$, *then*

(a) $a_{2i+1} = (-1)^i x_1$ *for* $i \geq 0$;

(b) $(a_{2i})_{i \geq 1} = \mathcal{F}(x_2, x_3, x_4, \ldots)$;

(c) $a_{2^k + j} = -a_{2^k - j}$ *for* $k \geq 1$, $0 \leq j \leq 2^k - 1$.

Proof. The easy proof by induction is left to the reader. ∎

If $e_i = \pm 1$ for $i \geq 1$, we say that $\mathcal{F}(e_1, e_2, \ldots)$ is a *paperfolding sequence* [13, 5]. In this case, the e_i are sometimes referred to as the *unfolding instructions*.

In the special case $e_i = 1$ for $i \geq 1$, we get the *regular paperfolding sequence*

$$(s_i)_{i \geq 1} = \mathcal{F}(1, 1, 1, \ldots) = (1, 1, -1, 1, 1, -1, -1, 1, 1, 1, -1, -1, 1, -1, -1, \ldots). \tag{7}$$

For this sequence, we have

$$s_n = (-1)^r \tag{8}$$

if $n = 2^a(2r + 1)$.

Now let $h(X) = X \sum_{i \geq 0} \varepsilon_i X^{-2^i}$, and assume $\varepsilon_0 = 1$, and $\varepsilon_i = \pm 1$ for $i \geq 1$. It is known [15] that

$$h(X) = [1, \ X\mathcal{F}(\varepsilon_1, \ -\varepsilon_2, \ -\varepsilon_3, \ -\varepsilon_4, \ \ldots)]. \tag{9}$$

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For example, we have

$$X \sum_{k \geq 0} X^{-2^k} = [1, \ X, \ -X, \ -X, \ -X, \ X, \ X, \ -X, \ -X, \ \ldots].$$

Similarly, we can define the *perturbed symmetry map* $p_x(w) = w \cdot x \cdot w^R$ [12]. Let $x_1, x_2, \ldots x_k$ each represent a *finite sequence* over an additive group G. Then we define

$$\mathcal{P}(x_1, x_2, \ldots, x_k) = p_{x_k}(p_{x_{k-1}}(\cdots(p_{x_2}(p_{x_1}(\lambda)))\cdots)).$$

As before, we can also consider the limiting sequence $\mathcal{P}(x_1, x_2, \ldots)$. It is known [4, Theorem 5] that if $g(X) = \sum_{i \geq 0} \varepsilon_i X^{-2^i}$, then

$$g(X) = [0, \ X - \varepsilon_1, \ \mathcal{P}(c_2, c_3, c_4, \ldots)], \tag{10}$$

where

$$c_2 = (X + \varepsilon_1 + \varepsilon_2, X + \varepsilon_1 - \varepsilon_2)$$

and

$$c_j = (X - \varepsilon_1 + \varepsilon_j, X - \varepsilon_1 - \varepsilon_j)$$

for $j \geq 3$. For example, we have

$$\sum_{k \geq 0} X^{-2^k} = [0, \ X - 1, \ X + 2, \ X, \ X, \ X - 2, \ X, \ X + 2, \ \ldots].$$

3   Continuants

We recall the basic definition of the multivariate polynomials known as *continuants*. Define $K_0() = 1$, $K_1(a_1) = a_1$, and $K_n(a_1, a_2, \ldots, a_n) = a_n K_{n-1}(a_1, \ldots, a_{n-1}) + K_{n-2}(a_1, \ldots, a_{n-2})$. It is also convenient to set $K_{-1}() = 0$. Then it is not difficult to see that

$$[a_0, a_1, \ldots, a_n] = \frac{K_{n+1}(a_0, a_1, \ldots, a_n)}{K_n(a_1, a_2, \ldots, a_n)}$$

for all $n \geq 0$ [9, §4.5.3].

We now give three technical lemmas on continuants which will be used later in the paper.

Lemma 2 *Let $n \geq 1$ and $a_1, a_2, \ldots, a_{n-1}$ be real numbers. Then*

$$K_{n+j}(a_1, a_2, \ldots, a_{n-1}, 0, -a_{n-1}, -a_{n-2}, \ldots, -a_{n-j}) = K_{n-j-2}(a_1, a_2, \ldots, a_{n-j-2})$$

for $0 \leq j \leq n - 1$.

Proof. The easy induction on j is left to the reader.   ∎

Lemma 3 *Let $n \geq 1$, and let $a_1, a_2, \ldots, a_n$ be integers. Then*

$$K_{n+j}(a_1, \ldots, a_{n-1}, a_n, -a_{n-1}, -a_{n-2}, \ldots, -a_{n-j}) \equiv$$

$$K_{n+j}(a_1, \ldots, a_{n-1}, 0, -a_{n-1}, -a_{n-2}, \ldots, -a_{n-j}) \pmod{a_n K_{n-1}(a_1, \ldots, a_{n-1})}$$

for $0 \leq j \leq n - 1$.

Proof. Let $q_{n-1} = K_{n-1}(a_1, a_2, \ldots, a_{n-1})$ and $q_{n-2} = K_{n-2}(a_1, a_2, \ldots, a_{n-2})$. Then we have

$$K_n(a_1, a_2, \ldots, a_{n-1}, a_n) = a_n q_{n-1} + q_{n-2}$$
$$K_n(a_1, a_2, \ldots, a_{n-1}, 0) = q_{n-2}.$$

Now for each j there exist integer constants c_1, c_2 such that

$$K_{n+j}(a_1, \ldots, a_{n-1}, a_n, -a_{n-1}, -a_{n-2}, \ldots, -a_{n-j}) = c_1 K_{n-1}(a_1, \ldots, a_{n-1}) + c_2 K_n(a_1, \ldots, a_{n-1}, a_n)$$

and

$$K_{n+j}(a_1, \ldots, a_{n-1}, 0, -a_{n-1}, -a_{n-2}, \ldots, -a_{n-j}) = c_1 K_{n-1}(a_1, \ldots, a_{n-1}) + c_2 K_n(a_1, \ldots, a_{n-1}, 0).$$

Hence, subtracting these last two equations, we get

$$K_{n+j}(a_1, \ldots, a_{n-1}, a_n, -a_{n-1}, -a_{n-2}, \ldots, -a_{n-j}) -$$

$$K_{n+j}(a_1, \ldots, a_{n-1}, 0, -a_{n-1}, -a_{n-2}, \ldots, -a_{n-j}) = c_2 a_n q_{n-1},$$

which proves the desired result. ∎

Lemma 4 *Let $k_i \geq 0$ for $i \geq 1$, and let*

$$c = (c_1, c_2, c_3, \ldots) = (-1, \ 0^{k_1}, \ 2, \ 0^{k_2}, \ -2, \ 0^{k_3}, \ 2, \ 0^{k_4}, \ -2, \ 0^{k_5}, \ 2, \ \ldots)$$

be an infinite sequence of integers. (The exponents on the 0's refer to the number of repetitions, and the 2's alternate in sign.) Then for all $n \geq 0$, $K_n(c_1, c_2, \ldots, c_n) = \pm 1$, and the same result holds for $-c = (-1, 0^{k_1}, -2, 0^{k_2}, 2, 0^{k_3}, -2, \ldots)$.

Proof. Let $q_n = K_n(c_1, c_2, \ldots, c_n)$. Then q_n is the denominator of the nth convergent of the continued fraction

$$[0, -1, 0^{k_1}, 2, 0^{k_2}, -2, 0^{k_3}, 2, 0^{k_4}, -2, 0^{k_5}, 2, \ldots].$$

By (2), it follows that

$$\frac{q_n}{q_{n+1}} = [\overbrace{\cdots, -2, 0^{k_2}, 2, 0^{k_1}, -1}^{n \text{ terms}}].$$

But

$$[2, 0^{k_1}, -1] = 1$$

for all $k_1 \geq 0$. It follows that

$$\frac{q_n}{q_{n+1}} = [\cdots, -2, 0^{k_2}, 1].$$

But

$$[-2, 0^{k_2}, 1] = -1$$

for all $k_2 \geq 0$. It now follows by induction that

$$\frac{q_n}{q_{n+1}} = \pm 1.$$

Since $q_0 = 1$, we conclude that $q_n = \pm 1$ for all $n \geq 0$. ∎

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4   The Constant Term

Let $g_\varepsilon(X)$ and $h_\varepsilon(X)$ be as defined above in Eqs. (5) and (6). Let p'_n/q'_n be the nth convergent to the continued fraction for $g_\varepsilon(X)$, and p_n/q_n be the nth convergent to the continued fraction for $h_\varepsilon(X)$. Then $(p'_n)_{n\geq 0}$, $(q'_n)_{n\geq 0}$, $(p_n)_{n\geq 0}$, and $(q_n)_{n\geq 0}$ are sequences of polynomials in X, and we are interested in their constant term.

We begin with the constant terms of p_n and q_n:

Theorem 5 *For $n \geq -1$ we have*

   (a) $p_n(0) = 1$;

   (b) $q_{2n}(0) = 1$;

   (c) $q_{2n+1}(0) = 0$.

Proof. Easy proof by induction on n, using the fact that $p_n = \pm X p_{n-1} + p_{n-2}$, and $q_n = \pm X q_{n-1} + q_{n-2}$.   ∎

We now turn to the constant terms of p'_n and q'_n.

Theorem 6 *For $n \geq 0$ we have $q'_n(0) = \pm 1$.*

Proof. Note that if $g_\varepsilon(X) = [a'_0(X), a'_1(X), \ldots]$, then by Eq. (10), we have $a'_i(0) = c_i$, where (c_i) is the sequence in Lemma 4.   ∎

Remark. Evidently the proof applies to *any* formal Laurent series such that the constant terms of the partial quotients of its continued fraction expansion is given by the sequence (c_i) of Lemma 4.

We now discuss p'_n and q'_n for a *particular* choice of ε; namely, when $\varepsilon_i = 1$ for all $i \geq 0$. First, we describe the partial quotients for the continued fraction expansion of $\sum_{i\geq 0} X^{-2^i}$:

Theorem 7 *Let $\sum_{i\geq 0} X^{-2^i} = [a'_0(X), a'_1(X), \ldots]$. Then $a'_1(X) = X - 1$, and $a'_n(X) = X - (-1)^{\lfloor n/2 \rfloor} + (-1)^n s_{\lfloor n/2 \rfloor}$, where $(s_n)_{n\geq 1}$ is the regular paperfolding sequence of Eq. (7).*

Proof. Follows from Eq. 10.   ∎

Now we discuss the denominators of the convergents to $\sum_{i\geq 0} X^{-2^i}$:

Theorem 8 *Let $q'_n(X)$ be the denominator of the nth convergent to $\sum_{i\geq 0} X^{-2^i}$. Then*

$$q'_n(0) = (-1)^{n + \lfloor n/2 \rfloor} s_1 s_2 \cdots s_n,$$

where $(s_i)_{i\geq 0}$ is the regular paperfolding sequence of Eq. (7).

Proof. By induction, using the fact that $q'_{n+1}(0) = a'_{n+1}(0) q'_n(0) + q'_{n-1}(0)$. (Recall that the sequence a'_{n+1} was described in Theorem 7.)   ∎

Now we discuss the numerators of the convergents to $\sum_{i\geq 0} X^{-2^i}$:

Theorem 9 *For $n \geq 0$ we have*

$$\frac{p_n'(0)}{q_n'(0)} = -\sum_{1 \leq i \leq n} s_i, \tag{11}$$

where $(s_i)_{i \geq 0}$ is the regular paperfolding sequence of Eq. (7).

Proof. By Eq. (4) we know that

$$\frac{p_n'(0)}{q_n'(0)} = \frac{1}{q_0'(0)q_1'(0)} - \frac{1}{q_1'(0)q_2'(0)} + \cdots + \frac{(-1)^{n-1}}{q_{n-1}'(0)q_n'(0)}.$$

Since $q_n'(0) = \pm 1$ by Theorem 6, it follows that $q_n'(0) = 1/q_n'(0)$, and we get

$$\frac{p_n'(0)}{q_n'(0)} = q_0'(0)q_1'(0) - q_1'(0)q_2'(0) + q_2'(0)q_3'(0) - \cdots + (-1)^{n-1}q_{n-1}'(0)q_n'(0). \tag{12}$$

Let us first consider the case where n is even, say $n = 2k$. Then Eq. (12) and Theorem 8 gives

$$
\begin{aligned}
\frac{p_{2k}'(0)}{q_{2k}'(0)} &= -a_2'(0) - a_4'(0) - \cdots - a_{2k}'(0) \\
&= -\sum_{1 \leq i \leq k} a_{2i}'(0) \\
&= \sum_{1 \leq i \leq k} (-1)^i - \sum_{1 \leq i \leq k} s_i \\
&= -\sum_{0 \leq i \leq k-1} s_{2i+1} - \sum_{1 \leq i \leq k} s_{2i} \\
&= -\sum_{1 \leq i \leq 2k} s_i.
\end{aligned}
$$

Now let us consider the case where n is odd, say $n = 2k+1$. Then by Eq. (12) and Theorem 8 we get

$$
\begin{aligned}
\frac{p_{2k+1}'(0)}{q_{2k+1}'(0)} &= \frac{p_{2k}'(0)}{q_{2k}'(0)} + q_{2k}'(0)q_{2k+1}'(0) \\
&= \frac{p_{2k}'(0)}{q_{2k}'(0)} + ((-1)^k s_1 s_2 \cdots s_{2k})((-1)^{k+1} s_1 s_2 \cdots s_{2k+1}) \\
&= \frac{p_{2k}'(0)}{q_{2k}'(0)} - s_{2k+1} \\
&= -\sum_{1 \leq i \leq 2k+1} s_i.
\end{aligned}
$$

The result now follows. ∎

Remark. It is easy to prove by induction that $\sum_{1 \leq k \leq n} s_k$ is always positive, and so we have $\operatorname{sgn} p_n'(0) = -q_n'(0)$.

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Remark. In true Apéry fashion, we have therefore established that the best approximations to $\sum_{k \geq 0} \frac{1}{0^{2^k}}$ are the *integers* $-|p'_n(0)|$.

Our result also implies that the sequence $p'_n(0)$ is 2-regular in the sense of Allouche and Shallit [1].

Exercise. Prove that $|p'_n(0)| = O(\log n)$. Also show that, for $k \geq 0$, the base-2 representation of the least index n such that $|p'_n(0)| = k$ is $\overbrace{10101\cdots}^{k \text{ symbols}}$.

Exercise. Show that the coefficient of the second-highest term in $q'_n(X)$ is $(-1)^n s_n$ for $n \geq 1$.

5 More on Convergents

In the previous section, we studied the constant term of the convergents to $g_\varepsilon(X) = \sum_{i \geq 0} \varepsilon_i X^{-2^i}$. Our results suggest studying the rest of the coefficients of the convergents. To aid intuition, here is a brief table of the convergents $p'_n(X)/q'_n(X)$ to $\sum_{i \geq 0} X^{-2^i}$:

n	$a'_n(X)$	$p'_n(X)$	$q'_n(X)$
0	0	0	1
1	$X-1$	1	$X-1$
2	$X+2$	$X+2$	X^2+X-1
3	X	X^2+2X+1	X^3+X^2-1
4	X	X^3+2X^2+2X+2	$X^4+X^3+X^2-1$
5	$X-2$	X^4-X^2-3	$X^5-X^4-X^2-X+1$
6	X	X^5+2X^2-X+2	$X^6-X^5+X^4+X-1$
7	$X+2$	$X^6+2X^5+X^4+2X^3+2X^2+1$	$X^7+X^6+X^4-1$
8	X	$X^7+2X^6+2X^5+2X^4+2X^3+2X^2+2$	$X^8+X^7+X^6+X^4-1$
9	$X-2$	$X^8-X^6-X^4-2X^2+2X-3$	$X^9-X^8-X^6+X^5-X^4-X+1$

Table 1: Convergents to $\sum_{i \geq 0} X^{-2^i}$.

The observant reader will note that the coefficients of $q'_n(X)$ lie in $\{0, 1, -1\}$ for $0 \leq n \leq 9$ and will naturally wonder if this is true for all n. We prove this (and more) below in Corollary 16. For the moment, however, we turn to the study of $h_\varepsilon(X) = Xg_\varepsilon(X)$ instead. This series is somewhat easier to handle and, as we will see in Section 6, the results we obtain for $h_\varepsilon(X)$ will also imply results for $g_\varepsilon(X)$.

Again, to build intuition, we provide a brief table of the convergents $p_n(X)/q_n(X)$ to $X \sum_{i \geq 0} X^{-2^i}$.

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n	$a_n(X)$	$p_n(X)$	$q_n(X)$
0	1	1	1
1	X	$X+1$	X
2	$-X$	$-X^2-X+1$	$-X^2+1$
3	$-X$	X^3+X^2+1	X^3
4	$-X$	$-X^4-X^3-X^2-2X+1$	$-X^4-X^2+1$
5	X	$-X^5-X^4-X^2+X+1$	$-X^5+X$
6	X	$-X^6-X^5-X^4-2X^3-X+1$	$-X^6-X^4+1$
7	$-X$	$X^7+X^6+X^4+1$	X^7
8	$-X$	$-X^8-X^7-X^6-2X^5-X^4-2X^3-2X+1$	$-X^8-X^6-X^4+1$
9	X	$-X^9-X^8-X^6-X^4-2X^2+X+1$	$-X^9-X^5+X$

Table 2: Convergents to $X\sum_{i\geq 0} X^{-2^i}$.

We begin our investigation by studying the denominators $q_n(X)$ of convergents to $h_\varepsilon(X)$. Unfortunately, the notation $q_n(X)$ is no longer sufficient, so we introduce the new notation $q_n^{(a_1,a_2,\ldots)}(X)$ to make the dependence on the partial quotients clear. More precisely, we define

$$q_n^{(a_1,a_2,\ldots)}(X) = K_n(a_1, a_2, a_3, \ldots, a_n).$$

(Since q_n does not depend on a_0, we do not specify it explicitly.)

Now let $e_i = \pm 1$ for $i \geq 1$, and define $e = (e_1, e_2, \ldots)$. By Eq. (9), we know that $q_n^{X\mathcal{F}(e)}(X)$ is the denominator of the nth convergent to $h_\varepsilon(X)$, where $\varepsilon_0 = 1$, $\varepsilon_1 = e_1$, and $\varepsilon_i = -e_i$ for $i \geq 2$. We now give a recursive formula for $q_n^{X\mathcal{F}(e)}(X)$ that expresses it as the sum of two polynomials, one with high-order terms and one with low-order terms.

Theorem 10 *Write $n = 2^k + j$, with $k \geq 0$ and $0 \leq j \leq 2^k - 1$. Let $e_i = \pm 1$ for $i \geq 1$. Define $e = (e_1, e_2, \ldots)$ and $e' = (e_1, e_2, \ldots, e_{k-1}, -e_k)$. Let $\mathcal{F}(e) = (a_1, a_2, \ldots)$. Then*

$$q_n^{X\mathcal{F}(e)}(X) = \begin{cases} e_1 X, & \text{if } n = 1; \\ -e_{k+1}X^{2^{k+1}-1}, & \text{if } k \geq 1 \text{ and } j = 2^k - 1; \\ -e_k e_{k+1}X^{2^k} q_j^{X\mathcal{F}(e')}(X) + q_{2^k-j-2}^{X\mathcal{F}(e)}(X), & \text{if } k \geq 1 \text{ and } 0 \leq j \leq 2^k - 2. \end{cases}$$

Proof. By induction on n. The result is clearly true for $n = 1$. Now assume it is true for all $n' < n$; we prove it for $n' = n$.

Write $n = 2^k + j$, with $k \geq 1$ and $0 \leq j \leq 2^k - 1$. We prove the identity by considering the high- and low-order terms of the polynomial $q_j^{X\mathcal{F}(e)}(X)$ separately. To do this, we consider the polynomial modulo X^{2^k}. First, we observe that

$$
\begin{aligned}
q_n^{X\mathcal{F}(e)}(X) &= K_n(a_1, \ldots, a_n) \\
&= K_n(a_1, \ldots, a_{2^k}, a_{2^k+1}, \ldots, a_{2^k+j}) \\
&= K_n(a_1, \ldots, a_{2^k}, -a_{2^k-1}, \ldots, -a_{2^k-j}) \quad \text{(by Theorem 1 (c))} \\
&\equiv K_n(a_1, \ldots, a_{2^k-1}, 0, -a_{2^k-1}, \ldots, -a_{2^k-j}) \pmod{a_{2^k}K_{n-1}(a_1, \ldots, a_{2^k-1})} \quad \text{(by Lemma 2)} \\
&\equiv K_n(a_1, \ldots, a_{2^k-1}, 0, -a_{2^k-1}, \ldots, -a_{2^k-j}) \pmod{X^{2^k}} \quad \text{(by the induction hypothesis)} \\
&= K_{2^k-j-2}(a_1, \ldots, a_{2^k-j-2}) \quad \text{(by Lemma 3)}
\end{aligned}
$$

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and so

$$q_n^{X\mathcal{F}(e)}(X) \equiv q_{2^k-j-2}^{X\mathcal{F}(e)}(X) \quad (\text{mod } X^{2^k}). \tag{13}$$

Now, recalling that $n = 2^k + j$, define

$$r_j(X) = \frac{q_{2^k+j}^{X\mathcal{F}(e)}(X) - q_{2^k-j-2}^{X\mathcal{F}(e)}(X)}{-e_k e_{k+1} X^{2^k}}$$

for $0 \leq j \leq 2^k - 1$. (We omit the superscript on the variable r_j, since it is the same in both of the terms in the numerator of the expression that defines it.)

By induction, we know that

$$q_{2^k-1}^{X\mathcal{F}(e)}(X) = -e_k X^{2^k-1},$$

so the sign of the leading coefficient of $q_{2^k}^{X\mathcal{F}(e)}$ is $-e_k e_{k+1}$. Hence $r_0(X) = 1$. Similarly, it is easily verified that $r_{-1}(X) = 0$. We claim that

$$r_j(X) = a_{2^k+j} r_{j-1}(X) + r_{j-2}(X)$$

for $1 \leq j \leq 2^k - 1$. From the definition of $r_j(X)$, it suffices to verify that both $q_{2^k+j}^{X\mathcal{F}(e)}(X)$ and $q_{2^k-j-2}^{X\mathcal{F}(e)}(X)$ satisfy this recurrence. The first is clear. For the second, observe that

$$q_{2^k-j} = a_{2^k-j} q_{2^k-j-1} + q_{2^k-j-2},$$

so

$$q_{2^k-j-2} = -a_{2^k-j} q_{2^k-j-1} + q_{2^k-j}.$$

But, by Theorem 1 (c), we have $a_{2^k-j} = -a_{2^k+j}$ for $1 \leq j \leq 2^k - 1$. It follows that

$$q_{2^k-j-2} = a_{2^k+j} q_{2^k-j-1} + q_{2^k-j},$$

as desired.

Now, by the folding property,

$$
\begin{aligned}
\left(a_{2^k+1}, a_{2^k+2}, \ldots, a_{2^{k+1}-1}\right) &= \left(-a_{2^k-1}, -a_{2^k-2}, \ldots, -a_1\right) \\
&= -X\mathcal{F}(e_1, e_2, \ldots, e_k)^R \\
&= -X(\mathcal{F}(e_1, e_2, \ldots, e_{k-1}), e_k, -\mathcal{F}(e_1, e_2, \ldots, e_{k-1})^R)^R \\
&= X(\mathcal{F}(e_1, e_2, \ldots, e_{k-1}), -e_k, -\mathcal{F}(e_1, e_2, \ldots, e_{k-1})^R) \\
&= X\mathcal{F}(e_1, e_2, \ldots, e_{k-1}, -e_k),
\end{aligned}
$$

so we have

$$r_j(X) = q_j^{X\mathcal{F}(e_1,e_2,\ldots,e_{k-1},-e_k)}(X)$$

for $0 \leq j \leq 2^k - 1$.

Finally, it remains to see that $q_{2^{k+1}-1}^{X\mathcal{F}(e)}(X) = -e_{k+1}X^{2^{k+1}-1}$. To see this, note that we have shown that

$$
\begin{aligned}
q_{2^{k+1}-1}^{X\mathcal{F}(e)}(X) &= -e_k e_{k+1} X^{2^k} q_{2^k-1}^{X\mathcal{F}(e')}(X) + q_{-1}(X) \\
&= -e_k e_{k+1} X^{2^k} q_{2^k-1}^{X\mathcal{F}(e')}(X) \\
&= (-e_k e_{k+1} X^{2^k})(e_k X^{2^k-1}) \\
&= -e_{k+1} X^{2^{k+1}-1},
\end{aligned}
$$

where we have used the fact that $q_{-1}(X) = 0$; induction, and the fact that $e_k^2 = 1$. This completes the proof. $\blacksquare$

Corollary 11 *All the coefficients of* $q_n^{X\mathcal{F}(e)}(X)$, *no matter what choice of signs is made, are in* $\{1, 0, -1\}$.

Proof. An easy induction on n, using the fact that Theorem 10 writes $q_n(X)$ as the sum of a polynomial with lowest nonzero term of degree X^{2^k} and one with highest nonzero term of degree X^{2^k-1}. $\blacksquare$

We now turn to a kind of converse to Corollary 11. It is easiest to phrase this converse in terms of something we call the *infinite continuant tree* T. This root of this infinite binary tree is labeled $K_0(0) = 1$. If z is a node of the tree labeled $K_{n-1}(a_1, \ldots, a_{n-1})$, then there is an edge labeled $+1$ connecting z to its left child, labeled $K_n(a_1, \ldots, a_{n-1}, X)$. There is also an edge labeled -1 connecting z to its right child, labeled $K_n(a_1, \ldots, a_{n-1}, -X)$. Figure 1 gives the first few levels of the tree T. Labels on the edges have been omitted.
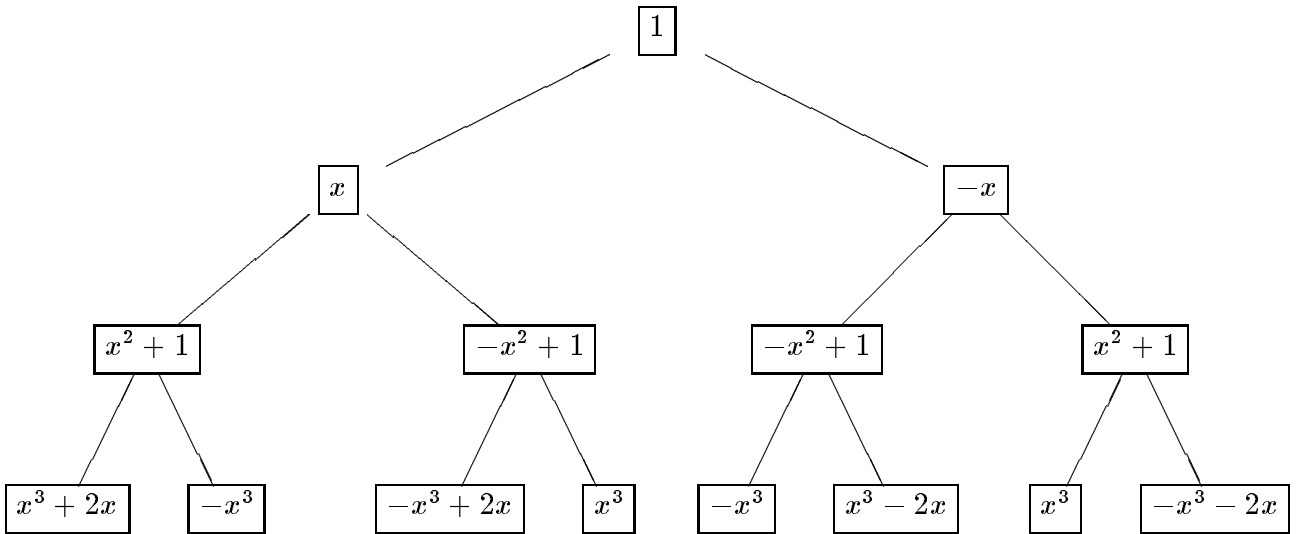

Figure 1: The first 4 levels of the continuant tree

By Theorem 11, any infinite path labeled with a paperfolding sequence reaches nodes with labels that have only coefficients $0, 1, -1$. The next theorem says that in T, *only* the paperfolding paths have this property.

Theorem 12 *Suppose we define a sequence of polynomials as follows: $q_{-1}(X) = 0$; $q_0(X) = 1$, and $q_i(X) = a_i X q_{i-1}(X) + q_{i-2}(X)$, where $a_i \in \{1, -1\}$. Furthermore, choose a_i successively so that the resulting polynomials $q_i(X)$ have coefficients in $\{-1, 0, 1\}$. Then $(a_1, a_2, \ldots)$ is a paperfolding sequence.*

Proof. Suppose, contrary to what we want to prove, that $a_1, a_2, \ldots$ is not a paperfolding sequence. Then there is some index, say n, such that $a_1, a_2, \ldots, a_n$ is not the prefix of any paperfolding sequence. Without loss of generality let n be the smallest such, so that $a_1, a_2, \ldots, a_{n-1}$ *is* the prefix of some paperfolding sequence.

By assumption, $q_n = a_n X q_{n-1} + q_{n-2}$ has only coefficients $0, 1, -1$, and by Theorem 10 so does $\hat{q}_n = -a_n X q_{n-1} + q_{n-2}$. Write $n = 2^k + j$, with $0 \le j < 2^k$. If n is not a power of 2, then by reducing modulo X^{2^k} and applying Theorem 10, we see that q_{n-1} has a nonzero term of the form $\pm X^{2^k - j - 1}$ and q_{n-2} has a nonzero term of the form $\pm X^{2^k - j}$. Since $\hat{q}_n = -a_n X q_{n-1} + q_{n-2}$, we see that these terms must cancel in $\hat{q}_n$, or else we would have a coefficient of ± 2 in the $X^{2^k - j}$ term in $\hat{q}_n$. But then $X^{2^k - j}$ has a coefficient of ± 2 in q_n, a contradiction.

It follows that n must be a power of 2. But then either choice $a_n = \pm 1$ corresponds to a valid prefix of a paperfolding sequence, a contradiction. ■

In Figure 2 below, we display the first few levels of the infinite continuant tree with only the paperfolding paths shown.

The continuant tree T has many other interesting properties, and we leave to the reader the pleasure of discovering some of them.
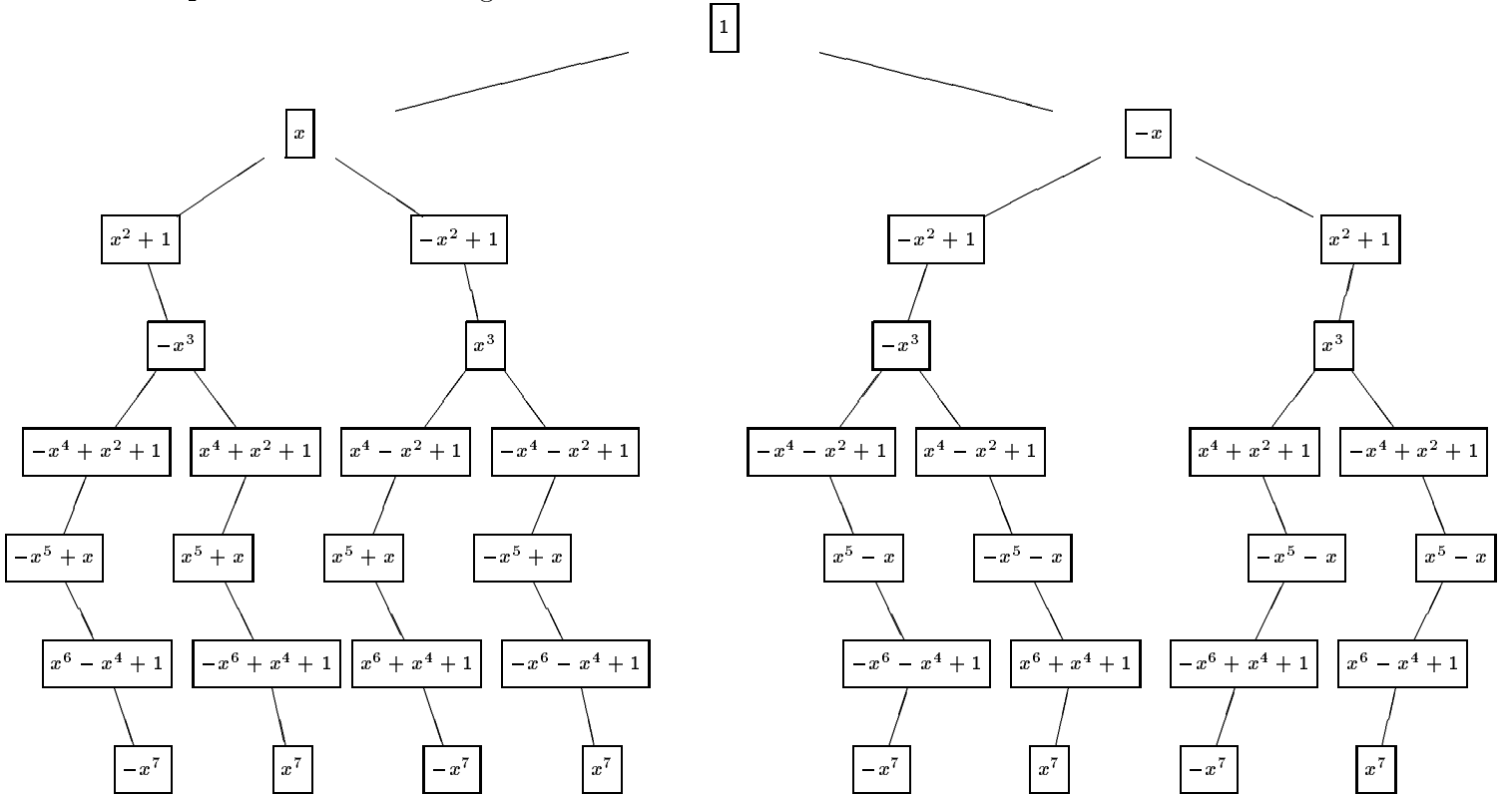

Figure 2: The first 8 levels of the continuant tree (only paperfolding paths shown)

The reader will notice that in any particular level of Figure 2, the polynomials are essentially the same; only the signs of the terms differ. We now prove that this pattern continues throughout the continuant tree.

First, we define the *support* $\operatorname{supp}(b)$ of a polynomial $b(X) = b_n X^n + \cdots + b_1 X + b_0$ to be the set $\{i \ : \ b_i \neq 0\}$, that is, the set of exponents corresponding to nonzero coefficients. For example, $\operatorname{supp}(X^5 - X) = \{1, 5\}$. We also define $[X^i]b(X)$ to be b_i, the coefficient of the X^i term in $b(X)$.

Corollary 13 *Let $\mathcal{F}(e_1, e_2, \ldots)$ be a paperfolding sequence, and define $a = X\mathcal{F}(e_1, e_2, \ldots)$. Then the set $\operatorname{supp}(q_n^a(X))$ is independent of $e_1, e_2, \ldots$, and is equal to*

$$\{n - 2i \ : \ 0 \leq i \leq n/2, \ \binom{n-i}{i} \equiv 1 \ (\mathrm{mod}\ 2)\}.$$

Proof. Since $e_1, e_2, \ldots$ is a paperfolding sequence, it follows that the coefficients of $q_n^a(X)$ are all $0, 1, -1$. Now, taking everything mod 2, it follows that

$$q_n^a(X) \equiv s_n(X) \ (\mathrm{mod}\ 2),$$

where $s_n(X)$ is the denominator of the nth convergent to

$$[1, X, X, X, \ldots].$$

But it is easy to prove by induction on n that

$$[X, X, X, \ldots] = \frac{t_{n+1}(X)}{t_n(X)},$$

where

$$t_n(X) = \sum_{0 \leq i \leq n/2} \binom{n-i}{i} X^{n-2i}.$$

Clearly $s_n(X) = t_n(X)$, and the result follows. ∎

Remark. If we define $a_{m,n} = [X^n]t_m(X) \bmod 2$, then it is not hard to see that

$$a_{m,n} = \begin{cases} 0, & \text{if } m \not\equiv n \ (\mathrm{mod}\ 2); \\ \binom{m+n}{m-n} \bmod 2, & \text{if } m \equiv n \ (\mathrm{mod}\ 2). \end{cases}$$

Thus $(a_{m,n})_{m,n \geq 0}$ is a 2-automatic double sequence in the sense of Salon [17, 18].

The size of the support of $q_n(X)$ can be computed using Theorem 10. Define u_n to be the number of nonzero coefficients in $q_n(X)$. Then we have

$$u_n = \begin{cases} 0, & \text{if } n = -1; \\ 1, & \text{if } n = 0, 1; \\ u_j + u_{2^k - j - 2}, & \text{if } n \geq 2 \text{ and } n = 2^k + j \text{ with } 0 \leq j < 2^k. \end{cases}$$

Here is a brief table of this sequence:

n	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
u_n	0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1	5

Table 3: The sequence (u_n).

This sequence is easily proved by induction to satisfy the identities

$$u_{2n} = u_n + u_{n-1}, \quad (n \geq 1)$$
$$u_{2n+1} = u_n, \quad (n \geq 0).$$

The sequence $(u_n)_{n \geq 0}$ is the Stern-Brocot sequence [23, 6], and is 2-regular in the sense of Allouche and Shallit [1]. It is Sloane and Plouffe's sequence M0141 [22].

6 Results through Approximation

In this section we obtain more results on the convergents to $g_\varepsilon(X)$ and $h_\varepsilon(X)$ through some very simple approximation results. Our main tool is the the following theorem:

Theorem 14 *Let $t(X)$ be a formal Laurent series with continued fraction expansion $[a_0, a_1, a_2, \ldots]$, let p_n/q_n be the nth convergent, and let p and q be polynomials. Then*

(a) $\deg(q_n t - p_n) = -\deg q_{n+1} < -\deg q_n$;

(b) *If $\deg(qt - p) < -\deg q$, then p/q is a convergent to t.*

Proof. See, for example, [14]. Note that in our statement of part (b) of the theorem, we have omitted the superfluous condition, specified in [14], that the polynomials p and q be relatively prime. ∎

Up to now, most of our results have been for $h_\varepsilon(X)$. We now show how to go from results for $h_\varepsilon(X)$ to results for $g_\varepsilon(X)$.

Theorem 15 *Let p_n/q_n be the nth convergent to $h_\varepsilon(X) = X \sum_{i \geq 0} \varepsilon_i X^{-2^i}$, and let p'_n/q'_n be the nth convergent to $g_\varepsilon(X) = \sum_{i \geq 0} \varepsilon_i X^{-2^i}$. Then*

$$p'_n = d_n(p_n - p_{n-1})/X$$
$$q'_n = d_n(q_n - q_{n-1})$$

where $d_n = \pm 1$.

Proof. From Theorem 14 (a), we obtain

$$\deg(q_n X g - p_n) = -\deg q_{n+1} = -(n+1); \tag{14}$$
$$\deg(q_{n-1} X g - p_{n-1}) = -\deg q_n = -n. \tag{15}$$

14

From Eqs. (14) and (15), it follows that

$$\deg((q_n - q_{n-1})Xg - (p_n - p_{n-1})) = -n,$$

and so

$$\deg((q_n - q_{n-1})g - (p_n - p_{n-1})/X) = -(n+1). \tag{16}$$

Since $p_i(0) = 1$ for all $i \geq -1$, it follows that $(p_n - p_{n-1})/X$ is a polynomial. Since $\deg q_n' = n$, it follows from Theorem 14 (b) that

$$\frac{(p_n - p_{n-1})/X}{q_n - q_{n-1}}$$

is the nth convergent to g. Since the degrees of numerator and denominator agree with the degrees of p_n' and q_n', it follows that

$$
\begin{aligned}
p_n' &= d_n(p_n - p_{n-1})/X; \\
q_n' &= d_n(q_n - q_{n-1})
\end{aligned}
$$

for some sequence (d_n) of ± 1's. ∎

Remark. It is easy to determine precisely what the sequence (d_n) is. Since the sign of the leading term of p_n' is clearly $+1$, and the sign of the leading term of p_n is $\prod_{1 \leq j \leq n} b_j$, where

$$(b_1, b_2, \ldots) = \mathcal{F}(\varepsilon_1, -\varepsilon_2, -\varepsilon_3, \ldots),$$

it follows that $d_n = \prod_{1 \leq j \leq n} b_j$.

Corollary 16 *The coefficients of $q_n'(X)$ lie in $\{-1, 0, 1\}$ for all $n \geq 0$.*

Proof. We know that $q_n' = \pm(q_n - q_{n-1})$, and it is easy to see $q_{2n}(X)$ is an even polynomial and $q_{2n+1}(X)$ is an odd polynomial. The result follows. ∎

7   Further Approximation Results

Let $\mathcal{E} = \{(\varepsilon_n)_{n \geq 0} \in \{-1, +1\}^{\mathbb{N}} : \varepsilon_0 = 1\}$. Define on $\mathcal{E}$ the map T as follows:

$$T((\varepsilon_n)_{n \geq 0}) = (\varepsilon_1 \varepsilon_n)_{n \geq 1} = (\varepsilon_1 \varepsilon_{n+1})_{n \geq 0}.$$

For $(\varepsilon_n)_{n \geq 0} \in \mathcal{E}$, let $g_\varepsilon = X \sum_{k \geq 0} \varepsilon_k X^{-2^k}$. We know that the continued fraction expansion of g_ε is of the form $[1, \pm X, \pm X, \cdots]$. Denote by $\frac{p_{\varepsilon,n}(X)}{q_{\varepsilon,n}(X)}$ its nth convergent. (Note that this notation is slightly different from the notation we introduced in Section 5. In particular, here the ε indicates the dependence of the convergent on a particular choice of signs in the associated series, while in Section 5, the superscript indicated the dependence of the convergent on the particular choice of partial quotients.)

Lemma 17 *The following properties hold:*

(a) $\deg q_{\varepsilon,n} = n$;

(b) $\deg \left(g_\varepsilon(X) - \frac{p_{\varepsilon,n}}{q_{\varepsilon,n}} \right) = -2n - 1$.

Proof. As the continued fraction expansion of g_ε is of the form $[1, \pm X, \pm X, \cdots]$, we have

$$\begin{aligned}
p_{\varepsilon,n+2}(X) &= \pm X p_{\varepsilon,n+1}(X) + p_{\varepsilon,n}(X); \\
q_{\varepsilon,n+2}(X) &= \pm X q_{\varepsilon,n+1}(X) + q_{\varepsilon,n}(X);
\end{aligned}$$

which proves our claim. ∎

Theorem 18 *The following relations hold between the polynomials $q_{\varepsilon,2n}(X)$, $q_{\varepsilon,2n+1}(X)$ and the polynomial $q_{T\varepsilon,n}(X)$:*

(a) $q_{\varepsilon,2n+1}(X) = \varepsilon_1 X q_{T\varepsilon,n}(X^2)$;

(b) $q_{\varepsilon,2n}(X) = (-1)^n (q_{T\varepsilon,n}(X^2) - q_{T\varepsilon,n-1}(X^2))$;

(c) *the polynomial $q_{\varepsilon,2n+1}(X)$ is odd;*

(d) *the polynomial $q_{\varepsilon,2n}(X)$ is even.*

Proof. We have

$$\deg \left(g_{T\varepsilon}(X^2) - \frac{p_{T\varepsilon,n}(X^2)}{q_{T\varepsilon,n}(X^2)} \right) = 2 \deg \left(g_{T\varepsilon}(X) - \frac{p_{T\varepsilon,n}(X)}{q_{T\varepsilon,n}(X)} \right) = -4n - 2. \tag{17}$$

On the other hand, we have

$$g_{T\varepsilon}(X^2) = X^2 \sum_{k \geq 0} \varepsilon_1 \varepsilon_{k+1} X^{-2^{k+1}} = \varepsilon_1 X (g_\varepsilon(X) - 1). \tag{18}$$

Combining Eqs. (17) and (18), we get

$$\deg \left(g_\varepsilon(X) - 1 - \varepsilon_1 \frac{p_{T\varepsilon,n}(X^2)}{X q_{T\varepsilon,n}(X^2)} \right) = -4n - 3. \tag{19}$$

Now, considering the even- and odd-indexed convergents of g_ε, we get

$$\begin{aligned}
\deg \left(g_\varepsilon(X) - \frac{p_{\varepsilon,2n}(X)}{q_{\varepsilon,2n}(X)} \right) &= -4n - 1; \\
\deg \left(g_\varepsilon(X) - \frac{p_{\varepsilon,2n+1}(X)}{q_{\varepsilon,2n+1}(X)} \right) &= -4n - 3.
\end{aligned} \tag{20}$$

Now, combining Eqs. (19) and (20), we get

$$\begin{aligned}
\deg \left(\frac{p_{\varepsilon,2n}(X)}{q_{\varepsilon,2n}(X)} - 1 - \varepsilon_1 \frac{p_{T\varepsilon,n}(X^2)}{X q_{T\varepsilon,n}(X^2)} \right) &= -4n - 1; \\
\deg \left(\frac{p_{\varepsilon,2n+1}(X)}{q_{\varepsilon,2n+1}(X)} - 1 - \varepsilon_1 \frac{p_{T\varepsilon,n}(X^2)}{X q_{T\varepsilon,n}(X^2)} \right) &\leq -4n - 3.
\end{aligned}$$

Hence

$$\deg(Xq_{T\varepsilon,n}(X^2)p_{\varepsilon,2n}(X) - Xq_{\varepsilon,2n}(X)q_{T\varepsilon,n}(X^2) - \varepsilon_1 p_{T\varepsilon,n}(X^2)q_{\varepsilon,2n}(X)) = 0;$$
$$\deg(Xq_{T\varepsilon,n}(X^2)p_{\varepsilon,2n+1}(X) - Xq_{\varepsilon,2n+1}(X)q_{T\varepsilon,n}(X^2) - \varepsilon_1 p_{T\varepsilon,n}(X^2)q_{\varepsilon,2n+1}(X)) \leq -1.$$

We thus have two polynomials, one of which has degree 0, hence is constant and equal to its value at 0, the second of which has degree at most -1, hence $-\infty$ and hence is the zero polynomial. Using the values at 0 of the q's and p's that we gave in Theorem 5, we get

$$Xq_{T\varepsilon,n}(X^2)p_{\varepsilon,2n}(X) - Xq_{\varepsilon,2n}(X)q_{T\varepsilon,n}(X^2) - \varepsilon_1 p_{T\varepsilon,n}(X^2)q_{\varepsilon,2n}(X) = -\varepsilon_1;$$
$$Xq_{T\varepsilon,n}(X^2)p_{\varepsilon,2n+1}(X) - Xq_{\varepsilon,2n+1}(X)q_{T\varepsilon,n}(X^2) - \varepsilon_1 p_{T\varepsilon,n}(X^2)q_{\varepsilon,2n+1}(X) = 0.$$

Solving this linear system in $q_{T\varepsilon,n}(X^2)$ and $p_{T\varepsilon,n}(X^2)$ and noticing that by Eq. (3), its determinant

$$\varepsilon_1 Xp_{\varepsilon,2n+1}(X)q_{\varepsilon,2n}(X) - p_{\varepsilon,2n}(X)q_{\varepsilon,2n+1}(X)$$

is equal to $\varepsilon_1 X$, we obtain

$$q_{\varepsilon,2n+1}(X) = \varepsilon_1 Xq_{T\varepsilon,n}(X^2);$$
$$p_{\varepsilon,2n+1}(X) = p_{T\varepsilon,n}(X^2) + \varepsilon_1 Xq_{T\varepsilon,n}(X^2).$$

This proves parts (a) and (c) of our theorem.

Now, using the recurrence relation satisfied by the q_n's and the fact that the partial quotients of g_ε are $\pm X$, we get

$$q_{\varepsilon,2n+1}(X) = \pm Xq_{\varepsilon,2n}(X) + q_{\varepsilon,2n-1}(X).$$

Replacing $q_{\varepsilon,2n+1}(X)$ (and $q_{\varepsilon,2n-1}(X)$) by the expressions in $q_{T\varepsilon,n}(X^2)$ (and $q_{T\varepsilon,n-1}(X^2)$) we have just obtained gives

$$\varepsilon_1 Xq_{T\varepsilon,n}(X^2) = \pm Xq_{\varepsilon,2n}(X) + \varepsilon_1 Xq_{T\varepsilon,n-1}(X^2);$$

that is,

$$q_{\varepsilon,2n}(X) = \pm\varepsilon_1(q_{T\varepsilon,n}(X^2) - q_{T\varepsilon,n-1}(X^2)).$$

Looking at the value at 0, we get

$$q_{\varepsilon,2n}(X) = (-1)^n(q_{T\varepsilon,n}(X^2) - q_{T\varepsilon,n-1}(X^2)).$$

This proves parts (b) and (d) of our theorem. ∎

Remark. Theorem 18 can be used to give a different proof of Corollary 11. Indeed, we deduce from Theorem 18 that if the coefficients of $q_{\varepsilon,j}$ are all $0, \pm 1$ for all $j \leq n$, then the coefficients of $q_{\varepsilon,2n}$ and $q_{\varepsilon,2n+1}$ have the same property: the different parities of $q_{\varepsilon,n}$ and $q_{\varepsilon,n-1}$ ensure that there is no "overlap" in the equation of Theorem 18(b).

We conclude the paper by proving a results about the automaticity of the double sequence $([X^n]q_{m,\varepsilon}(X))_{m,n\geq 0}$.

Theorem 19 *Let $(\varepsilon_n)_{n\geq 0}$ be a sequence in $\mathcal{E}$. As before, let $\frac{p_{\varepsilon,n}(X)}{q_{\varepsilon,n}(X)}$ be the nth convergent of the continued fraction expansion of $g_\varepsilon = X\sum_{k\geq 0}\varepsilon_k X^{-2^k}$. Let $(a^\varepsilon_{m,n})_{m,n\geq 0}$ be the double sequence defined by $a_{m,n} = [X^n]q_{m,\varepsilon}(X)$. Then the double sequence $(a^\varepsilon_{m,n})_{m,n\geq 0}$ is 2-automatic in the sense of Salon [17, 18] if and only if the sequence $\varepsilon = (\varepsilon_j)_{j\geq 0}$ is ultimately periodic.*

Proof. Suppose first that the sequence $(a^\varepsilon_{m,n})_{m,n\geq 0}$ is 2-automatic. Then [17, 18] the sequence $(a^\varepsilon_{n,n})_{n\geq 0}$ is a one-dimensional 2-automatic sequence. Hence, by a classical result, the sequence

$$(a^\varepsilon_{2^j-1,2^j-1})_{j\geq 0}$$

is ultimately periodic.

But from Theorem 18 we have

$$a^\varepsilon_{2m+1,2n+1} = \varepsilon_1 a^{T\varepsilon}_{m,n}.$$

By induction, it follows that

$$a^\varepsilon_{2^j-1,2^j-1} = \varepsilon_1(T\varepsilon)_1\cdots(T^{j-1}\varepsilon)_1 a^{T^j\varepsilon}_{0,0}.$$

Now observe that $a^{T^j\varepsilon}_{0,0} = 1$ and $(T^j\varepsilon)_n = \varepsilon_j\varepsilon_{n+j}$. Hence

$$\varepsilon_1(T\varepsilon)_1\cdots(T^{j-1}\varepsilon)_1 = \varepsilon_j = a^\varepsilon_{2^j-1,2^j-1}.$$

Since $(a^\varepsilon_{2^j-1,2^j-1})_{j\geq 0}$ is ultimately periodic, the sequence $(\varepsilon_j)_{j\geq 0}$ is also ultimately periodic.

Now we prove the converse. Suppose that the sequence $(\varepsilon_j)_{j\geq 0}$ is ultimately periodic. Theorem 18 gives the following relations:

$$\begin{aligned}
a^\varepsilon_{2m,2n} &= (-1)^m\left(a^{T\varepsilon}_{m,n} - a^{T\varepsilon}_{m-1,n}\right); \\
a^\varepsilon_{2m,2n+1} &= 0; \\
a^\varepsilon_{2m+1,2n} &= 0; \\
a^\varepsilon_{2m+1,2n+1} &= \varepsilon_1 a^{T\varepsilon}_{m,n}.
\end{aligned}$$

Denote by $V^\varepsilon_{m,n}$ for $m\geq 1$ the vector

$$V^\varepsilon_{m,n} = \begin{pmatrix} a^\varepsilon_{m,n} \\ a^\varepsilon_{m-1,n} \\ (-1)^m a^\varepsilon_{m,n} \\ (-1)^m a^\varepsilon_{m-1,n} \end{pmatrix}.$$

We then obtain

$$\begin{aligned}
V^\varepsilon_{2m,2n} &= A_{0,0}V^{T\varepsilon}_{m,n}; \\
V^\varepsilon_{2m,2n+1} &= \varepsilon_1 A_{0,1}V^{T\varepsilon}_{m,n}; \\
V^\varepsilon_{2m+1,2n} &= A_{1,0}V^{T\varepsilon}_{m,n}; \\
V^\varepsilon_{2m+1,2n+1} &= \varepsilon_1 A_{1,1}V^{T\varepsilon}_{m,n};
\end{aligned}$$

where the matrices $A_{i,j}$ are given by

$$A_{0,0} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad A_{0,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};$$

$$A_{1,0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}; \qquad A_{1,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, examining the set of subsequences of the double sequence $(V_{m,n}^{\varepsilon})_{m,n}$ defined by

$$\mathcal{N} = \{(V_{2^j m + r, 2^j n + s}^{T^u \varepsilon})_{m,n \geq 0} : j \geq 0, \ 0 \leq r,s \leq 2^j - 1, \ u \geq 0\}$$

we see from the above relations that this set is finite if $\varepsilon = T^j \varepsilon$ for some j, i.e., if the sequence ε is ultimately periodic. Hence the 2-kernel of the sequence $(V_{m,n}^{\varepsilon})_{m,n}$ is also finite, which means that this sequence is 2-automatic. Finally, its first component $(a_{m,n}^{\varepsilon})_{m,n}$ is also 2-automatic. ■

Remark. The same proof works for the coefficients of the polynomial $q'_{\varepsilon,n}$, but the corresponding result for the coefficients of $q'_{\varepsilon,n}$ can also be deduced from Theorem 15 and from the property that $(d_n)_n$ is 2-automatic if and only if $(\varepsilon_n)_n$ is ultimately periodic.

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