On *abab*-free and *abba*-free set partitions

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Set partitions

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Abstract

These are partitions of $[l] = \{1, 2, \ldots, l\}$ into n blocks such that no four term subsequence of $[l]$ induces the mentioned pattern and each k consecutive numbers of $[l]$ fall into different blocks. These structures are motivated by Davenport-Schinzel sequences. We summarize and extend known enumerative results for the pattern $p = abab$ and give an explicit formula for the number $p(abab, n, l, k)$ of such partitions. Our main tool are generating functions. We determine the corresponding generating function for $p = abba$ and $k = 1, 2, 3$. For $k = 2$ there is a connection with the number of directed animals. We solve exactly two related extremal problems.

1 Introduction and notation

A *partition* P of $[l] = \{1, 2, \dots, l\}$ is a collection $(B_1, B_2, \dots, B_n)$ of nonempty disjoint subsets of $[l]$, called *blocks*, whose union is $[l]$ and which are listed in the increasing order of their least elements. We define $|P| = l$ and $\|P\| = n$. Empty partition is denoted by $\emptyset$. Any partition P can be written in the *canonical sequential form* $P = a_1a_2 \dots a_l$ where $i \in B_{a_i}$. One can use any set of n symbols to express P this way. We call it *sequential form* and we call the set of symbols *alphabet* of P. For instance, 123242151 is the canonical sequential form of $P_0 = (\{1, 7, 9\}, \{2, 4, 6\}, \{3\}, \{5\}, \{8\})$. One of possible sequential forms is *ctrtdtcwc*, the alphabet is $\{c, t, r, d, w\}$. We are interested in enumeration of pattern-free partitions and therefore we will use often the sequential form.

A partition P is *k-regular*, $k \geq 1$, if $x, y \in B_i, x > y$, implies $x - y \geq k$. In other words, each k or less consecutive elements in the sequence are mutually different. The partition P_0 is not 3-regular but is 2-regular. 1-regularity poses no restriction. We say that P is *abab-free* if $x, y \in B_i$ and $z, t \in B_j$ for no four numbers $x < z < y < t$ and two different blocks B_i, B_j. Similarly, P is *abba-free* if $x, y \in B_i$ and $z, t \in B_j$ for no four numbers $x < z < t < y$ and two different blocks. In other words, no four term subsequence of the type *abab*, resp. *abba*, is present. It is easy to check that P_0 above is *abab*-free but not *abba*-free.

Suppose $p = abab$ or $p = abba$. By $p(p, n, l, k)$ we denote the cardinality of the set $\mathcal{P}(p, n, l, k)$ of k-regular and p-free partitions of $[l]$ with n blocks. $P(p, k)$ stands for the bivariate generating function

$$P(p, k) = P(p, k)(x, y) = \sum_{n, l \geq 0} p(p, n, l, k) x^n y^l.$$

By $p(p, n, \cdot, k)$, resp. $\mathcal{P}(p, n, \cdot, k)$, we mean $\sum_{l \geq 0} p(p, n, l, k)$, resp. $\bigcup_{l \geq 0} \mathcal{P}(p, n, l, k)$. Similarly for n replaced by the dot. Obviously $p(p, n, \cdot, 1) = \infty$ but it is not difficult to see that $p(p, n, \cdot, k) < \infty$ for $k \geq 2$. We define, for $k \geq 2$ and $n \geq 0$, $Ex(p, n, k)$ to be the maximum l such that $\mathcal{P}(p, n, l, k)$ is nonempty.

The sets $\mathcal{P}(abab, n, l, 1)$ appeared first in Kreweras [11] under the name of *noncrossing partitions*. The sets $\mathcal{P}(abab, n, \cdot, 2)$ were introduced by Davenport and Schinzel [3] when they studied $Ex(abab, n, 2)$ as a special case of a more general extremal function. The function $Ex(abab, n, 2)$ is often denoted as $\lambda_2(n)$ and is a special case of maximum lengths of *Davenport-Schinzel sequences* (we determine $Ex(abab, n, k)$ in Theorem 2.2). What is $\lambda_3(n)$ then? $Ex(ababa, n, 2)$, this function is far more difficult to handle. See [7], [15], [1], and [8] for more information and referrences.

The next section contains strenghtenings and generalizations of several known enumerative results concerning $\mathcal{P}(abab, \cdot, \cdot, \cdot)$. We determine the generating function $P(abab, k)$ and use it to generalize in Theorem 2.5 an identity of Simion and Ullman and to derive a general explicit formula for $p(abab, n, l, k)$. Nice formulas for these numbers are summarized in Theorem 2.7. Various specializations lead to Catalan, Motzkin, Narayana, and Schröder numbers. In the third section we determine in Theorem 3.1 the function $Ex(abba, n, k)$ and in Theorem 3.5 we derive an identity for $P(abba, k)$. Then we proceed to determine $P(abba, k)$ for $k = 1, 2, 3$. A specialization leads to numbers of directed animals with one root. In the last section we pose several problems.

2 *abab*-free partitions

The set of k-regular partitions of length $< k - 1$ is simply $\mathcal{C}(k) = \{\emptyset, x_1, x_1x_2, \dots, x_1x_2 \dots x_{k-2}\}$. The symbol X^j means the cartesian product $X \times X \times \dots \times X$ j times. Here $A \times \emptyset = A$. Consider the mapping

$$F : \bigcup_{j \geq 1} (\mathcal{P}(abab, \cdot, \cdot, k) \backslash \mathcal{C}(k))^{j-1} \times \mathcal{P}(abab, \cdot, \cdot, k) \to \mathcal{P}(abab, \cdot, \cdot, k) \backslash \{\emptyset\}$$

defined by $F(u_1, u_2, \ldots, u_j) = xu_1xu_2x\ldots xu_j$ where the partitions u_i are interpreted as sequences with disjoint alphabets and x is a completely new symbol. The following easy lemma is crucial for handling *abab*-free partitions.

Lemma 2.1 *F is a bijection and if $F(u_1, u_2, \ldots, u_j) = u$ then $\sum \|u_i\| = \|u\| - 1$ and $\sum |u_i| = |u| - j$.*

Proof. It is easy to see that F is defined correctly and preserves lengths and numbers of blocks in the stated manner. Take a $u \in \mathcal{P}(abab, \cdot, \cdot, k)$, $u \neq \emptyset$, and consider the unique decomposition $u = xu_1xu_2x\ldots xu_j$ given by the occurrences of the first symbol. Note that the alphabets of u_is are disjoint. Obviously $F(u_1, u_2, \ldots, u_j) = u$ and we see that F is bijective. $\Box$

The following theorem generalizes the result $Ex(abab, n, 2) = 2n - 1$ of Davenport and Schinzel [3].

Theorem 2.2 *Suppose $k \geq 2$. For $0 \leq n \leq k-1$ we have $Ex(abab, n, k) = n$. For $n \geq k-1$ we have $Ex(abab, n, k) = 2n - k + 1$ and, for $k \geq 3$, only one partition realizing this length:*

$$u(n, k) = a_1a_2\ldots a_{n-k+1}b_1b_2\ldots b_{k-1}a_{n-k+1}a_{n-k}\ldots a_2a_1.$$

Proof. The first equality is trivial. We prove the rest by induction on n. For $n = k-1$ it is true. We show first $Ex(abab, n, k) \leq 2n - k + 1$. Suppose $n > k - 1$ and take a $u \in \mathcal{P}(abab, n, \cdot, k)$. If no symbol in u repeats we are done. Otherwise consider the shortest interval I in u starting and ending with the same symbol. Clearly $|I| \geq k + 1$ and, except for the end elements, no symbol in I repeats. The inner symbols of I cannot apppear elsewhere. Deleting the first two elements of I we get a partition v in $\mathcal{P}(abab, n-1, \cdot, k)$. So $|u| = |v| + 2 \leq 2n - 2 - k + 1 + 2 = 2n - k + 1$ and we conclude that $Ex(abab, n, k) = 2n - k + 1$.

Now suppose, in addition, that u attains the maximum length. Consider the decomposition $u = xu_1xu_2x\ldots xu_j$ of Lemma 2.1. $j = 1$ is impossible for then x could be added to the end of u. So $j \geq 2$. If u_j is nonempty and has no repetition then it can be added before u in the opposite order. If u_j is nonempty and has a repetition then x again can be added to the end of u. So u_j is empty. If $j > 2$ we get a contradiction $|u| = \sum_{i=1}^{j-1} |xu_i| + 1 \leq \sum_{i=1}^{j-1}(2\|xu_i\| - k) + 1 = 2n + 2(j-2) - (j-1)k + 1 = 2n - (j-1)(k-2) - 1 < 2n - k + 1$. So $j = 2$ and $u_2 = \emptyset$. Applying the induction assumption on u_1 we conclude that $u = u(n, k)$. $\Box$

For $k = 2$ the longest partition is not unique, actually $p(abab, n, 2n-1, 2) = \binom{2n-2}{n-1}/n$. This was proved first by Mullin and Stanton [12]. The following is both generalization and simplification of the argument of Gardy and Gouyou-Beauchamps [5] ($k = 2$).

Theorem 2.3 *For any $k \geq 1$,*

$$P(abab, k)(x, y) = \frac{1}{2y}\left(1 + y + yC(k) - xy - \sqrt{(1 + y + yC(k) - xy)^2 - 4y(1 + yC(k))}\right)$$

where $C(k) = C(k)(x, y) = 1 + xy + (xy)^2 + \ldots + (xy)^{k-2}$ ($C(2) = 1, C(1) = 0$) is the generating function for $\mathcal{C}(k)$.

Proof. Lemma 2.1 translates directly to generating functions:

$$P(abab, k) = 1 + x\sum_{j\geq 1} y^j(P(abab, k) - C(k))^{j-1}P(abab, k) = 1 + \frac{xyP(abab, k)}{1 + yC(k) - yP(abab, k)}.$$

Thus we have the quadratic equation $yP(abab, k)^2 - (1 + y + yC(k) - xy)P(abab, k) + 1 + yC(k) = 0$. Taking $P(abab, k)(0, 0) = 1$ into account we get the above solution. $\Box$

Some specializations of $P(abab,k)(x,y)$ generate standard sequences of numbers. Several special cases of $P(abab,k)(x,y)$ were also investigated before. Setting $k=1$ and $x=1$ we get the generating function $\frac{1}{2y}\left(1-\sqrt{1-4y}\right)$ of the sequence $\{p(abab,\cdot,l,1)\}_{l\geq 1}=\{1,2,5,14,42,132,429,\ldots\}$ of notorious *Catalan numbers*, A0108 in [E]. For $k=2$ and $y=1$ we get the generating function $\frac{1}{2}\left(3-x-\sqrt{1-6x+x^2}\right)$ of $\{p(abab,n,\cdot,2)\}_{n\geq 1}=\{1,2,6,22,90,394,1806,\ldots\}$. These are twice the *Schröder numbers*, A1003 in [E], which appeared first in [14]. The generating function $(P(abab,2)(x,1)-1+x)/2$ was derived in [12]. The specialization $k=2$ and $x=1$ yields $\frac{1}{2y}\left(1+y-\sqrt{1-2y-3y^2}\right)$ generating $\{p(abab,\cdot,l,2)\}_{l\geq 1}=\{1,1,2,4,9,21,51,\ldots\}$ which are *Motzkin numbers*, A1006 in [E], see [4]. For $k\geq 4$ the sequences $\{p(abab,n,\cdot,k)\}_{n\geq 1}$ seem new, for instance $\{p(abab,n,\cdot,4)\}_{n\geq 3}=\{1,2,5,13,35,97,275,\ldots\}$. For $k=3$ we get Catalan number once again. $\{p(abab,\cdot,l,k)\}_{l\geq 1}$ for $k\geq 3$ are not new, $\{p(abab,\cdot,l,3)\}_{l\geq 3}=\{1,2,4,8,17,37,82,\ldots\}$ is the sequence A4148 in [E]. These sequences were investigated in [17] by Stein and Waterman who, motivated by the secondary structure of the molecules of nucleic acids, introduced there the sets $\mathcal{P}(abab,\cdot,l,k)$. They mentioned without proof the result of C. J. Everett which we restate as the second half of the following theorem. We omit the proof as well.

Theorem 2.4

$$\lim_{k\to\infty}\lim_{n\to\infty}p(abab,n,\cdot,k)^{1/n}=\frac{3+\sqrt{5}}{2}\quad\textit{and}\quad\lim_{k\to\infty}\lim_{l\to\infty}p(abab,\cdot,l,k)^{1/l}=2.$$

The following theorem refines the identity of [16] where the version with two parameters k and l can be found (the proof there is combinatorial).

Theorem 2.5 *For any $n,l\geq 1$, $k\geq 2$, it is true that $p(abab,n,l,k)=p_{\leq 2}(abab,n-1,l-1,k-1)$. The subscript ≤ 2 means that we consider only the partitions with all blocks of size at most 2. Briefly, $xyP_{\leq 2}(abab,k-1)=P(abab,k)-1$.*

Proof. The generating function $P_{\leq 2}(abab,k)(x,y)$ is defined in the obvious manner. The relation for it differs from the one for $P(abab,k)$ only in that j may now attain only the values 1 and 2. So $P_{\leq 2}(abab,k)=1+x(yP_{\leq 2}(abab,k)+y^2(P_{\leq 2}(abab,k)-C(k))P_{\leq 2}(abab,k))$ and we get the equation $y(xyP_{\leq 2}(abab,k))^2-(1+xy^2C(k)-xy)(xyP_{\leq 2}(abab,k))+xy=0$. Thus

$$xyP_{\leq 2}(abab,k-1)(x,y)=\frac{1}{2y}\left(1+xy^2C(k-1)-xy-\sqrt{(1+xy^2C(k-1)-xy)^2-4xy^2}\right).$$

Taking $xy^2C(k-1)=yC(k)-y$ into account and comparing with the expression in Theorem 2.3 we get $xyP_{\leq 2}(abab,k-1)=P(abab,k)-1$. The identity is verified. □

Example

$$\mathcal{P}(abab,\cdot,5,2)=\{12345,12343,12342,12341,12324,12321,12314,12134,12131\}$$

and

$$\mathcal{P}_{\leq 2}(abab,\cdot,4,1)=\{1122,1123,1223,1233,1234,1221,1231,1232,1213\}.$$

To give an explicit formula for $p(abab,n,l,k)$ we need first to recall a well known bijection. It matches the elements of the sets $\mathcal{P}_{=2}(abab,n,2n,1)$ and $\mathcal{T}(n+1)$. Here $=2$ indicates partitions with all blocks of size 2 and $\mathcal{T}(n)$ is the set of all rooted plane trees with n vertices. Recursively: one vertex tree corresponds to $\emptyset$ and a general T corresponds to $x_1u_1x_1x_2u_2x_2\ldots x_ju_jx_j$ where u_i corresponds to the ith (counted from left) principal subtree of T and j is the degree of the root of T. The sequences u_i have disjoint alphabets and the symbols x_i are new and mutually different.

Recall that $|\mathcal{T}(n+1)| = c_n = \binom{2n}{n}/(n+1)$ is the nth Catalan number. Recall the formula

$$n(a,b) = n(a,a-b) = \frac{1}{a-b}\binom{a-1}{b}\binom{a-2}{b-1} = \frac{1}{a-1}\binom{a-1}{b}\binom{a-1}{b-1}$$

of Narayana [13] for the number of rooted plane trees with a vertices and b leaves.

Theorem 2.6 *For $k \geq 2$ and $n \leq l \leq \max(2n-k+1, n)$ we have*

$$p(abab, n, l, k) = \sum_{b=1}^{*} \frac{1}{l-n+1-b}\binom{l-n}{b}\binom{l-n-1}{b-1}\binom{l-1-b(k-2)}{2l-2n}$$

where $* = \min(l-n, \lfloor \frac{2n-l-1}{k-2} \rfloor)$ and the empty sum is equal to 1.

Proof. By Theorem 2.5 it is enough to count the number $p_{\leq 2}(abab, n-1, l-1, k-1)$ of partitions $u \in \mathcal{P}_{\leq 2}(abab, n-1, l-1, k-1)$. Each such u has $s = 2n-l-1$ singletons, symbols with one occurrence, and $d = l-n$ doubletons with two occurrences. The doubleton part of u corresponds, by the bijection, to a tree $T \in \mathcal{T}(d+1)$. By the $k-1$-regularity inside of each doubleton of u corresponding to a leaf of T there are at least $k-2$ singletons, in particular $b \leq (2n-l-1)/(k-2)$ for the number b of leaves of T. Besides this requirement singletons may be located arbitrarily in the $2d+1$ gaps of the doubleton part. The number of such u is therefore

$$\sum_{b=1}^{*} n(d+1, b)\binom{2d+1+s-b(k-2)-1}{s-b(k-2)}.$$

This is the general formula. □

In several instances one can give closed formulas.

Theorem 2.7 *For $n, l \geq 1$,*

$$p(abab, n, l, 1) = n(l+1, n) = \frac{1}{l-n+1}\binom{l}{n}\binom{l-1}{n-1},$$

$$p(abab, n, l, 2) = c_{l-n}\binom{l-1}{2l-2n} = \frac{1}{l-n+1}\binom{2l-2n}{l-n}\binom{l-1}{2l-2n},$$

$$p(abab, n, l, 3) = n(n, l-n+1) = \frac{1}{l-n+1}\binom{n-1}{2n-l-1}\binom{n-2}{2n-l-2}.$$

Thus $p(abab, n, l, 1) = p(abab, l-n+1, l, 1)$, $p(abab, n, l, 3) = p(abab, n, 3n-2-l, 3)$, $p(abab, n, l, 3) = p(abab, l-n+1, n-1, 1)$, and

$$p(abab, \cdot, n-1, 1) = p(abab, n, 2n-1, 2) = p(abab, n, \cdot, 3) = c_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}.$$

Proof. The generating function for Narayana numbers $n(a,b)$ is

$$N(x,y) = \sum_{a,b \geq 1} n(a,b)x^a y^b = \frac{1-x+xy-\sqrt{(1-x+xy)^2-4xy}}{2}$$

where we put $n(1,1) = 1$. This formula can be easily derived by considerations similar to those in the proof of Theorem 2.3 and is well known. Consider the first three formulas. The formulas for $k=1$ and $k=3$ are consequences of the identities $P(abab, 1)(x,y) = N(y,x)/y - x + 1$ and $P(abab, 3)(x,y) =$

$N(xy,y)/y+1$ which can be readily checked. The formula for $k=2$ is a special case of Theorem 2.6 since for $k=2$

$$\sum_{b=1}^{*} n(d+1,b)\binom{2d+1+s-1}{s} = \binom{l-1}{s}\sum_{b=1}^{d} n(d+1,b) = \binom{l-1}{2l-2n}.c_d.$$

The remaining formulas follow from the symetry $n(a,b)=n(a,a-b)$ and from $\sum_b n(a,b)=c_{a-1}$. □

The formula for $p(abab,n,l,1)$ is contained implicitly already in the Narayana's result since one can prove it by an easy bijection matching partitions with trees. The formula for $p(abab,n,l,2)$ was derived in [5] directly extracting the coefficient from $P(abab,2)$. Although our counting relies on generating functions too, it indicates a bijective proof which is worked out in [9]. We have not seen the formula for $p(abab,n,l,3)$ before.

The closed formulas for $k=2,3$ are indicated by the presence of only small prime factors in the numbers $p(abab,n,l,k)$ when calculated by the general formula of Theorem 2.6. For $k\geq 4$ we get typicly factorizations as $p(abab,20,26,4)=2.13.330641$ or $p(abab,20,30,5)=5.31.2003$ which seems to exclude simple closed forms.

A sequence of numbers is called *unimodal* if it can be split into two parts, the initial one nondecreasing and the final one nonincreasing. The sequences $\{p(abab,n,l,1)\}_{n=1}^{l}$ and $\{p(abab,n,l,3)\}_{l=n}^{2n-2}$ are unimodal and symmetric. Examining the ratio $p(abab,n,l,2)/p(abab,n,l+1,2)$ one can prove easily that $\{p(abab,n,l,2)\}_{l=n}^{2n-1}$ is also unimodal and attains its maximum for $l=\lfloor n(1+\sqrt{1-1/n}/\sqrt{2})\rfloor$. Similarly, $\{p(abab,n,l,k)\}_{n=*}^{l}$, $*=\lceil (l+k-1)/2\rceil$, $k=2,3$, are unimodal for any $l\geq 2$.

Conjecture 2.8 *We conjecture that the sequences* $\{p(abab,n,l,k)\}_{l=n}^{2n-k+1}$ *and* $\{p(abab,n,l,k)\}_{n=*}^{l}$ *are unimodal for any* $n,l\geq k-1$ *and* $k\geq 2$.

3 *abba*-free partitions

Theorem 3.1 *Let* $k\geq 2$. *For* $1\leq n\leq k-1$ *again* $Ex(abba,n,k)=n$. *For* $n\geq k$ *we have* $Ex(abba,n,k)=2n+\lfloor\frac{n-1}{k-1}\rfloor-1$. *The longest partition is unique iff* $n-1$ *is divisible by* $k-1$. *In particular* $Ex(abba,n,2)=3n-2$ *and the longest partition*

$$1212323434545\ldots(n-1)n(n-1)n$$

is unique for any $n\geq 1$.

Proof. We prove first by induction on n the general upper bound. It is true for $n=k$ giving the value $2k$. Let $v\in\mathcal{P}(abba,n,\cdot,k)$ and $n>k$.

Claim 1 *One can suppose that no symbol appears in* v *more than three times.*

In the contrary case take four occurrences of a symbol a and consider the second and the third of them. A symbol $b\neq a$ must appear between them and b may have only one occurrence in v, for otherwise v is not *abba*-free. We delete the b-appearance plus possibly one a-appearance, the k-regularity is not violated. By induction $|v|\leq 2(n-1)+\lfloor\frac{n-2}{k-1}\rfloor-1+2\leq 2n+\lfloor\frac{n-1}{k-1}\rfloor-1$ and we are done in this case.

Let S_2 be the set of the symbols which appear in v at most twice and let S_3 consist of those appearing exactly three times. Let $|S_2|=n_2$ and $|S_3|=n_3$. Thus $n=n_2+n_3$.

Claim 2 $n_3(2k-4)+2\leq 2n_2-2(k-1)$

By a 3-*interval* we mean an interval I in v which begins and ends with an a-occurrence and which has one a-occurrence inside. There are n_3 3-intervals, one for each $a\in S_3$, no two of them are comparable by inclusion and no three of them intersect.

For any 3-interval I corresponding to an $a\in S_3$ there are at least $2k-2$ distinct symbols appearing in I which are distinct to a. Only at most 2 of those symbols can belong to S_3 and hence any I contributes by at least $2k-4$ elements to S_2. On the other hand any $x \in S_2$ can appear only in at most two 3-intervals. This gives roughly the inequality in Claim 2, the corrections $+2$ and $-2(k-1)$ are caused by the first and by the last 3-interval — each contributes by at least $2k-3$ elements to S_2 and for each there are at least $k-1$ elements of S_2 which appear only in it.

Therefore $n_2 \geq n_3(k-2)+k = (n-n_2)(k-2)+k$ and $n_2 \geq n - \frac{n-1}{k-1}+1$. Finally,

$$|v| \leq 3n_3 + 2n_2 = 3n - n_2 \leq 2n + \frac{n-1}{k-1} - 1.$$

To prove that this cannot be improved we express $n \geq k$ in the form $n-1 = m(k-1)+i, 0 \leq i < k-1$, and we consider the sequence (partition) $v(n,k) = B_1B_2 \dots B_{m-1}B_m$ where the jth segment B_j, $1 \leq j \leq m-1$, is of the form

$$B_j = j\ x_1^j x_2^j \dots x_{k-2}^j\ (j+1)j\ x_1^j x_2^j \dots x_{k-2}^j$$

and the mth segment is of the form

$$B_m = m\ x_1^m \dots x_{k-2}^m\ (m+1)m\ y_1y_2 \dots y_i x_1^m \dots x_{k-2}^m\ (m+1)y_1y_2 \dots y_i.$$

The n element alphabet here is

$$\{1,2,\dots,m+1,y_1,y_2,\dots,y_i\} \cup \{x_q^p \mid p = 1 \dots m, q = 1 \dots k-2\}.$$

An easy check reveals that the k-regular $v(n,k)$ is *abba*-free and that the length of v is

$$m(2k-1)+2i+1 = 2(n-1)+m+1 = 2n + \lfloor \frac{n-1}{k-1} \rfloor - 1.$$

Thus $Ex(abba,n,k) = 2n + \lfloor \frac{n-1}{k-1} \rfloor - 1$. If $i > 0$ then the symbols $y_1, \dots, y_i$ can be placed in $v(n,k)$ differently than it is indicated above and we get several longest partitions.

It remains to prove that for $n-1$ divisible by $k-1$ there is no other longest partition than $v(n,k)$. For $n = k$ this is true. Let $n-1 > k-1$ be divisible by $k-1$ and let $u \in \mathcal{P}(abba,n,\cdot,k)$ be of the maximum length and in the canonical form. Since the length is maximum we have only symbols appearing two times or three times and no singletons. The sequence u starts with $u = 12\dots k \dots$ and each of the symbols $1,2,\dots,k-1$ appears in u only twice, in the contrary case we would have singletons. Thus $u = 12\dots k-1\ k \dots 1 \dots 2 \dots$. The second 1 must follow immediately after k, in the contrary case we could delete 1's without violating k-regularity and get a sequence longer than $Ex(abba,n-1,k)$. The case $u = 12\dots k1x\dots 2\dots$ reduces by the switching $u = 12\dots kx1\dots 2\dots$ to the previous case. So $u = 123\dots k123\dots k\dots$. Now k must appear three times for otherwise by deleting the initial segment of length $2k$ we would decrese n by k but l only by $2k$. Deleting the intial segment of length $2k-1$ and applying the induction assumption on the rest we conclude that $u = v(n,k)$. □

To enumerate the sets $\mathcal{P}(abba,n,l,k)$ we start with definitions and with an analogy of Lemma 2.1. Again, the subscript ≤ 2 indicates partitions with no block of size 3 or more. The set $\mathcal{I}(k)$ (resp. $\mathcal{E}(k)$) of *initial segments* (resp. *end segments*) consists of all partitions u where $u \in \mathcal{P}_{\leq 2}(abba,\cdot,\cdot,k)$ and the last (resp. the first) element of u is a doubleton. *Middle segments* $\mathcal{M}(k)$ are partitions $u \in \mathcal{P}_{\leq 2}(abba,\cdot,\cdot,k)$ such that the first and the last elements of u differ and are doubletons. Finally, *simple segments* $\mathcal{S}(k)$ are k-regular partitions u beginning and ending with a in which no symbol, except for a, repeats. Consider the mapping

$$G : \mathcal{I}(k) \times \mathcal{S}(k) \times \bigcup_{j\geq 1}(\mathcal{M}(k) \times \mathcal{S}(k))^{j-1} \times \mathcal{E}(k) \to \mathcal{P}(abba,\cdot,\cdot,k) \backslash \mathcal{P}_{\leq 2}(abba,\cdot,\cdot,k)$$

defined by $G(u_1,u_2,\dots,u_{2j+1}) = u = u_1u_2\dots u_{2j+1}+$ identification. This means that u_is are concatenated as sequences with disjoint alphabets and then the neighboring end elements of these segments are identified.

Lemma 3.2 *G is a bijection and $\sum |u_i| = |u| + 2j$, $\sum \|u_i\| = \|u\| + 2j$.*

Proof. The mapping G is defined correctly and preserves lengths and numbers of symbols in the stated manner. Take a $u \in \mathcal{P}(abba, \cdot, \cdot, k) \backslash \mathcal{P}_{\leq 2}(abba, \cdot, \cdot, k)$. Consider the splitting $u = v_1 a v_2 a \dots a v_m, m \geq 4$, of u by the occurrences of the first symbol a which appears more than twice. Obviously $v_1 a v_2 a \in \mathcal{I}(k)$ and $a v_3 a \dots a v_{m-2} a \in \mathcal{S}(k)$. If in $a v_{m-1} a v_m$ no symbol appears more than twice we are done since then $a v_{m-1} a v_m \in \mathcal{E}(k)$. Otherwise let $a v_{m-1} a v_m = a w_1 b w_2 b \dots b w_r$ be the splitting where b is the first symbol appearing $r \geq 3$ times and w_1 contains one a-appearance and one b-appearance. Then $a w_1 b \in \mathcal{M}(k)$ and $b w_2 b \dots b w_{r-2} b \in \mathcal{S}(k)$. Now we are left with the last segment $b w_{r-1} b w_r$. Continuing this way until the last segment falls into $\mathcal{P}_{\leq 2}(abba, \cdot, \cdot, k)$ we get a unique decomposition of u into segments. These segments have disjoint alphabets, except for the symbols $a, b, \dots$, and decompose u as described in the definition of G. Therefore G is bijective. □

We introduce the generating functions $S(k)(x, y)$, $I(k)(x, y)$, $E(k)(x, y)$, and $M(k)(x, y)$ which count the numbers of simple segments, initial segments, end segments, and middle segments with a given length and number of blocks, respectively. Clearly $I(k) = E(k)$.

Lemma 3.3 *For any $k \geq 1$,*

$$P(abba, k) = \frac{I^2(k)S(k)}{x^2y^2 - M(k)S(k)} + P_{\leq 2}(abba, k).$$

Proof. Translating the decomposition Lemma 3.2 we get

$$P(abba, k) = P_{\leq 2}(abba, k) + I(k)S(k)[\sum_{j \geq 1} (M(k)S(k))^{j-1}(xy)^{-2j}]I(k).$$

The rest is a routine simplification using the geometric series formula. □

Lemma 3.4 *For any $k \geq 1$,*

1. $S(k)(x, y) = \frac{xy(1-xy)}{1-xy-y(xy)^{k-1}}$.
2. $I(k)(x, y) = E(k)(x, y) = (1 - xy)P_{\leq 2}(abba, k)(x, y) - 1$.
3. $M(k)(x, y) = (1 - xy)^2 P_{\leq 2}(abba, k)(x, y) - \frac{y(xy)^k}{1-xy} - 1 + xy$.

Proof. To build up a simple segment means to take a sequence of $m \geq 1$ a's, to put $k - 1$ (mutually different) singletons into each of the $m - 1$ gaps and then to add $r \geq 0$ new singletons into these gaps. Hence

$$S(k) = \sum_{m \geq 1} xy^m (xy)^{(m-1)(k-1)} \sum_{r \geq 0} \binom{m - 1 + r - 1}{r} (xy)^r.$$

The inner sum equals, by a well known identity, $1/(1 - xy)^{m-1}$. Using the geometric series formula we get the expression.

The number of initial segments of length l with n blocks equals to $p_{\leq 2}(abba, n, l, k) - p_{\leq 2}(abba, n - 1, l - 1, k)$, we are subtracting the partitions ending with a singleton. We have to subtract also the empty partition.

Similarly, the number of middle segments of length l with n blocks is $p_{\leq 2}(abba, n, l, k) - 2p_{\leq 2}(abba, n - 1, l - 1, k) + p_{\leq 2}(abba, n - 2, l - 2, k) - 1$ (modulo some adjustment for very small numbers n, l) which corresponds to the subtraction of the partitions beginning or ending with a singleton and the only partition beginning and ending with the same symbol. □

Putting it all together we get the following unexpected result.

Theorem 3.5 *For any $k \geq 1$,*

$$P(abba, k) = \frac{(1-2xy)P_{\leq 2}(abba, k) - 1}{(1-xy)^2 P_{\leq 2}(abba, k) - 1}.$$

Proof. Just substitute the expressions from Lemma 3.4 into the equation of Lemma 3.3. The terms with k will disappear during simplifications. □

It is surprising that the relation between $P_{\leq 2}(abba, k)$ and $P(abba, k)$ is independent on k. Theorem 3.5 is a counterpart of the relation $xyP_{\leq 2}(abab, k-1) = P(abab, k) - 1$ of Theorem 2.5.

We proceed to determine the functions $P_{\leq 2}(abba, k)$ and $P(abba, k)$ for $k = 1, 2, 3$. We know $P_{\leq 2}(abba, 1)$ already:

Lemma 3.6

$$P_{\leq 2}(abba, 1) = P_{\leq 2}(abab, 1) = \frac{P(abab, 2) - 1}{xy} = \frac{1 - xy - \sqrt{(1-xy)^2 - 4xy^2}}{2xy^2}.$$

Proof. The ultimate equality is a consequence of Theorem 2.3 and the penultimate equality is an instance of Theorem 2.5. We show by a simple bijection that $p_{\leq 2}(abba, n, l, 1) = p_{\leq 2}(abab, n, l, 1)$ for any $n, l \geq 0$ which proves the first equality.

We start with a bijection between $\mathcal{P}_{=2}(abba, n, 2n, 1)$ and $\mathcal{T}(n+1)$. Empty sequence is represented by a single vertex. Let $u \in \mathcal{P}_{=2}(abba, n, 2n, 1)$. The root of the tree T representing u will have degree m where $v = 12\ldots m$, $u = vw$, is the maximal initial interval of u without repetitions. Consider the same decomposition $u' = v'w'$, $v' = m+1\ldots m+r$, of the sequence u' that arises from u by deleting the $2m$ appearances of $1, \ldots, m$. Note that w starts with 1 and decomposes into $w = 1w_1 2w_2 \ldots mw_m$.

T is defined as follows. Suppose that the tree U representing u' has the principal subtrees, from left to right, $U_1, U_2, \ldots, U_r$, with roots $r(1), r(2), \ldots, r(r)$. Let $|v' \cap w_i| = l_i$, $l_1 + \ldots + l_m = r$, and let $l_0 = 0$. We delete the root of U and we join the l_j vertices $r(l_0 + \ldots + l_{j-1} + 1), \ldots, r(l_0 + \ldots + l_{j-1} + l_j)$, $j = 1, 2, \ldots, m$, to a new vertex v_j. Finally, we join the vertices v_j to a common vertex, the root of T. It is not difficult to check that this is indeed a bijection.

Now it is easy to give a bijection between $\mathcal{P}_{\leq 2}(abba, n, l, 1)$ and $\mathcal{P}_{\leq 2}(abab, n, l, 1)$. Let u lie in the former set. Consider the doubleton part of u and the tree T corresponding to it by the bijection we have just described. Replace the doubleton part by the sequence $v \in \mathcal{P}_{=2}(abab, \cdot, \cdot, 1)$ corresponding to T by the bijection described before Theorem 2.6. □

Theorem 3.7

$$P(abba, 1) = \frac{(1-xy)^2 - y - x^2y^3 P_{\leq 2}(abab, 1)}{(1-xy)^3 - y} = \frac{2 - 2y - 5xy + 3x^2y^2 + xy\sqrt{(1-xy)^2 - 4xy^2}}{2(1-xy)^3 - 2y}$$

$$P(abba, 1)(1, y) = \frac{1 - 3y + y^2 - y^3 P_{\leq 2}(abab, 1)(1, y)}{1 - 4y + 3y^2 - y^3} = \frac{-2 + 7y - 3y^2 - y\sqrt{1 - 2y - 3y^2}}{-2 + 8y - 6y^2 + 2y^3}$$

Proof. By the proof of Theorem 2.5 and by the previous lemma, the function $P_{\leq 2}(abba, 1)$ satisfies the quadratic equation $xy^2P^2 + (xy-1)P + 1 = 0$. Thus the identity

$$((1-xy)^2 P - 1)\left(P\,\frac{xy^2}{(1-xy)^2} + \frac{xy^2}{(1-xy)^4} + \frac{1}{xy-1}\right) = -1 - \frac{1}{xy-1} - \frac{xy^2}{(1-xy)^4}$$

by which we rationalize the denominator of the expression in Theorem 3.5. Simplifying and substituting the explicit form of $P_{\leq 2}(abba, 1)$ we get the final result. The second formula arises by specialization. □

$P(abba, 1)(1, y)$ generates the sequence $\{p(abba, \cdot, l, 1)\}_{l \geq 1} = \{1, 2, 5, 14, 41, 123, 374, \ldots\}$ which seems new.

Lemma 3.8

$$P_{\leq 2}(abba,2) = \frac{P_{\leq 2}(abba,1)+1}{2+xy^2-xy}$$

Proof. Take a $u \in \mathcal{P}_{\leq 2}(abba,\cdot,\cdot,1) \backslash \mathcal{P}_{\leq 2}(abba,\cdot,\cdot,2)$ and consider the first violation of the 2-regularity $u = vaaw$. Thus $v \in \mathcal{P}_{\leq 2}(abba,\cdot,\cdot,2)$ and v and w have disjoint alphabets. Translated to generating functions, $P_{\leq 2}(abba,1) = P_{\leq 2}(abba,2).xy^2.P_{\leq 2}(abba,1) + P_{\leq 2}(abba,2)$. The solution for $P_{\leq 2}(abba,2)$ is

$$P_{\leq 2}(abba,2) = \frac{P_{\leq 2}(abba,1)}{xy^2 P_{\leq 2}(abba,1)+1}.$$

Rationalizing the denominator as in the proof of Theorem 3.7 we get the desired relation. □

Setting $y = 1$ in the previous lemma we get the following identity.

Consequence 3.9 *For any $n \geq 1$ it is true that $p_{\leq 2}(abba,n,\cdot,2) = p_{\leq 2}(abba,n,\cdot,-2)$. The minus sign indicates partitions which are* not *2-regular.*

Example

$$\mathcal{P}_{\leq 2}(abba,3,\cdot,2) = \{123, 1213, 12123, 12132, 121323, 1231, 1232, 12312, 12313, 123123, 12323\}$$

and

$$\mathcal{P}_{\leq 2}(abba,3,\cdot,-2) = \{1123, 1223, 1233, 11233, 12233, 11223, 112233, 12133, 121233, 11232, 112323\}.$$

Theorem 3.10

$$P(abba,2) = \frac{1-x(2y+3y^2+y^3)+x^2(y^2+y^3)-x^2y^3P_{\leq 2}(abab,1)}{1-x(3y+3y^2+y^3)+x^2(3y^2+2y^3)-x^3y^3}$$

$$P(abba,2)(1,y) = \frac{1-2y-2y^2-y^3P_{\leq 2}(abab,1)(1,y)}{1-3y}$$

$$P(abba,2)(x,1) = \frac{1-6x+2x^2-x^2P_{\leq 2}(abab,1)(x,1)}{1-7x+5x^2-x^3}$$

Proof. The expression for $P_{\leq 2}(abba,2)$ from the previous lemma is substituted in the formula of Theorem 3.5. The denominator of the resulted fraction is rationalized as in Theorem 3.7. Specializations lead to the other two formulas. □

The function $P(abba,2)(x,1)$ generates $\{p(abba,n,\cdot,2)\}_{n\geq 1} = \{1, 3, 15, 85, 501, 3007, 18235, \ldots\}$ which seems new. The sequence $\{p(abba,\cdot,l,2)\}_{l\geq 2} = \{1, 2, 5, 13, 35, 96, 267, \ldots\}$ generated by $P(abba,2)(1,y)$ is the sequence A5773 in [E]. Recall that a *directed animal with one root* is a finite set X of lattice points in the plane containing the origin and such that each point of X can be reached from the origin by a path lying completely in X and making only east or north unit steps. For more details consult [6].

Consequence 3.11 *For any $l \geq 2$ it is true that $p(abba,\cdot,l,2)$ is the same as the number of directed animals with one root and $l-1$ points.*

Proof. Simplifying the formula for $P(abba,2)(1,y)$ further we get a compact expression

$$P(abba,2)(1,y) = 1 + \frac{y}{2}\left(1 + \sqrt{\frac{1+y}{1-3y}}\right)$$

which equals $yQ + 1 + y$ where Q is the generating function for directed animals with one root, see [6]. $\square$

Lemma 3.12

$$P_{\leq 2}(abba,3) = \frac{P_{\leq 2}(abba,2)}{(1-xy^2)^2 + xy^2(2 - xy + x^2y^3 - x^2y^4)P_{\leq 2}(abba,2)}$$

Proof. The idea is the same as in Lemma 3.8. Take a $u \in \mathcal{P}_{\leq 2}(abba,\cdot,\cdot,2)\backslash\mathcal{P}_{\leq 2}(abba,\cdot,\cdot,3)$ and consider the first violation of the 3-regularity by $u = vabaw$. Thus $v \in \mathcal{P}_{\leq 2}(abba,\cdot,\cdot,3)$ and v and w have disjoint alphabets. Now we have to distinguish three possibilities. For the sake of brevity we use P for $P_{\leq 2}(abba,3)$ and Q for $P_{\leq 2}(abba,2)$.

1) b is a singleton. The number of such u is counted by the coefficient in Px^2y^3Q.

2) b appears once more in w. The number of such u is counted by the coefficient in $P(x^2y^4Q + xy^2E(2))$. The first term counts the u's with the structure $u = vababw'$. If the second b does not follow immediately after the second a then bw is an end segment (see the beginning of Section 3) and such u's are counted by the second term.

3) b appears in v. Consider the interval I spanned by the two b appearances. Clearly $|I| \geq 4$. In the case $|I| > 4$ we are done as well as in the case when $|I| = 4$ but the other symbol in I different from a, say c, is a singleton. The bad situation is when $u = v'bcabaw$ and c appears in v'. Then consider the interval J spanned by the two c's. The bad situation is when $u = v''cdbcabaw$ and d appears in v''. Continuing this way we get a unique decomposition $u = v^*a_1sa_2a_1a_3a_2a_4a_3\ldots abaw$ where either $|s| \geq 2$ or s is a singleton. In the former case $v^*a_1sa_1$ is an initial segment in $\mathcal{I}(3)$ and such u's are accounted for in $I(3)[\sum_{m\geq 1}(xy^2)^m]Q$. In the latter case we have the splitting $v^*\ a_1sa_2a_1a_3a_2a_4a_3\ldots aba\ w$ of u into three segments with disjoint alphabets and so we account for such u in $P[\sum_{m\geq 2}(xy^2)^m]xyQ$.

We have the equation $Q = P[1 + x^2y^3Q + x^2y^4Q + xy^2(1-xy)Q - xy^2 + \frac{x^3y^5}{1-xy^2}Q + (1-xy)\frac{xy^2}{1-xy^2}Q] - \frac{xy^2}{1-xy^2}Q$ which solves for P by the stated formula. $\square$

Theorem 3.13

$$P(abba,3) = \frac{1 - x(2y+4y^2) + x^2(y^2+2y^4-y^5) + x^3(y^4-y^5+2y^7) - x^4(y^7-y^8+y^9) - (x^2y^3 - 2x^3y^5 + x^4y^7)F}{1 - x(3y+4y^2) + x^2(3y^2+3y^3+2y^4-y^5) - x^3(y^3+3y^5-2y^7) + x^4(y^6-y^7+y^8-y^9)}$$

where $F = P_{\leq 2}(abab,1) = (1 - xy - \sqrt{(1-xy)^2 - 4xy^2})/2xy^2$. *The specializations are*

$$P(abba,3)(x,1) = \frac{1 - 6x + 2x^2 + 2x^3 - x^4 - (x^2 - 2x^3 + x^4)P_{\leq 2}(abab,1)(x,1)}{1 - 7x + 7x^2 - 2x^3} \quad \textit{and}$$

$$P(abba,3)(1,y) = \frac{1 - 2y - 3y^2 + 3y^4 - 2y^5 + y^7 + y^8 - y^9 - (y^3 - 2y^5 + y^7)P_{\leq 2}(abab,1)(1,y)}{1 - 3y - y^2 + 2y^3 + 2y^4 - 4y^5 + y^6 + y^7 + y^8 - y^9}$$

Proof. This is again only a manipulation with rational functions. First we substitute in the expression of Lemma 3.12 the formula for $P_{\leq 2}(abba,2)$ from Lemma 3.8 and express this way $P_{\leq 2}(abba,3)$ in terms of $P_{\leq 2}(abab,1)$:

$$P_{\leq 2}(abba,3) = \frac{m_1(x,y) + m_2(x,y)P_{\leq 2}(abab,1)}{m_3(x,y) + m_4(x,y)P_{\leq 2}(abab,1)}$$

where $m_1(x,y) = 1 + xy - xy^2 + x^2y^3$, $m_2(x,y) = -1 + 2xy + 2xy^2 - x^2y^3 + x^3y^5 - x^3y^6$, $m_3(x,y) = m_1(x,y) - x^2y^2$, and $m_4(x,y) = m_2(x,y) - x^2y^2$. Rationalizing the denominator we get the stated formula. $\square$

The first specialization generates the sequence $\{p(abba,n,\cdot,3)\}_{n\geq 2} = \{1,4,19,95,448,2553,13537,\ldots\}$ and the second one the sequence $\{p(abba,\cdot,l,3)\}_{l\geq 3} = \{1,2,5,14,38,102,276,\ldots\}$, both of them seem new. Now we list the beginnings of the expansions of the functions $P(abba,k)(x,y)$ for $k=1,2,3$.

$$P(abba,1)(x,y) = 1 + xy + (x+x^2)y^2 + (x+3x^2+x^3)y^3 + (x+6x^2+6x^3+x^4)y^4 +$$
$$+(x+9x^2+20x^3+10x^4+x^5)y^5 + (x+12x^2+44x^3+50x^4+15x^5+x^6)y^6 +$$
$$+(x+15x^2+77x^3+154x^4+105x^5+21x^6+x^7)y^7 + (x+18x^2+119x^3+350x^4+434x^5+196x^6+28x^7+x^8)y^8 +$$
$$+(x+21x^2+170x^3+663x^4+1260x^5+1050x^6+336x^7+36x^8+x^9)y^9 +$$
$$+(x+24x^2+230x^3+1120x^4+2907x^5+3822x^6+2268x^7+540x^8+45x^9+x^{10})y^{10} + \ldots$$

$$P(abba,2)(x,y) = 1 + yx + (y^2+y^3+y^4)x^2 + (y^3+3y^4+6y^5+4y^6+y^7)x^3 + (y^4+6y^5+20y^6+29y^7 +$$
$$+21y^8+7y^9+y^{10})x^4 + (y^5+10y^6+50y^7+119y^8+154y^9+111y^{10}+45y^{11}+10y^{12}+y^{13})x^5 +$$
$$+(y^6+15y^7+105y^8+364y^9+714y^{10}+837y^{11}+605y^{12}+274y^{13}+78y^{14}+13y^{15}+y^{16})x^6 +$$
$$+(y^7+21y^8+196y^9+924y^{10}+2520y^{11}+4257y^{12}+4642y^{13} +$$
$$+3354y^{14}+1638y^{15}+545y^{16}+120y^{17}+16y^{18}+y^{19})x^7 + \ldots$$

$$P(abba,3)(x,y) = 1 + yx + y^2x^2 + (y^3+y^4+y^5+y^6)x^3 + (y^4+3y^5+6y^6+7y^7+2y^8)x^4 +$$
$$+(y^5+6y^6+20y^7+34y^8+25y^9+8y^{10}+y^{11})x^5 + (y^6+10y^7+50y^8+124y^9+157y^{10}+106y^{11}+36y^{12}$$
$$+4y^{13})x^6 + (y^7+15y^8+105y^9+364y^{10}+687y^{11}+748y^{12}+465y^{13}+148y^{14}+19y^{15}+y^{16})x^7 + \ldots$$

4 Concluding remarks

We demonstrated in the paper that the structure $\mathcal{P}(p,n,l,k)$ leads to interesting extremal and enumerative results, we emphasized here the latter. Our solution for the pattern $p = abba$ is not completely satisfactory since we gave the explicit formula for $P(abba,k)$ only for the first three values of k.

Problem 1 What can be said about the generating function $P(abba,k)(x,y)$ for $k \geq 4$?

A field for exploration opens when one tries other patterns p. Methods yielding strong upper bounds on $Ex(p,n,k)$ were developed in [8], [10] but we do not know many nontrivial exact values of this function.

Problem 2 What is $Ex(abcabc,n,k), k \geq 3$? It is not too difficult to give the upper bound $6n$ on $Ex(abcabc,n,3)$ but we do not know the exact value. What can be said about the numbers $p(abcabc,n,l,k)$?

Consider the pattern *ababa*. It contains three appearances of a, thus each partition from $\mathcal{P}_{\le 2}(\cdot,\cdot,\cdot)$ avoids it. In consequence the numbers $p(ababa,\cdot,l,k)$ and $p(ababa,n,\cdot,k)$ grow superexponentially for any fixed k and exponential rather than ordinary generating function is in place. The function $Ex(ababa,n,2)$ grows superlinearly (see [7]) and it seems very difficult to describe completely the structure of *ababa*-free sequences. Any enumerative result concerning $p = ababa$ would be of great interest.

Problem 3 What can be said about the numbers $p(ababa,n,l,k,)$?

We omitted here the first order asymptotics of the numbers $p(p,n,\cdot,k)$ and $p(p,\cdot,l,k)$, $p = abab, abba$. Knowing the explicit form of the generating function, the asymptotics can be found more or less routinely by methods described in [2]. The reader may wish to consult [17] where the asymptotics of the numbers $p(abab,\cdot,l,k)$, $k = 1,2,3$ is worked out this way.

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