Aesthetics for the Working Mathematician

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BLAKE

● Songs of Innocence and Experience (1825)

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ABSTRACT.

"If my teachers had begun by telling me that mathematics was pure play with presuppositions, and wholly in the air, I might have become a good mathematician. But they were overworked drudges, and I was largely inattentive, and inclined lazily to attribute to incapacity in myself or to a literary temperament that dullness which perhaps was due simply to lack of initiation."
(George Santayana)

"Persons and Places", 1945, pp. 238-9

● Most research mathematicians neither think deeply about nor are terribly concerned about either pedagogy or the philosophy of mathematics. Nonetheless, as I hope to indicate, aesthetic notions have always permeated (pure and applied) mathematics.

● I shall argue for aesthetics before utility.

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● Through examples, I aim to illustrate how and what that means at the research mine face. I also will argue that the opportunities to tie research and teaching to aesthetics are almost boundless — at all levels of the curriculum. This is in part due to the increasing power and sophistication of visualization, geometry, algebra and other mathematical software.

"The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful." (Henri Poincaré)

● The transparencies, and other resources, for this presentation are available at
www.cecm.sfu.ca/personal/jborwein/talks.html
www.cecm.sfu.ca/personal/jborwein/mathcamp00.html
and
www.cecm.sfu.ca/ loki/Papers/Numbers/

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GAUSS

Gauss once confessed,

"I have the result, but I do not yet know how to get it."

("Asimov's Book of ... Quotations," p. 115)

• One of Gauss's greatest discoveries, in 1799, was the relationship between the lemniscate sine function and the arithmetic-geometric mean iteration. This was based on a purely computational observation. The young Gauss wrote in his diary that the result

"will surely open up a whole new field of analysis."

◇ He was right, as it pried open the whole vista of nineteenth century elliptic and modular function theory.

• Gauss's specific discovery, based on tables of integrals provided by Stirling (1692-1770), was that the reciprocal of the integral

$$\frac{2}{\pi} \int_0^1 \frac{dt}{\sqrt{1 - t^4}}$$

agreed numerically with the limit of the rapidly convergent iteration given by $a_0 := 1$, $b_0 := \sqrt{2}$ and computing

$$a_{n+1} := \frac{a_n + b_n}{2}, \qquad b_{n+1} := \sqrt{a_n b_n}$$

◇ The sequences a_n, b_n have a common limit
$$1.1981402347355922074\ldots.$$

• Which is familiar, which is elegant — then and now?

◇ Aesthetic criteria change: 'closed forms' versus 'recursion'. 'Biology envy' replaces 'the blind watchmaker'.

GAUSS and HADAMARD

The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it. (J. Hadamard, 1865-1963)

In Borel, "Lecons sur la theorie des fonctions," 1928.

• Perhaps the greatest mathematician to think deeply and seriously about cognition in mathematics (*"... in arithmetic, until the seventh grade, I was last or nearly last."*).

◇ Author of "The psychology of invention in the mathematical field" (1945) and co-prover of the Prime Number Theorem (1896):

"The number of primes less than n tends to ∞ as does $\frac{n}{\log n}$."

AESTHETIC(s) in WEBSTER

aesthetic, adj 1. pertaining to a sense of the beautiful or to the science of aesthetics.

2. having a sense of the beautiful; characterized by a love of beauty.

3. pertaining to, involving, or concerned with pure emotion and sensation as opposed to pure intellectuality.

4. a philosophical *theory or idea of what is aesthetically valid at a given time and place*: the clean lines, bare surfaces, and sense of space that bespeak the machine-age aesthetic.

5. aesthetics.

6. Archaic. the study of the nature of sensation.

Also, esthetic. Syn 2. discriminating, cultivated, refined.

aesthetics, noun 1. the branch of philosophy dealing with such notions as the beautiful, the ugly, the sublime, the comic, etc., as applicable to the fine arts, with a view to establishing the meaning and validity of critical judgments concerning works of art, and the principles underlying or justifying such judgments.

2. *the study of the mind and emotions in relation to the sense of beauty*.

- **JMB**: (unexpected) simplicity or organization in apparent complexity or chaos.

† We need to integrate this into mathematics education — to capture minds not only for utilitarian reasons. Detachment is important, — curtains, stages and picture frames.

INSIGHT − demands speed ≡ parallelism

- For rapid verification.

- For validation; proofs *and* refutations. For 'monster barring'.

† What is 'easy' changes — merging disciplines, levels and collaborators.

- Marry theory & practice, history & philosophy, proofs & experiments.

- Match elegance and balance to utility and economy.

- In analysis, algebra, geometry & topology.

AND GOALS

- Towards an Experimental M*a*thodology — philosophy and practice.

- Intuition is acquired — mesh computation and mathematics.

- Visualization — three is a lot of dimensions (pictures and sounds).

- 'Caging' and 'Monster-barring' (Lakatos).

    - <u>graphic checks</u>: compare
      $2\sqrt{y} - y$ and $\sqrt{y}\ln(y)$, $0 < y < 1$

    - <u>randomized checks</u>: equations, linear algebra, primality

PART of OUR 'METHODOLOGY'

1. (*High Precision*) computation of object(s).

2. *Pattern Recognition* of *Real Numbers* (Inverse Calculator and 'RevEng')*, or *Sequences* (Salvy & Zimmermann's 'gfun', Sloane and Plouffe's Encyclopedia).

3. Extensive use of 'Integer Relation Methods': *PSLQ & LLL* and FFT.†

    - Exclusion bounds are especially useful.

    - Great test bed for "Experimental Math".

4. Some automated theorem proving (Wilf-Zeilberger etc).

*ISC space limits: from 10Mb in 1985 to 10Gb today.
†Top Ten "Algorithm's for the Ages," Random Samples, Science, Feb. 4, 2000.

FOUR EXPERIMENTS

• 1. **Kantian** example: generating "the classical non-Euclidean geometries (hyperbolic, elliptic) by replacing Euclid's axiom of parallels (or something equivalent to it) with alternative forms."

• 2. The **Baconian** experiment is a contrived as opposed to a natural happening, it "is the consequence of 'trying things out' or even of merely messing about."

• 3. **Aristotelian** demonstrations: "apply electrodes to a frog's sciatic nerve, and lo, the leg kicks; always precede the presentation of the dog's dinner with the ringing of a bell, and lo, the bell alone will soon make the dog dribble."

• 4. The most important is **Galilean**: "a critical experiment − one that discriminates between possibilities and, in doing so, either gives us confidence in the view we are taking or makes us think it in need of correction."

◇ It is also the only one of the four forms which will make Experimental Mathematics a serious enterprise.

• From Peter Medawar's *Advice to a Young Scientist*, Harper (1979).

MILNOR

"If I can give an abstract proof of something, I'm reasonably happy. But if I can get a concrete, computational proof and actually produce numbers I'm much happier. I'm rather an addict of doing things on the computer, because that gives you an explicit criterion of what's going on. I have a visual way of thinking, and I'm happy if I can see a picture of what I'm working with."
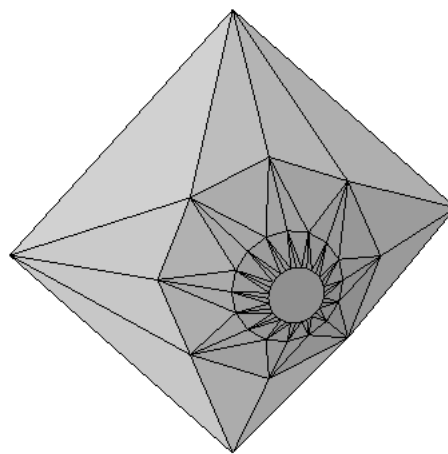
. . .

• Consider the following images of zeroes of 0/1 polynomials: `www.cecm.sfu.ca/interfaces/`

◇ But symbols are often more reliable than pictures.

A MISLEADING PICTURE

• LetsDoMath : `www.mathresources.com`

◇ Challenging students honestly? (Life)
◇ Making things tangible (Platonic solids)

De MORGAN & SYLVESTER

"Considerable obstacles generally present themselves to the beginner, in studying the elements of Solid Geometry, from the practice which has hitherto uniformly prevailed in this country, of never submitting to the eye of the student, the figures on whose properties he is reasoning, but of drawing perspective representations of them upon a plane. ... I hope that I shall never be obliged to have recourse to a perspective drawing of any figure whose parts are not in the same plane."

(Augustus De Morgan, 1806-71, First LMS President.)

● Adrian Rice, "What Makes a Great Mathematics Teacher?" MAA Monthly, June 1999, p. 540.

SYLVESTER'S THEOREM

† <u>JavaViewLib</u> : `www.cecm.sfu.ca/interfaces/` is Polthier's modern version of Felix Klein's (1840-1928) models.
† A modern version of Euclid: <u>Cinderella.de</u> : `personal/jborwein/circle.html` & Sketchpad.

"The early study of Euclid made me a hater of geometry."

(James Joseph Sylvester, 1814-97, Second LMS President)
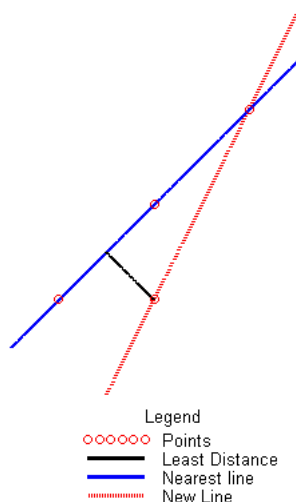
● In D. MacHale, "Comic Sections" (1993)

But discrete (now 'computational') geometry was different:

THEOREM. Given N non-collinear points in the plane there is a *proper* line through only two points.*

*Posed in *The Educational Times* **59** (1893).

KELLY'S "PROOF FROM 'THE BOOK' "

Legend
○○○○○○ Points
▬▬ Least Distance
▬▬ Nearest line
············ New Line

◇ It was forgotten for 50 years?

● First solved ("badly") by Gallai (1943). Also by Erdos who named 'the book'.

◇ Kelly's proof was published by Coxeter (1948)!

● Two more examples from the book:

— Niven's 1947 proof that π is irrational (`personal/jborwein/pi.pdf`); and

— Snell's law — travelling between Physics and the Calculus. (To or from?)

"Recent Discoveries about the Nature of Mind.

In recent years, there have been revolutionary advances in cognitive science — advances that have a profound bearing on our understanding of mathematics.* Perhaps the most profound of these new insights are the following:

1. The embodiment of mind. The detailed nature of our bodies, our brains and our everyday functioning in the world structures human concepts and human reason. This includes mathematical concepts and mathematical reason.

*More serious curricular insights should come from neuro-biology (Dehaene et al., "Sources of Mathematical Thinking: Behavioral and Brain-Imaging Evidence," *Science*, May 7, 1999).

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2. The cognitive unconscious. Most thought is unconscious — not repressed in the Freudian sense but simply inaccessible to direct conscious introspection. We cannot look directly at our conceptual systems and at our low-level thought processes. This includes most mathematical thought.

3. Metaphorical thought. For the most part, human beings conceptualize abstract concepts in concrete terms, using ideas and modes of reasoning grounded in sensory-motor systems. The mechanism by which the abstract is comprehended in terms of the concept is called *conceptual metaphor*. Mathematical thought also makes use of conceptual metaphor, as when we conceptualize numbers as points on a line."

- "Where Mathematics Comes From," Basic Books, 2000. (p. 5)

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- They later observe:

   "What is particularly ironic about this is that it follows from the empirical study of numbers as a product of mind that it is natural for people to believe that numbers are not a product of mind!" (Lakoff and Nunez, p. 81)

   . . .

- Compare a more traditional view:
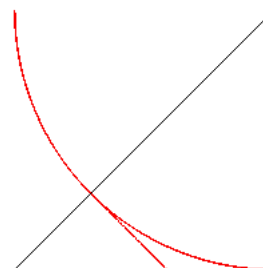
   "The price of metaphor is eternal vigalence." (Arturo Rosenblueth and Norbert Wiener)

      Quoted by R. C. Leowontin
   in *Science* p. 1264, Feb 16, 2001
      (The *Human Genome* Issue)

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- *A. Irrationality.*

- Tom Apostol's new geometric proof* of the irrationality of $\sqrt{2}$.

PROOF. Consider the *smallest* right-angled isoceles integral with integer sides:

◇ the smaller triangle is integral ⋯

*MAA, November 2000, pp. 241-242

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• *B. Rationality.*

◇ $\sqrt{2}$ also makes things rational:

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} =$$
$$\sqrt{2}^{\left(\sqrt{2}\cdot\sqrt{2}\right)} = \sqrt{2}^2 = 2.$$

• Hence there are irrational numbers α and β with α^β rational. But which ones?

† Compare: $\alpha := \sqrt{2}$, $\beta := 2\ln_2(3)$, which Maples says yields $\alpha^\beta = 3$.

† There are eight possible (ir)rational triples:

$$\alpha^\beta = \gamma.$$

Even Maple knows

• A. $\pi \neq \frac{22}{7}$.

$$\int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,dx = \frac{22}{7} - \pi,$$

$$\ldots$$

but struggles with

• *B. The sophomore's dream.*

$$\int_0^1 \frac{1}{x^x}\,dx = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

PARTIAL FRACTIONS & CONVEXITY

• We consider a network *objective function* p_N given by

$$p_N(\vec{q}) = \sum_{\sigma \in S_N} \left(\prod_{i=1}^{N} \frac{q_{\sigma(i)}}{\sum_{j=i}^{N} q_{\sigma(j)}}\right)\left(\sum_{i=1}^{N} \frac{1}{\sum_{j=i}^{N} q_{\sigma(j)}}\right)$$

summed over *all* $N!$ permutations; so a typical term is

$$\left(\prod_{i=1}^{N} \frac{q_i}{\sum_{j=i}^{N} q_j}\right)\left(\sum_{i=1}^{N} \frac{1}{\sum_{j=i}^{n} q_j}\right) .$$

◇ For $N = 3$ this is

$$q_1 q_2 q_3 \left(\frac{1}{q_1+q_2+q_3}\right)\left(\frac{1}{q_2+q_3}\right)\left(\frac{1}{q_3}\right)$$

$$\times\left(\frac{1}{q_1+q_2+q_3} + \frac{1}{q_2+q_3} + \frac{1}{q_3}\right) .$$

• We wish to show p_N is *convex* on the positive orthant. First we try to simplify the expression for p_N.

• The *partial fraction decomposition* gives:

$$p_1(x_1) = \frac{1}{x_1},$$
$$p_2(x_1, x_2) = \frac{1}{x_1} + \frac{1}{x_2} - \frac{1}{x_1+x_2},$$
$$p_3(x_1, x_2, x_3) = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$$
$$- \frac{1}{x_1+x_2} - \frac{1}{x_2+x_3} - \frac{1}{x_1+x_3}$$
$$+ \frac{1}{x_1+x_2+x_3}.$$

So we predict the 'same' for $N = 4$ and are rewarded with:

CONJECTURE. For each $N \in \mathbb{N}$

$$p_N(x_1, \ldots, x_N) := \int_0^1 \left(1 - \prod_{i=1}^{N} (1 - t^{x_i})\right)\frac{dt}{t}$$

is convex, indeed 1/concave.

• One can check $N < 5$ via a large symbolic Hessian computation. But not $N = 5$!

PROOF. A year later, analysis of *joint expectations* gave a convex integrand:

$$p_N(\vec{x}) = \int_{\mathbb{R}^n_+} e^{-(y_1 + \cdots + y_n)} \max\left(\frac{y_1}{x_1}, \ldots, \frac{y_n}{x_n}\right) dy$$

⋄ See *SIAM Electronic Problems and Solutions*.

• Computing adds reality, making concrete the abstract, and some hard things simple.

† <u>Pascal's Triangle</u> : `www.cecm.sfu.ca/interfaces/`

BERLINSKI

"The computer has in turn changed the very nature of mathematical experience, suggesting for the first time that mathematics, like physics, may yet become an empirical discipline, a place where things are discovered because they are seen."

. . .

"The body of mathematics to which the calculus gives rise embodies a certain swashbuckling style of thinking, at once bold and dramatic, given over to large intellectual gestures and indifferent, in large measure, to any very detailed description. But the era in thought that the calculus made possible is coming to an end. *Everyone feels this is so and everyone is right."*

π and FRIENDS

A: (*A quartic algorithm* (1984).) Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$(1) \qquad y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}}$$

$$(2) \qquad \begin{aligned} a_{k+1} &= a_k(1 + y_{k+1})^4 \\ &\quad - 2^{2k+3} y_{k+1}(1 + y_{k+1} + y_{k+1}^2) \end{aligned}$$

Then a_k converges *quartically* to $1/\pi$.

⋄ 19 pairs of simple algebraic equations (1, 2) that *fit on one page* differ from π only after 700 billion digits. After 17 years, this still gives me an aesthetic buzz!

• Used since 1986, with Salamin-Brent scheme, by Bailey (LBL) and Kanada (Tokyo).

• In 1997, Kanada computed over 51 billion digits on a Hitachi supercomputer (18 iterations, 25 hrs on 2^{10} cpu's). His present world record is 2^{36} digits in April 1999.

⋄ A billion (2^{30}) digit computation has been performed on a single Pentium II PC in under 9 days.

⋄ The 50 billionth decimal digit of π *or* of $\frac{1}{\pi}$ is 04<u>2</u> !

• And after 18 billion digits, 0123456789 has finally appeared (Brouwer's famous intuitionist example *now* converges!).

<u>Details at</u>: www.cecm.sfu.ca/personal/jborwein/ pi_cover.html.

B: (*'Pentium farming' for binary digits.*) Bailey, P. Borwein and Plouffe (1996) discovered a series for π (and some other *polylogarithmic constants*) which a startlingly allows one to compute hex–digits of π *without* computing prior digits.

• The algorithm needs very little memory and no multiple precision. The running time grows only slightly faster than linearly in the order of the digit being computed.

• The key, found by 'PSLQ' (below) is:

$$\pi = \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

• Knowing an algorithm would follow they spent several months hunting for such a formula (PSLQ).

◇ Once found, easy to prove in Mathematica, Maple or by hand.

◇ A most successful case of

REVERSE MATHEMATICAL ENGINEERING

$\cdots$

This is entirely practicable, God reaches her hand deep into π:

$\cdots$

• (Sept 97) Fabrice Bellard (INRIA) used a variant of this formula to compute 152 binary digits of π, starting at the *trillionth position* (10^{12}). This took 12 days on 20 work-stations working in parallel over the Internet.

PERCIVAL ON THE WEB

• (August 98) Colin Percival (SFU, age 17) finished a similar "embarrassingly parallel" computation of *five trillionth bit* (using 25 machines at about 10 times the speed). In *Hex*:

$$\underline{0}7E45733CC790B5B5979$$

The binary digits of π starting at the 40 trillionth place are

$$\underline{0}0000111110011111.$$

• (September 00) The quadrillionth bit is '0' (using 250 cpu years on 1734 machines from 56 countries).

Starting at the $999, 999, 999, 999, 997$th bit of π one has:

$$1110001100010000101101011000000110$$

FORM FOLLOWS FUNCTION

• A century after biology started to think physically:

> *"The waves of the sea, the little ripples on the shore, the sweeping curve of the sandy bay between the headlands, the outline of the hills, the shape of the clouds, all these are so many riddles of form, so many problems of morphology, and all of them the physicist can more or less easily read and adequately solve."*

(D'Arcy Thompson, "On Growth and Form" 1917)

• In Philip Ball's "The Self-Made Tapestry: Pattern Formation in Nature,"

`http://scoop.crosswinds.net/books/tapestry.html`

• How will mathematics follow?

> *"The idea that we could make biol-ogy mathematical, I think, perhaps is not working, but what is happening, strangely enough, is that maybe math-ematics will become biological!"*

(Greg Chaitin, Interview, 2000)

• Consider

- simulated annealing ('folding')

- genetic algorithms ('scheduling')

- neural networks ('training')

- DNA computation ('traveling')

- quantum computing ('sorting').

• The number of *additive partitions* of n, $p(n)$, is generated by

$$P(q) := \prod_{n \geq 1} (1 - q^n)^{-1}.$$

◇ Thus $p(5) = 7$ since

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1$$

$$= 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1.$$

QUESTION. How hard is $p(n)$ to compute — in 1900 (for MacMahon) and in 2000 (for Maple)?

$$\dots$$

• Algorithmic analysis uncovers *Euler's pentagonal number theorem*:

$$\boxed{\prod_{n \geq 1} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n+1)n/2}.}$$

◇ Ramanujan used MacMahon's table to find remarkable and deep congruences such as

$$p(5n+4) \equiv 0 \mod 5, \quad p(7n+5) \equiv 0 \mod 7$$

$$\text{and}$$

$$p(11n + 6) \equiv 0 \mod 11,$$

from data like

$$\begin{aligned} P(q) \quad &= 1 + q + 2\,q^2 + 3\,q^3 + 5\,q^4 + 7\,q^5 + 11\,q^6 \\ &+ 15\,q^7 + 22\,q^8 + 30\,q^9 + 42\,q^{10} + 56\,q^{11} \\ &+ 77\,q^{12} + 101\,q^{13} + 135\,q^{14} + 176\,q^{15} \\ &+ 231\,q^{16} + 297\,q^{17} + 385\,q^{18} + 490\,q^{19} \\ &+ 627\,q^{20} + 792\,q^{21} + 1002\,q^{22} + 1255\,q^{23} \\ &+ \cdots \end{aligned}$$

◇ We can recognize the *pentagonal numbers* in *Sloane's* on-line 'Encyclopedia of Integer Sequences'. And much more: `www.research.att.com/` `personal/njas/sequences/eisonline.html`.

• Keith Devlin: *Mathematics: he Science of Patterns* (1997).

• G. N. Watson, discussing his response to such formulae of the wonderful Indian mathematical genius Ramanujan (1887-1920), describes:

> *"a thrill which is indistinguishable from the thrill I feel when I enter the Sagres-tia Nuovo of the Capella Medici and see before me the austere beauty of the four statues representing 'Day,' 'Night,' 'Evening,' and 'Dawn' which Michelan-gelo has set over the tomb of Guiliano de'Medici and Lorenzo de'Medici."*

(G. N. Watson, 1886-1965)

One of these is his remarkable formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

Each term of this series produces an additional *eight* correct digits in the result. Gosper used this formula to compute 17 million terms of the continued fraction for π in 1985.

• That said, Ramanujan prefers explicit forms such as

$$\frac{\log(640320^3)}{\sqrt{163}} = 3.14159265358979\underline{30164} \approx \pi.$$

◇ The number e^{π} is the easiest transcendental to fast compute (by elliptic methods). One 'differentiates' $e^{-t\pi}$ to obtain π (the AGM).

"All physicists and a good many quite respectable mathematicians are contemptuous about proof." (G.H. Hardy, 1877-1947)

◇ Hardy's "A Mathematician's Apology" is a spirited defense of beauty over utility: *"Beauty is the first test. There is no permanent place in the world for ugly mathematics."*
His *"Real mathematics ... is almost wholly 'useless'"* has been overplayed and is dated: *"If the theory of numbers could be employed for any practical and obviously honourable purpose ..."*

◇ The Apology is one of Amazon's best sellers.

• "Hardy asked *'What's your father doing these days. How about that esthetic measure of his?'* I replied that my father's book was out. He said, *'Good, now he can get back to real mathematics'.*" (Garret Birkhoff).

• Hardy, in "Ramanujan, Twelve Lectures ...," page 15, gives 'Skewes number' as a *"striking example of a false conjecture"*. The integral

$$\text{li}\, x = \int_0^x \frac{dt}{\log t}$$

is a very good approximation to $\pi(x)$, the number of primes not exceeding x. Thus,
$\text{li}\, 10^8 = 5,761,455$ while $\pi(10^8) = 5,762,209$.

• It was conjectured that

$$\text{li}\, x > \pi(x)$$

and indeed it so for many x. Skewes (1933) showed the first explicit crossing $10^{10^{10^{34}}}$ — now reduced merely to 10^{1167}.

The USES of LLL and PSLQ

• A vector $(x_1, x_2, \cdots, x_n)$ of reals *possesses an integer relation* if there are integers a_i not all zero with

$$0 = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

PROBLEM: Find a_i if such exist. If not, obtain lower bounds on the size of possible a_i.

• ($n = 2$) *Euclid's algorithm* gives solution.

• ($n \geq 3$) Euler, Jacobi, Poincaré, Minkowski, Perron, others sought method.

• *First general algorithm* in 1977 by Ferguson & Forcade. Since '77: **LLL** (in Maple), HJLS, PSOS, **PSLQ** ('91, *parallel* '99).

• Integer Relation Detection was recently ranked among "the 10 algorithms with the greatest influence on the development and practice of science and engineering in the 20th century." J. Dongarra, F. Sullivan, *Computing in Science & Engineering* **2** (2000), 22–23.

Also: Monte Carlo, Simplex, Krylov Subspace, QR Decomposition, Quicksort, ..., FFT, Fast Multipole Method.

ALGEBRAIC NUMBERS

Compute α to sufficiently high precision ($O(n^2)$) and apply LLL to the vector

$$(1, \alpha, \alpha^2, \cdots, \alpha^{n-1}).$$

- Solution integers a_i are coefficients of a polynomial likely satisfied by α.

- If no relation is found, exclusion bounds are obtained.

FINALIZING FORMULAE

◇ If we know or suspect an identity exists integer relations are very powerful.

- (*Machin's Formula*) We try `lin_dep` on

  $[\arctan(1), \arctan(1/5), \arctan(1/239)]$

  and recover [1, -4, 1]. That is,

  $$\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239}).$$

  (Used on all serious computations of π from 1706 (100 digits) to 1973 (1 million).)

- (*Dase's Formula*). We try `lin_dep` on

  $[\pi/4, \arctan(1/2), \arctan(1/5), \arctan(1/8)]$

  and recover [-1, 1, 1, 1]. That is,

  $$\frac{\pi}{4} = \arctan(\frac{1}{2}) + \arctan(\frac{1}{5}) + \arctan(\frac{1}{8}).$$

  (Used by Dase to compute 200 digits of π in his head.⋯)

JOHANN MARTIN ZACHARIAS DASE

• History at: www-history.mcs.st-andrews.ac.uk

"Zacharias Dase (1824-1861) had incredible calculating skills but little mathematical ability. He gave exhibitions of his calculating powers in Germany, Austria and England. While in Vienna in 1840 he was urged to use his powers for scientific purposes and he discussed projects with Gauss and others.

Dase used his calculating ability to calculate to 200 places in 1844. This was published in Crelle's Journal for 1844. Dase also constructed 7 figure log tables and produced a table of factors of all numbers between 7 000 000 and 10 000 000.

Gauss requested that the Hamburg Academy of Sciences allow Dase to devote himself full-time to his mathematical work but, although they agreed to this, Dase died before he was able to do much more work."

KUHN

"*The issue of paradigm choice can never be unequivocally settled by logic and experiment alone.*

⋯

in these matters neither proof nor error is at issue. The transfer of allegiance from paradigm to paradigm is a conversion experience that cannot be forced."

(Thomas Kuhn)

• In *Who got Einstein's Office?* by Ed Regis. A 1986 history of the Institute for Advanced Study.

And PLANCK

"… a new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents die and a new generation grows up that's familiar with it."

(Albert Einstein quoting Max Planck)

● From "The Quantum Beat," by F.G. Major, Springer (1998)

HERSH

● Whatever the outcome of these discourses, mathematics is and will remain a uniquely human undertaking. Indeed Reuben Hersh's arguments for a humanist philosophy of mathematics, as paraphrased below, become more convincing in our setting:

1. Mathematics is human. It is part of and fits into human culture. It does not match Frege's concept of an abstract, timeless, tenseless, objective reality.

2. Mathematical knowledge is fallible. As in science, mathematics can advance by making mistakes and then correcting or even re-correcting them. The "fallibilism" of mathematics is brilliantly argued in Lakatos' *Proofs and Refutations*.

3. There are different versions of proof or rigor. Standards of rigor can vary depending on time, place, and other things. The use of computers in formal proofs, exemplified by the computer-assisted proof of the four color theorem in 1977, is just one example of an emerging nontraditional standard of rigor.

4. Empirical evidence, numerical experimentation and probabilistic proof all can help us decide what to believe in mathematics. Aristotelian logic isn't necessarily always the best way of deciding.

5. Mathematical objects are a special variety of a social-cultural-historical object. Contrary to the assertions of certain post-modern detractors, mathematics cannot be dismissed as merely a new form of literature or religion. Nevertheless, many mathematical objects can be seen as shared ideas, like Moby Dick in literature, or the Immaculate Conception in religion.

◇ From "Fresh Breezes in the Philosophy of Mathematics", *American Mathematical Monthly*, August-Sept 1995, 589–594.

● The recognition that "quasi-intuitive" analogies may be used to gain insight in mathematics can assist in the learning of mathematics. And honest mathematicians will acknowledge their role in discovery as well.

We should look forward to what the future will bring.

SANTAYANA

"When we have before us a fine map, in which the line of the coast, now rocky, now sandy, is clearly indicated, together with the winding of the rivers, the elevations of the land, and the distribution of the population, we have the simultaneous suggestion of so many facts, the sense of mastery over so much reality, that we gaze at it with delight, and need no practical motive to keep us studying it, perhaps for hours altogether. A map is not naturally thought of as an aesthetic object...

† My earliest, and still favourite, encounter with aesthetics.*

*Jerry Fodor: "... it is no doubt important to attend to the eternally beautiful and true. But it is more important not to be eaten." In Kieran Egan's, *Getting it Wrong from the Beginning*).

And yet, let the tints of it be a little subtle, let the lines be a little delicate, and the masses of the land and sea somewhat balanced, and we really have a beautiful thing; a thing the charm of which consists almost entirely in its meaning, but which nevertheless pleases us in the same way as a picture or a graphic symbol might please. Give the symbol a little intrinsic worth of form, line and color, and it attracts like a magnet all the values of things it is known to symbolize. It becomes beautiful in its expressiveness." (George Santayana)

● From "The Sense of Beauty," 1896.

A FEW CONCLUSIONS

● Draw your own! − perhaps ⋯

● Proofs are often out of reach − understanding, even certainty, is not.

● Packages can make concepts accessible (Maple, Cinderella).

● Progress is made 'one funeral at a time' (Niels Bohr (?)).

● 'We are Pleistocene People' (Kieran Egan).

● 'You can't go home again' (Thomas Wolfe).

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● These and other references are available at www.cecm.sfu.ca/preprints/

◇ Quotations at jborwein/quotations.html