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Arrays, Numeration Systems and Frankenstein Games

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Abstract. We define an infinite array $\mathcal{A}$ of nonnegative integers based on a linear recurrence, whose second row provides basis elements of an exotic ternary numeration system. Using the numeration system we explore many properties of $\mathcal{A}$. Further, we propose and analyze a family *Frankenstein* of 2-player pebbling games played on a semi-infinite strip, and present a winning strategy based on certain subarrays of $\mathcal{A}$. Though the strategy looks easy, it is actually computationally hard. The numeration system is then used to decide whether the family has an efficient strategy or not.

1. Introduction

Consider a doubly infinite array (matrix) $\mathcal{A} = \{A_j^n : 0 \le j,\ n \le \infty\}$ of nonnegative integers whose first few entries are displayed in Table 1. To define its formation rule, we introduce a little notation.

Denote by $\mathbb{Z}$, $\mathbb{Z}^0$ and $\mathbb{Z}^+$ the set of integers, nonnegative integers and positive integers respectively. If S is any *proper* subset of $\mathbb{Z}^0$, i.e., $S \ne \mathbb{Z}^0$, denote by $\operatorname{mex} S$ the least nonnegative integer in the complement of S with respect to $\mathbb{Z}^0$, i.e., the least nonnegative integer not occurring in S. Note that the mex of the empty set is 0. The term mex, introduced in [BCG1982], stands for Minimum EXcluded value.

For $n \in \mathbb{Z}^0$, the entries of the array are defined as follows.

$$A_0^n = \operatorname{mex}\{A_j^i : 0 \le i < n,\ j \ge 0\}, \tag{1}$$

Table 1: A doubly infinite array of nonnegative integers.

n	A_0^n	A_1^n	A_2^n	A_3^n	A_4^n	A_5^n	A_6^n	
0	0	0	0	0	0	0	0	
1	1	3	8	21	55	144	377	
2	2	6	16	42	110	288	754	
3	4	11	29	76	199	521	1364	
4	5	14	37	97	254	665	1741	…
5	7	19	50	131	343	898	2351	
6	9	24	63	165	432	1131	2961	
7	10	27	71	186	487	1275	3338	
8	12	32	84	220	576	1508	3948	
9	13	35	92	241	631	1652	4225	
10	15	40	105	275	720	1885	4935	
		⋮						

$$A_1^n = 2A_0^n + n \ (n \geq 0), \quad A_j^n = 3A_{j-1}^n - A_{j-2}^n \ \ (j \geq 2, \ n \geq 0). \tag{2}$$

It can be seen, by induction on n, that the set on the right hand side of (1) is indeed a proper subset of $\mathbb{Z}^0$.

We further introduce a special ternary numeration system $\mathcal{U}$. Its basis elements are defined by $u_0 = 1$, $u_1 = 3$, $u_i = 3u_{i-1} - u_{i-2}$ $(i \geq 2)$.

Theorem I. *Every positive integer n has a unique representation over $\mathcal{U}$, in the form $n = \sum_{i \geq 0} d_i u_i$, where the digits d_i assume values in $\{0, 1, 2\}$, subject to the following special condition: if for some $0 \leq j < l$, $d_j = d_l = 2$, then there exists k satisfying $j < k < l$ (so actually $l - j \geq 2$), such that $d_k = 0$.*

Theorem I is a special case of Theorem 4, stated and proved in [Fra1985, §4]. The representation of the first few positive integers over $\mathcal{U}$ is given in Table 2. We write the representation of n both in terms of its basis elements, $n = \sum_{i=0}^m d_i u_i$, and in its "ternary" form $n = d_m \dots d_0$, the same as is customary for more conventional numeration systems, such as decimal or binary ($528 = 8 \times 10^0 + 2 \times 10^1 + 5 \times 10^2$). Table 2 shows, for example, that $41 = 1211$; and $42 = 2000$ rather than 1212, because of the special condition. Similarly, $55 = 10000$, not 2112.[1]

[1] Some of my best friends are nonsemitic, among them referees and readers of my articles. A number of them have commented to me that in a table such as Table 2, the basis elements $1, 3, 8, 21, 55$ should be written from left to right rather than from right to left. I disagree. The "ternary" number $n = d_m \dots d_0$, now easily readable from the table, would be reversed! There is a discrepancy in nonsemitic languages, often ignored, between text, including mathematical

Table 2: A special ternary representation of integers n.

55	21	8	3	1	n	21	8	3	1	n
	1	1	0	2	31				1	1
	1	1	1	0	32				2	2
	1	1	1	1	33			1	0	3
	1	1	1	2	34			1	1	4
	1	1	2	0	35			1	2	5
	1	1	2	1	36			2	0	6
	1	2	0	0	37			2	1	7
	1	2	0	1	38		1	0	0	8
	1	2	0	2	39		1	0	1	9
	1	2	1	0	40		1	0	2	10
	1	2	1	1	41		1	1	0	11
	2	0	0	0	42		1	1	1	12
	2	0	0	1	43		1	1	2	13
	2	0	0	2	44		1	2	0	14
	2	0	1	0	45		1	2	1	15
	2	0	1	1	46		2	0	0	16
	2	0	1	2	47		2	0	1	17
	2	0	2	0	48		2	0	2	18
	2	0	2	1	49		2	1	0	19
	2	1	0	0	50		2	1	1	20
	2	1	0	1	51	1	0	0	0	21
	2	1	0	2	52	1	0	0	1	22
	2	1	1	0	53	1	0	0	2	23
	2	1	1	1	54	1	0	1	0	24
1	0	0	0	0	55	1	0	1	1	25
1	0	0	0	1	56	1	0	1	2	26
1	0	0	0	2	57	1	0	2	0	27
1	0	0	1	0	58	1	0	2	1	28
1	0	0	1	1	59	1	1	0	0	29
1	0	0	1	2	60	1	1	0	1	30

formulas, and "digital" numbers. Though all of these are both written and read from left to right, the basis elements of the latter, which are usually implicit but here explicit, nevertheless increase from right to left. (There is an even greater discrepancy when embedding formulas and digital numbers in semitic language texts, but it is well-known and acknowledged. Moreover, word processors have long since learned to overcome it; human beings still have difficulties with it.)

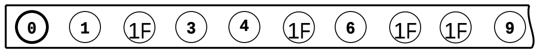

FIGURE 1.1. A position in *Frankenstein* with 1Fr. coins.

Lastly, we define a two-person pebbling game called *Frankenstein*[2], played on a semi-infinite strip with a finite number of pebbles, say coins, at most one per square. The squares are numbered with the nonnegative integers $0, 1, 2, \ldots$ from the left end of the strip, as in Fig. 1. There is a hole at square 0: a coin landing on it falls through the hole, disappearing from the play. The empty strip is denoted by ϕ. A single coin on the strip is a *spinster*. A legal move is to shift a number of coins from their present squares to *any* unoccupied squares with a lower number (a left shift), avoiding a spinster: we never permit a spinster position. Every move of ≥ 2 coins involves a *sequential* shifting of coins: an arbitrary coin is first shifted. Then a coin to its left is shifted, then a coin to its left, and so on. Every coin is shifted *at most once* in a single move. Also new coins can be created. Specifically, the moves from a position with say k $(k \geq 2)$ coins on squares

$$X = (x_0, \ldots, x_{k-1}), \quad 0 < x_0 < \cdots < x_{k-1}, \tag{3}$$

are of two types.

I (a) Shift a positive number of at most $k-1$ tokens, at least one of them to a *positive* numbered square. (b) A coin on precisely one square m may be shifted to 0 and new coins be placed on the unoccupied squares $j_1, \ldots, j_\ell$ if and only if $0 < \sum_{i=1}^{\ell} j_i < m$. A move consists of either (a) or (b) (or both).

II Shift all of the tokens by say, $0 < n_0 \leq \cdots \leq n_{k-1}$ squares, either preserving k or resulting in Φ. Moreover, n_{k-1} should not be too large; namely,

$$n_{k-1} \leq 2n_{k-2} + n_{k-3} + \cdots + n_0. \tag{4}$$

The player first unable to move loses, and the opponent wins. Notice that in every position there is at most one coin per square, and the only end position is Φ. A spinster is never permitted. In a type **II** move, either all coins are removed, or none. The number of coins can decrease or increase during play; but the sum of the occupied square numbers decreases at each move. Therefore play ends, and no game position is repeated.

Examples.

(i) Let $X = (1, 3)$. A move $X \to (0, 0)$ is inconsistent with (4). Also a move to 1 or 3 is not permitted, since they are spinsters (and also by the second part of **I**(a)). Thus the only possible move is to $(1, 2)$. Then player II can move to $(0, 0)$, winning.

(ii) From the position $(1, 3, 7)$ player I can move to Φ winning instantly, because the move $7 \to 0$, $3 \to 0$, $1 \to 0$ satisfies 4 (with equality).

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[2]The game is played with coins called Francs (in Belgium or France) and Franks (in Switzerland). Alternatively, it may be played with pebbles or stones. Hence the name of the game.

(iii) Given the initial position $X = (1,3,8)$. A move $X \to (1,3)$ is not permitted by the second part of **I**(a). It can be seen that if only the coin at 8 is shifted, then player II can move to Φ in the next move. We leave it to the reader to verify that X is a position from which player II can win, either by moving directly to Φ or by moving first to $(1,3)$.

(iv) The winning move $(6,8,100) \to (1,3,8)$ involves (a): $100 \to 1$, $6 \to 3$ (or $100 \to 3$, $6 \to 1$).

(v) The winning move $(8,19) \to (1,3,8)$ is of type (b): $19 \to (1,3)$.

(vi) Show that $(55,56,200) \to (1,3,8,21,55)$ is a winning move $\big($involving both (a): $200 \to 8$ and (b): $(56 \to (1,3))\big)$.

(vii) Verify that player II can win from the position $(2,6)$.

We shall show that certain subarrays of the array $\mathcal{A}$ are the so-called "losing positions" of Frankenstein. For proving this it is helpful to use some of the properties of $\mathcal{A}$. Essentially, $\mathcal{A}$ is a splitting of $\mathbb{Z}^+$, but to state the result precisely, some further notions will first be introduced.

Define the operators L (*L*eft shift) and R (*R*ight shift) on representations over $\mathcal{U}$: if $n = \sum_{i=0}^m d_i u_i$ for some $n \in \mathbb{Z}^+$, then $L(n) = \sum_{i=0}^m d_i L(u_i) = \sum_{i=0}^m d_i u_{i+1}$; and $R(n) = \sum_{i=1}^m d_i R(u_i) = \sum_{i \ge 1} d_i u_{i-1}$ is defined if $d_0 = 0$. In other words, if $n = d_m \dots d_0$, then $L(n) = d_m \dots d_0 0$, and, if $i \ge 1$ (i.e., $d_0 = 0$), then $R(d_m \dots d_1 0) = d_m \dots d_1$. In particular: $L(u_i) = u_{i+1}$ $(i \ge 0)$; and $R(u_i) = u_{i-1}$ $(i \ge 1)$.

The j-th column of $\mathcal{A}$, excluding the 0 in the first row, is denoted by $A_j = \bigcup_{n=1}^{\infty} A_j^n$, $j \ge 0$; and the n-th row is $A^n = \bigcup_{j=0}^{\infty} A_j^n$, $n \ge 1$. If $n = \sum_{i \ge 0} d_i u_i$ with $d_0 \ne 0$, we say that n is *reduced*. A reduced number n has no right shift. The *golden section* is the positive root ϕ of the polynomial equation $x^2 - x - 1 = 0$, so $\phi = (1+\sqrt{5})/2$ and $\phi^2 = \phi + 1$.

In §2 we prove,

Theorem 1. *The array $\mathcal{A}$ is a splitting of $\mathbb{Z}^+$: every positive integer appears precisely once in $\mathcal{A}$. Moreover, for every $j \ge 0$, the column A_j consists precisely of all positive integers whose representation ends in j 0s. In particular, A_0^n is reduced for all $n \in \mathbb{Z}^+$.*

The proof leans heavily on properties of the special ternary numeration system $\mathcal{U}$, which are also explored in §2. The system $\mathcal{U}$ is even more useful: the winning strategy for Frankenstein, based on subarrays of $\mathcal{A}$, is inefficient (exponential). The system $\mathcal{U}$ enables one to decide whether there is or there isn't a different, efficient (polynomial) strategy. This is taken up in §4. Some further remarkable properties of $\mathcal{A}$ are listed in Theorem 2, also proved in §2.

Let $f_0 = 1$, $f_1 = 2$, $f_n = f_{n-1} + f_{n-2}$ $(n \ge 2)$ be the sequence of Fibonacci numbers. (It is easily seen that the numeration basis elements u_i defined above, which constitute the second row of $\mathcal{A}$, are precisely the "even" Fibonacci numbers, i.e., $u_i = f_{2i}$ for all $i \ge 0$. Also the other rows of $\mathcal{A}$ are "even Fibonacci numbers" with different initial conditions, but these facts are not needed here.)

Theorem 2.

(i) *For $j, n \in \mathbb{Z}^+$, $A_j^n = \lfloor A_{j-1}^n \phi^2 \rfloor + 1 = L(A_{j-1}^n)$.*

(ii) *For all $n \ge 1$, $A_0^n = \lfloor (n-1)\phi \rfloor + 1$ is reduced.*

(iii) *For all* $n \geq 0$, *all* $j \geq 0$ *we have,* $A_j^{n+1} - A_j^n \in \{f_{2j}, f_{2j+1}\}$, *and for fixed* j, *each of* f_{2j}, f_{2j+1} *is assumed for infinitely many* n. *Moreover, for all* n *for which* $A_0^{n+1} - A_0^n = f_0$ *(respectively* f_1*), we also have for all* j, $A_j^{n+1} - A_j^n = f_{2j}$ *(respectively* f_{2j+1}*).*

(iv) *Let* $j \geq 1$. *There are no real numbers* α, γ, *such that for all* $n \geq 1$, $A_j^n = \lfloor n\alpha + \gamma \rfloor$.

Properties of $\mathcal{A}$ are also presented in Lemmas 1 and 2 in §2. The formulation of a winning strategy for Frankenstein needs a few technical concepts, so is best postponed to §3, where the precise result is stated and proved. A sum up is presented in the final §5.

2. Some Properties of the Array

We begin with a simple result.

Lemma 1. *For all* $j, n \in \mathbb{Z}^+$, $A_j^n = 2A_{j-1}^n + A_{j-2}^n + \cdots + A_0^n + n$.

Proof. Induction on j, for arbitrary but fixed n. By the first part of (2), the assertion holds for $j = 1$. Suppose it holds for some $j \geq 1$. By the second part of (2),

$$A_{j+1}^n = 2A_j^n + (A_j^n - A_{j-1}^n) = 2A_j^n + A_{j-1}^n + \cdots + A_0^n + n. \quad \blacksquare$$

The following is the main lemma used for proving both Theorem 1 and Theorem 2.

Lemma 2. *Let* $n \geq 1$, $\mathcal{S}_n = \bigcup_{m<n} \bigcup_{j=0}^{\infty} A_j^m$. *In every row* A^n *of* $\mathcal{A}$, *the element* A_0^n *is the smallest reduced element not in* $\mathcal{S}_n$, *and* A_{j+1}^n *is the left shift of* A_j^n *for all* $j \geq 0$.

Proof. Since $u_1 = 2u_0 + 1$ and $u_i = 3u_{i-1} - u_{i-2}$ $(i \geq 2)$, the same as the recurrence (2), and $A_0^1 = 1 = u_0$, the row A^1 consists of the basis elements of $\mathcal{U}$, for which the statement clearly holds. Suppose it holds for all $m < n$ $(n \geq 2)$. If A_0^n would not be reduced, then $R(A_0^n)$ would be a smaller element than A_0^n. Moreover, $R(A_0^n) \notin \mathcal{S}_n$, otherwise also $A_0^n = LR(A_0^n)$ would be in $\mathcal{S}_n$, by the induction hypothesis, contradicting (1). Thus A_0^n is the smallest reduced element not in $\mathcal{S}_n$.

Let $A_0^{n-1} = \sum_{i \geq 0} d_i u_i$ be the representation of A_0^{n-1} over $\mathcal{U}$. By the induction hypothesis, $A_1^{n-1} = L(A_0^{n-1}) = \sum_{i \geq 0} d_i u_{i+1}$ and A_0^{n-1} is reduced. In particular, $d_0 \neq 0$. We consider two cases.

(i) There exists $j \geq 1$ such that $d_i = 1$ for all $i < j$, but $d_j = 0$. Then $A_0^{n-1} + 1 = \sum_{i \geq 1} d_i u_i + (d_0 + 1)u_0$ is reduced (by Theorem I, with least significant digit 2), so the first part of the proof implies $A_0^{n-1} + 1 = A_0^n$. Now,

$$\begin{aligned} A_1^n &= 2A_0^n + n = 2(A_0^{n-1} + 1) + n = 2A_0^{n-1} + (n-1) + 3 = A_1^{n-1} + 3 \\ &= \sum_{i \geq 0} d_i u_{i+1} + u_1 = \sum_{i \geq 1} d_i u_{i+1} + (d_0 + 1)u_1 = L(A_0^n). \end{aligned}$$

(ii) There exists $j \geq 0$ such that $d_i = 1$ for all $i < j$, but $d_j = 2$. By Theorem I, $d_{j+1} \leq 1$. By Lemma 1 with $n = 1$, $A_0^{n-1} + 1 = \sum_{i \geq j+2} d_i u_i + (d_{j+1}+1)u_{j+1}$ is not reduced, but $A_0^{n-1} + 2 = \sum_{i \geq j+2} d_i u_i + (d_{j+1}+1)u_{j+1} + u_0 = A_0^n$ is reduced. Then by Lemma 1 ($n = 1$),

$$\begin{aligned}
A_1^n &= 2A_0^n + n = 2(A_0^{n-1} + 2) + n = 2A_0^{n-1} + (n-1) + 5 \\
&= A_1^{n-1} + 5 = \sum_{i \geq 0} d_i u_{i+1} + 5 = \sum_{i \geq j+2} d_i u_{i+1} + \sum_{i=0}^{j+1} d_i u_{i+1} + 5 \\
&= \sum_{i \geq j+2} d_i u_{i+1} + d_{j+1} u_{j+2} + (2u_{j+1} + u_j + \cdots + u_1) + (u_0 + 1 + 3) \\
&= \sum_{i \geq j+2} d_i u_{i+1} + (d_{j+1}+1)u_{j+2} + u_1 = L(A_0^n).
\end{aligned}$$

It remains only to show that $A_{j+1}^n = L(A_j^n)$ for all $j \geq 1$. Suppose we already showed this for all $j < m$. For $m = 1$ this was just done. So consider A_{m+1}^n. Let $A_{m-1}^n = \sum_{i \geq 0} d_i u_i$ be the representation of A_{m-1}^n. By the induction hypothesis and (2),

$$A_{m+1}^n = 3A_m^n - A_{m-1}^n = \sum_{i \geq 0} d_i(3u_{i+1} - u_i) = \sum_{i \geq 0} d_i u_{i+2} = L(A_m^n). \quad \blacksquare$$

Proof of Theorem 1. By Lemma 2, $A_{j+1}^n = L(A_j^n)$. Therefore the representation of A_{j+1}^n has one additional 0 at its tail end than that of A_j^n. Since the representations of positive integers over $\mathcal{U}$ are unique (Theorem I), all entries in $\mathcal{A}$ are indeed distinct. Finally, every positive integer appears in $\mathcal{A}$ in view of (1). $\blacksquare$

For proving the left shift part of Theorem 2(i), we prove, more generally,

Lemma 3. *Let $m \in \mathbb{Z}^+$, $n = \lfloor m\phi^2 \rfloor + 1$. Then $n = L(m)$.*

Proof. Let $m = \sum_{i=0}^r d_i u_i$ be the representation of m, for suitable $r \in \mathbb{Z}^0$. We have to show: $n = \sum_{i=0}^r d_i u_{i+1}$. It suffices to show that $0 < m\phi^2 + 1 = \sum_{i=0}^r d_i u_{i+1} + \rho$, for some $0 < \rho < 1$. So it suffices to show that $0 < \sum_{i=0}^r d_i(u_{i+1} - u_i\phi^2) < 1$.

The characteristic equation of the second recurrence of (2) is $x^2 - 3x + 1 = 0$, with solutions $\phi^2 = (3 + \sqrt{5})/2$ and conjugate $\phi^{-2} = (3 - \sqrt{5})/2$. From this it follows that for $n \geq 0$,

$$u_n = \frac{\phi^{2n+2} - \phi^{-(2n+2)}}{\sqrt{5}} \,. \tag{5}$$

Then $u_{i+1} - u_i\phi^2 = \phi^{-2i}(1 - \phi^{-4})/\sqrt{5} > 0$. Note that, due to the special condition of Theorem I, $\sum_{i=0}^r d_i(u_{i+1} - u_i\phi^2)$ is largest when $d_0 = 2$ and $d_i = 1$ for all $i \geq 1$. Thus,

$$0 < \sum_{i=0}^r d_i(u_{i+1} - u_i\phi^2) < \frac{(1 - \phi^{-4})}{\sqrt{5}}(1 + \sum_{i=0}^{\infty} \phi^{-2i}) = \frac{(1 - \phi^{-4})}{\sqrt{5}}\phi^2 = 1. \quad \blacksquare$$

For proving (iv) of Theorem 2, we prove a technical result.

Lemma 4. *Let $\alpha > 0$, γ be real numbers. Letting $N_n = \lfloor (n+1)\alpha + \gamma \rfloor - \lfloor n\alpha + \gamma \rfloor$, we have*

$$\lfloor \alpha \rfloor \le N_n \le \lceil \alpha \rceil. \tag{6}$$

Moreover, each of the values $\lfloor \alpha \rfloor$ and $\lceil \alpha \rceil$ is assumed for infinitely many n.

Proof. The definition of N_n implies (6) directly. If $\alpha = p/q$ with gcd $(p, q) = 1$ is rational, then we may clearly assume, without loss of generality, that $\gamma = r/q$ ($p \in \mathbb{Z}^0$, $q \in \mathbb{Z}^+$, $r \in \mathbb{Z}$). The congruence $xp \equiv q - r \pmod{q}$ has a solution $x = n_0$, $0 \le n_0 < q$, so $n_0 p = kq - r$ for some $k \in \mathbb{Z}$. It is then easily verified that $N_{n_0 - 1} = \lceil \alpha \rceil$, and $N_{n_0} = \lfloor \alpha \rfloor$. Since the above congruence has the general solution $n = n_0 + sq$, $s \in \mathbb{Z}$, each of the values $\lfloor \alpha \rfloor$ and $\lceil \alpha \rceil$ is assumed infinitely often.

If α is irrational, then the fractional values $(n\alpha)$ are dense in $(0, 1)$ (Kronecker's Theorem; see e.g., [HaWr1989], Ch. 23). Hence each of $\lfloor \alpha \rfloor$ and $\lceil \alpha \rceil$ is assumed infinitely often also in this case. ■

Proof of Theorem 2. From Lemma 3 we have, in particular, $\lfloor A^n_{j-1}\phi^2 \rfloor + 1 = L(A^n_{j-1})$. By Lemma 2, this is also the same as A^n_j for all $j \ge 1$, proving (i).

Since $\phi^{-1} + \phi^{-2} = 1$, it follows from Theorem II of [Fra1969] that if $S = \bigcup_{n=1}^{\infty}(\lfloor n\phi \rfloor + 1)$, $T = \bigcup_{n=1}^{\infty}(\lfloor n\phi^2 \rfloor + 1)$, then S, T are 2-*upper complementary*, i.e., $S \cup T = \mathbb{Z}^+ \setminus \{1\}$ and $S \cap T = \emptyset$. By Lemma 3, T contains only non reduced numbers. Hence S consists of precisely all the reduced numbers > 1, and T of all the non reduced numbers. Replacing n by $n - 1$, (ii) follows from (1).

For establishing (iii), we use induction on j. For $j = 0$, the claim follows directly from Lemma 4 and (ii), with $\alpha = \phi$, $\gamma = 1$. For $j = 1$ we have by (2), $A^{m+1}_1 - A^m_1 = 2(A^{m+1}_0 - A^m_0) + 1 \in \{2f_0 + 1, 2f_1 + 1\} = \{f_2, f_3\}$; and f_2 (respectively f_3) is assumed precisely when $2(A^{m+1}_0 - A^m_0) = f_0$ (f_1 respectively). Suppose it holds for all $i < j$ ($j \ge 2$). By (2), $A^{m+1}_j - A^m_j = 3(A^{m+1}_{j-1} - A^m_{j-1}) - (A^{m+1}_{j-2} - A^m_{j-2})$. This is either $3f_{2j-2} - f_{2j-4} = f_{2j}$ or f_{2j+1}, according to whether in the previous column $(j-1)$ the result was $f_{2(j-1)}$ or f_{2j-1}. We have demonstrated the validity of (iii).

By Lemma 4, a necessary condition for the existence of real α, γ with α positive and irrational such that $A_j = \lfloor n\alpha + \gamma \rfloor$, is that $A^{n+1}_j - A^n_j \in \{\lfloor \alpha \rfloor, \lceil \alpha \rceil\}$. In particular, $A^{n+1}_j - A^n_j$ has to assume two consecutive integer values. But by (iii), the two assumed values are f_{2j} and f_{2j+1}; which are consecutive if and only if $j = 0$. This proves (iv). ■

Remark. Theorem 2 can be used to give an independent proof of Theorem 1, because the former implies, using the uniqueness of representation (Theorem I), that all entries of $\mathcal{A}$ are distinct, $A_{j+1} = L(A_j)$, and also every positive integer is assumed.

3. A Winning Strategy for Frankenstein

Informally, a position u in a game such as Frankenstein is called a P-position, if the *P*revious player can win, i.e., the player who moved to u. It is an N-position, if the *N*ext player can win, i.e., the player moving from u. The position Φ is a P-position, since player I (the player called upon to move from the the given position), cannot even make a move, so the opponent, player II, wins by default. By $F(u)$ we denote the set of all immediate followers of u, i.e., the set of all positions reachable from u by a single move . Note that $F(u) = \emptyset$ if u is a *leaf*, i.e., an end position.

Denote by $\mathcal{P}$ the set of all P-positions, and by $\mathcal{N}$ the set of all N-positions. The informal definition of P- and N-positions implies,

$$u \in \mathcal{P} \iff F(u) \subseteq \mathcal{N}, \quad u \in \mathcal{N} \iff F(u) \cap \mathcal{P} \neq \emptyset. \tag{7}$$

All of these things can be done formally. See [Fra≥2001].

For the sake of compactness of discussion, we will be talking about reducing integers, rather than shifting coins on squares numbered with those integers. In terms of this convention, we state the main result of this section.

Theorem 3. *The P-positions of the game Frankenstein are given by*

$$\mathcal{P} = \bigcup_{n=0}^{\infty} \bigcup_{k=2}^{\infty} (A_0^n, \ldots, A_{k-1}^n).$$

Proof. Let $W = \bigcup_{n=0}^{\infty} \bigcup_{k=2}^{\infty} (A_0^n, \ldots, A_{k-1}^n)$. As was pointed out in §1, the empty strip Φ is a leaf, i.e., $F(\Phi) = \emptyset$, and so is a P-position by (7). It turns out that in view of (7), it suffices to demonstrate the following two properties for all positions.

(A) Every move from a position in W produces a position not in W.
(B) From every position not in W there exists a move to a position in W.

(A) Let $(A_0^n, \ldots, A_{k-1}^n) \in W$. For a move of type **I**, there is a number A_j^n which remains fixed, and a number L which is either reduced or replaced by a collection of smaller numbers. In either case, the resulting position contains A_j^n and a number $L \neq A_i^n$ for all $i \geq 0$, so it is not in W.

Now consider a move of type **II**. Suppose there is a move $X = (A_0^n, \ldots, A_{k-1}^n) \to (A_0^m, \ldots, A_{j-1}^m) \in W$. If $m > n$ $\big($such as $(2, 6, 16) \to (4, 11)\big)$, the move involves $A_0^n \to 0$, contrary to the requirement of preserving k. Clearly we cannot have $m = n$. So $m < n$, $j \leq k$. Suppose first that $j < k$. If $m = 0$ (so $(A_0^m, \ldots, A_{j-1}^m) = \Phi$), we have, using Lemma 1,

$$A_{k-1}^n = 2A_{k-2}^n + \sum_{i=0}^{k-3} A_i^n + n > 2A_{k-2}^n + \sum_{i=0}^{k-3} A_i^n, \tag{8}$$

contradicting condition (4). This contradiction holds a fortiori if $m > 0$, because then the terms to the right of A_{k-1}^n in (8) are even smaller, but the left side is still A_{k-1}^n if $j < k$. We conclude that $j = k$.

The presumed move is thus $(A_0^n, \ldots, A_{k-1}^n) \to (A_0^m, \ldots, A_{k-1}^m)$. By Lemma 1,

$$\begin{aligned} A_{k-1}^n - A_{k-1}^m &= 2(A_{k-2}^n - A_{k-2}^m) + \sum_{i=0}^{k-3} (A_i^n - A_i^m) + n - m \\ &> 2(A_{k-2}^n - A_{k-2}^m) + \sum_{i=0}^{k-3} (A_i^n - A_i^m). \end{aligned} \tag{9}$$

This contradicts (4), since Theorem 2(iii) implies that for every $n > m > 0$ and all $j \ge 0$, $A^n_{j+1} - A^m_{j+1} > A^n_j - A^m_j$, so $A^n_{k-1} - A^m_{k-1} = \max_{0 \le i \le k-1}(A^n_i - A^m_i)$.

(B) Given a position $X = (x_0, \ldots, x_{k-1}) \notin W$ of the form (3), with $k \ge 2$. We show that there is a single move to a position in W. By complementarity (Theorem 1), $x_0 = A^n_{j-1}$ for some $j, n \in \mathbb{Z}^+$.

Assume first $j > 1$. Since $k \ge 2$, there is $x_1 > x_0$. By Lemma 1, $x_1 > x_0 = 2A^n_{j-2} + \sum_{i=0}^{j-3} A^n_i + n$. If $k \ge j$, we reduce $(x_1, \ldots, x_{j-1}) \to (A^n_0, \ldots, A^n_{j-2})$, and put $x_\ell \to 0$ for all $\ell \ge j$, if any. If $k < j$, we reduce $(x_1, \ldots, x_{k-2}) \to (A^n_0, \ldots, A^n_{k-3})$, and then split a suitably reduced x_{k-1} into $(A^n_{k-2}, \ldots, A^n_{j-2})$. In particular, if $k = 2$, then x_1 is reduced and split into $(A^n_0, \ldots, A^n_{j-2})$. We have made a type **I** move to $(A^n_0, \ldots, A^n_{j-1}) \in W$.

We may thus assume $x_0 = A^n_0$. Then there exists $j \ge 2$ such that $x_i = A^n_i$ for $i < j-1$, but $x_{j-1} \ne A^n_{j-1}$. If $x_{j-1} > A^n_{j-1}$, move $x_{j-1} \to A^n_{j-1}$ and put $x_i \to 0$ for all $i > j-1$.

So we may assume $x_{j-1} < A^n_{j-1}$. We consider the following cases.

(i) $j = k$, so $x_{j-1} = x_{k-1}$. We have $x_{k-1} = A^n_{k-1} - t$ for some $t \ge 1$. We claim that $X = (A^n_0, \ldots, A^n_{k-2}, x_{k-1}) \to (A^{n-t}_0, \ldots, A^{n-t}_{k-2}, A^{n-t}_{k-1}) \in W$ is a legal type **II** move for $t < n$; and $X \to \Phi$, for $t \ge n$.

For $t < n$ we have by (9) (with $m = n-t$), $x_{k-1} - A^{n-t}_{k-1} = A^n_{k-1} - A^{n-t}_{k-1} - t > A^n_{k-2} - A^{n-t}_{k-2}$. Then by Lemma 1,

$$x_{k-1} - A^{n-t}_{k-1} = A^n_{k-1} - A^{n-t}_{k-1} - t = 2(A^n_{k-2} - A^{n-t}_{k-2}) + \sum_{i=0}^{k-3}(A^n_i - A^{n-t}_i),$$

which satisfies (4). If $t \ge n$, then by Lemma 1, $x_{k-1} \le A_{k-1} - n = 2A^n_{k-2} + \sum_{i=0}^{k-3} A^n_i$, so $X \to \Phi$ satisfies (4).

(ii) $j < k$. (Recall that $x_{j-1} < A^n_{j-1}$.) We first dispose of two subcases.

a. If there is $r > j-1$ with $x_r > A^n_{j-1}$ and $x_i \ne A^n_{j-1}$ for all $i \ge 0$, then make the type **I** move $x_r \to A^n_{j-1}$ and $x_i \to 0$ for all $i \ge j-1$, $i \ne r$, resulting in $(A^n_0, \ldots, A^n_{j-1}) \in W$.

b. If $x_i \le A^n_{j-1}$ for all $i > j-1$, then $X = (A^n_0, \ldots, A^n_{j-2}, x_{j-1}, \ldots, x_{k-1}) \to \Phi$ is a legal move. Indeed, $x_{j-1} > A^n_{j-2}$; and Lemma 1 implies $A^n_{j-2} \ge n$. Hence,

$$x_{k-1} \le A^n_{j-1} = 2A^n_{j-2} + \sum_{i=0}^{j-3} A^n_i + n < 2x_{j-1} + \sum_{i=0}^{j-3} A^n_i \le 2x_{k-2} + \sum_{i=0}^{k-3} x_i,$$

is a legal type **II** move by (4).

So we may assume that X has the form

$$X = (A^n_0, \ldots, A^n_{j-2}, x_{j-1}, \ldots, A^n_{j-1}, \ldots, A^n_{j+s}, x_t, \ldots, x_{k-1}),$$

where each A^n_{j+i} appears for all $i \le s$, $s \ge -1$, and possibly also some intermediate $x_i \ne A^n_r$, but A^n_{j+s+1} does not appear. Here are the two final subcases.

c. $x_{k-1} > A^n_{j+s+1}$. Then move, $x_{k-1} \to A^n_{j+s+1}$, $x_{j-1}, \ldots, x_t, \ldots, x_{k-2} \to 0$ (type **I** move), resulting in the position $(A^n_0, \ldots, A^n_{j+s+1}) \in W$.

d. $x_{k-1} < A^n_{j+s+1}$. Then $X \to \Phi$ is a legal type **II** move. Indeed, $x_{j-1} > A^n_{j-2} \ge n$, so

$$x_{k-1} < A^n_{j+s+1} = 2A^n_{j+s} + \sum_{i=0}^{j+s-1} A^n_i \le 2x_{k-2} + \sum_{i=0}^{k-3} x_i.$$

We have shown that $W = \mathcal{P}$. ■

4. Does Frankenstein have a Polynomial Strategy?

The *statement* of Theorem 3 enables one to decide whether any given position X of the form (3) of Frankenstein is a P-position or an N-position, and the *proof* clearly indicates a winning move from any N-position. These two things together constitute a winning strategy for the game.

Given any position X of the form (3) of Frankenstein. To decide whether $X \in \mathcal{P}$ or $X \in \mathcal{N}$, we have to compute the entries of $\mathcal{A}$ only up to the first encounter of x_0. Thus it is readily seen that Theorem 2(ii) implies that A^n_j has to be computed only for $n \le x_0(\phi - 1)$; and (5) implies that $j < \frac{1}{2}\log_\phi(\sqrt{5}(x_0+1)) - 1$. So the array has to be computed only up to $\Theta(x_0)$, which implies a strategy computation linear in x_0, which looks good.

However, the input size for Frankenstein is $\Theta(\sum_{i=0}^{k-1} \log x_i)$. So unless either k or x_{k-1} are exponentially larger than x_0, the indicated strategy is actually exponential. But only the construction of the table needs exponential time and, in fact, exponential space. The rest of the algorithm embodied in the proof of Theorem 3 is polynomial. A winning strategy is polynomial only if both of its parts are polynomial.

It follows from [Fra1985] that the computation of the representation of a positive integer N over the numeration system $\mathcal{U}$ can be done by a greedy Euclidean algorithm, namely always dividing the remainder r (initially: $r = N$), by the largest basis element $u_n \le r$. This is a polynomial process. In particular, expressing a game position X of the form (3) over $\mathcal{U}$ can be done in polynomial time. It can then be observed in linear time whether or not x_0 is reduced, and all the other steps of the winning algorithm indicated in the proof of Theorem 3 can also be done in polynomial time. Thus the numeration system $\mathcal{U}$ actually enables us to formulate a polynomial strategy for Frankenstein — not only to decide whether it has or doesn't have one.

The game Frankenstein proposed here belongs to the family of *succinct* games, i.e., their input size is logarithmic. Normally an extra effort is required for showing that such games have a polynomial strategy. Different families of succinct games seem to require different methods of strategy computations.

For example, in *octal* games, invented by Guy and Smith [GuSm1956], a linearly ordered string of beads may be split and or reduced according to rules encoded in octal. See also [BCG1982, Ch. 4], [Con1976, Ch. 11]. The standard method for showing that an octal game is polynomial, is to demonstrate that its *Sprague-Grundy* function (the 0s of which constitute the set of P-positions) is periodic.

Periodicity has been established for a number of octal games. Some of the periods and or preperiods may be very large; see [GaPl1989]. Another way to establish polynomiality is to show that the Sprague-Grundy function values obey some other simple rule, such as forming an arithmetic sequence, as for Nim.

For the present class of pebbling games, polynomiality was established by a non-standard method. An arithmetic procedure, based on a class of special numeration systems, was the key to polynomiality. In [Fra1998] a game was proposed and analysed, and another numeration system was used there to establish polynomiality. For Wythoff's game [Wyt1907], [Cox1953], [YaYa1967], the Zeckendorf numeration system [Zec1972] can be used to establish polynomiality. But for Wythoff's game, this can be done also using the integer value function. From Theorem 2(iv) it follows that this cannot be done for Frankenstein. In [Fra1998] it was also proved that the integer value function cannot be used to establish polynomiality for the game defined there. But the question remains whether there or here, there is some *polynomial* algorithm not based on numeration systems.

5. Epilogue

We recap the main properties of the array $\mathcal{A}$.

(a)

$$A_0^n = \operatorname{mex}\{A_j^i : 0 \le i < n,\ j \ge 0\}\ (n \ge 0),$$

$A_1^n = 2A_0^n + n\ (n \ge 0), \quad A_j^n = 3A_{j-1}^n - A_{j-2}^n\ \ (j \ge 2,\ n \ge 0).$ (The definition.)

(b) For all $j, n \in \mathbb{Z}^+$, $A_j^n = 2A_{j-1}^n + A_{j-2}^n + \cdots + A_0^n + n$. (Lemma 1.)

(c) $\mathcal{A}$ is a splitting of $\mathbb{Z}^+$: every positive integer appears precisely once in $\mathcal{A}$. Moreover, for every $j \ge 0$, the column A_j consists precisely of all positive integers whose representation ends in j 0s. In particular, A_0^n is reduced for all $n \in \mathbb{Z}^+$. (Theorem 1.)

(d) (i) For $j, n \in \mathbb{Z}^+$, $A_j^n = \lfloor A_{j-1}^n \phi^2 \rfloor + 1 = L(A_{j-1}^n)$. (ii) For all $n \ge 1$, $A_0^n = \lfloor (n-1)\phi \rfloor + 1$ is reduced. (iii) For all $n \ge 0$, all $j \ge 0$ we have, $A_j^{n+1} - A_j^n \in \{f_{2j}, f_{2j+1}\}$, and for fixed j, each of f_{2j}, f_{2j+1} is assumed for infinitely many n. Moreover, for all n for which $A_0^{n+1} - A_0^n = f_0$ (respectively f_1), we also have for all j, $A_j^{n+1} - A_j^n = f_{2j}$ (respectively f_{2j+1}). (iv) Let $j \ge 1$. There are no real numbers α, γ, such that for all $n \ge 1$, $A_j^n = \lfloor n\alpha + \gamma \rfloor$.

The numeration system $\mathcal{U}$ was used both for proving the most important of these properties, and for deciding the polynomiality question of the strategy of Frankenstein.

The reason our title contains the term "arrays", whereas we have presented only a single array, is that we allude to an infinite family of arrays, based on some linear recurrence of the form

$$u_0 = 1, \quad u_n = b_1 u_{n-1} + \cdots + b_m u_m, \tag{10}$$

where the b_i are constants, except that $b_1 = b_1(n)$ may depend on n, with given initial integer values $u_{-m+1}, \ldots, u_{-1}$. If

$$1 \le b_m \le \cdots \le b_1, \tag{11}$$

then there is also an associated numeration system [Fra1985]. Replacing in (10) the elements u_j by columns A_j the recurrence is used to construct $\mathcal{A}$ (with possibly a special construction for the first initial values of j).

A first — to my knowledge — "Fibonacci array" has been defined in [Sto1977]. Other "Stolarsky arrays" were defined in papers such as [Kim1995] and [FrKi1994], and there are infinitely many such arrays. But we have not seen any applications of these arrays. Perhaps the present use for a winning strategy to a new class of games is the first application? Is there a natural infinite family of combinatorial games, matching the infinite family of arrays? And what's the nature of these arrays and their uses if (11) is violated?

It seems that the array defined here was not given before. Its antidiagonal hasn't appeared in [Slo1998] until we sent it in there recently; and its columns A_j and its rows A^n do not seem to appear in it for $j > 1$ and $n > 3$. As we remarked just prior to the statement of Theorem 2, the rows of the present array are "even Fibonacci numbers".

Several comments can be made about recurrences such as (2). We shall briefly relate to two items.

(I) The second recurrence of (2) can be considered to be the recurrence of the convergents of the *quasiregular* (or *semiregular*— halbregelmässig) continued fraction

$$3 + \cfrac{-1}{3 + \cfrac{-1}{3 + \cfrac{-1}{\ddots}}}$$

In [Per1950, Ch. 5] it is shown that every quasiregular continued fraction converges. In the present case it converges to ϕ^2. Many of the above properties of $\mathcal{A}$ can be deduced from this observation; also other properties not mentioned above, such as $u_n^2 - u_{n-1}u_{n+1} = 1$ for all n, and somewhat more complicated identities for elements in the other rows of $\mathcal{A}$.

(II) In [BBDD1998], the authors quote [BSS1993]: "...the recurrence $f_{n+1} = 6f_n - f_{n-1}$ cries out for a combinatorial interpretation. Finding this interpretation is an open problem." [BBDD1998] gives such an interpretation. We remark that in [Fra1985, §4] a class of regular (simple) continued fractions is defined whose convergents satisfy recurrences including the above. In particular, the numerators of the even-indexed convergents of the simple continued fraction

$$\sqrt{2} = [1, 2, 2, \ldots] = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}$$

constitute the sequence $1, 7, 41, 239, \ldots$ with initial values $f_1 = 1$, $f_2 = 7$ considered in [BBDD1998]. Needless to say that each such recurrence also defines an exotic numeration system. Perhaps these facts constitute a "combinatorial interpretation".

The game Frankenstein is superficially reminiscent of the game of *Welter*, analyzed in [Con1976, Ch. 13]. The terminology "spinster" was introduced there. Welter is played on a semi-infinite strip with a finite number of coins, at most one per square, and the squares are numbered with the nonnegative integers $0, 1, 2, \ldots$ from the left end of the strip. A move consists of selecting a single coin and shifting it to an unoccupied square with lower number. The player first unable to move loses, and the opponent wins. The winning strategy is intricate. Moreover, it seems very difficult to generalize Welter. The game proposed here is not a generalization of Welter, but the moves are reminiscent of several moves of Welter taken simultaneously.

Acknowledgment

At the FUN conference I presented a paper entitled "Heap games and numeration systems". I sent it for publication two weeks before the conference, since at that time there were no plans for a special conference issue, to the best of my knowledge. It appeared in expanded form, with a modified title, in [Fra1998]. The present paper, though new in content, is nevertheless close in spirit to the earlier one. I thank the editors for considering it for the special issue.

References

1. [BBDD1998] E. Barcucci, S. Brunetti, A. Del Lungo and F. Del Ristoro [1998], A combinatorial interpretation of the recurrence $f_{n+1} = 6f_n - f_{n-1}$, *Discrete Math.* **190**, 235–240.

2. [BCG1982] E.R. Berlekamp, J.H. Conway and R.K. Guy [1982], *Winning Ways* (two volumes), Academic Press, London.

3. [BSS1993] J. Bonin, L. Shapiro and R. Simion [1993], Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths, *J. Statist. Plann. Inference* **34**, 35–55.

4. [Con1976] J.H. Conway [1976], *On Numbers and Games*, Academic Press, London.

5. [Cox1953] H.S.M. Coxeter [1953], The golden section, phyllotaxis and Wythoff's game, *Scripta Math.* **19**, 135–143.

6. [Fra1969] A.S. Fraenkel [1969], The bracket function and complementary sets of integers, *Canadian J. Math.* **21**, 6–27.

7. [Fra1985] A.S. Fraenkel [1985], Systems of numeration, *Amer. Math. Monthly* **92**, 105–114.

8. [Fra1998] A.S. Fraenkel [1998], Heap games, numeration systems and sequences, *Ann. of Combinatorics* **2**, 197–210.

9. [Fra≥2001] A.S. Fraenkel [≥2001], *Adventures in Games and Computational Complexity*, to appear in *Graduate Studies in Mathematics*, Amer. Math. Soc., Providence, RI.

10. [FrKi1994] A.S. Fraenkel and C. Kimberling [1994], Generalized Wythoff arrays, shuffles and interspersions, *Discrete Mathematics* **126**, 137–149.

11. [GaPl1989] A. Gangolli and T. Plambeck [1989], A note on periodicity in some octal games, *Internat. J. Game Theory* **18**, 311–320.

12. [GuSm1956] R.K. Guy and C.A.B. Smith [1956], The G-values of various games, *Proc. Camb. Phil. Soc.* **52**, 514–526.

13. [HaWr1989] G.H. Hardy [1989], *An Introduction to the Theory of Numbers*, 5th ed., Clarendon Press, Oxford.

14. [Kim1995] C. Kimberling [1995], Stolarsky interspersions, *Ars Combinatoria* **39**, 129–138.

15. [Per1950] O. Perron [1950], *Die Lehre von den Kettenbrüchen*, Chelsea, New York.

16. [Slo1998] N.J.A. Sloane [1998], Sloane's On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences/ .

17. [Sto1977] K.B. Stolarsky [1977], A set of generalized Fibonacci sequences such that each natural number belongs to exactly one, *Fibonacci Quart.* **15**, 224.

18. [Wyt1907] W.A. Wythoff [1907], A modification of the game of Nim, *Nieuw Arch. Wisk.* **7**, 199–202.

19. [YaYa1967] A.M. Yaglom and I.M. Yaglom [1967], *Challenging Mathematical Problems with Elementary Solutions*, translated by J. McCawley, Jr., revised and edited by B. Gordon, Vol. II, Holden-Day, San Francisco.

20. [Zec1972] E. Zeckendorf [1972], Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, *Bull. Soc. Roy. Sci. Liège* **41**, 179–182.