Polyominoes with maximum convex hull

Sascha Kurz

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Bachelor thesis in mathematics
adviced by Prof. Dr. A. Kerber
University of Bayreuth
and by Prof. Dr. H. Harborth
Technical University of Braunschweig

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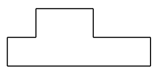
Preface

The first time I came along with **polyominoes** was in 1998 when I read a do-it-yourself story about a little worm named Heiner Würmeling [104]. I will tell a short version of this story in own words:

Heiner Würmeling, his wife Amelia and baby Wermentrude just recovered their procession into a new lair. But Amelia was not amused seeing the bath room. ”Heiner! Come to me!” Heiner reluctantly wormed one's way towards the bath room leaving his comfortable armchair.“My dear, what's wrong?” ”Didn't the builder promised to tile the whole bath? NOTHING, NOTHING is done yet and in the corner there's standing a big box with tiles!” ”I'll phone him.” The builder apologized ”Sorry chief, we had a little problem. Have a look at the funny tiles your wife ordered, we can't match them without leaving holes.” Heiner mashed ”That's ridiculous! Why didn't you form a rectangle?” ”That's exactly what we tried, but with no success.”. ”Ridiculous! I'll do it by myself!”.

Can you do it? Here is the tile  .

Two days later Heiner gave up and called his friend Albert Wurmstein working for the patent office. Hearing Heiner's story Albert said after a great time of thinking ”Your tile is some kind of a polyomino, that's a plane figure of equal sized squares neighbored edge-to-edge. In 1969 Klarner defined the order of a polyomino as the minimum number of copies of a polyomino filling a rectangle." Albert told Heiner that his polyomino has order 78 and gives him a solution to fill a rectangle. Heiner run into the kitchen and proudly said "Amelia, I have solved the tile problem. All I have to do is taking 78 tiles and build a rectangle." "Wish you a lot of fun", Amelia replied. Heiner went into the bathroom and had a look at the description of the box

Combinatoric Ceramic Factory
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During the winter semester 2002/2003 I took part in a course named Discrete Geometry lectured by Prof. Dr. H. Harborth at the Technical University of Braunschweig. A lot of unsolved problems concerning the field of Discrete Geometry were treated in this course. Of those I took two problems about polyominoes which I was able to solve. The solution to the first problem is recently submitted [77] and the second problem is the topic of this bachelor thesis.

Beside from proving a few theorems about maximum convex hulls of polyominoes we find it interesting to give the known exact numbers of some kinds of polyominoes in the appendix. And I also like to give an overview of the literature about enumerating polyominoes in the bibliography.

Acknowledgements

I would like to thank Prof. Dr. Adalbert Kerber and Prof. Dr. Heiko Harborth for looking after this thesis. For reading the manuscript I am thankful to Sonja Ertel, Christel Jantos, Nina Jantos, Matthias Koch, Heike Kurz, Frank Liczba, Armin Rund, Tobias Schneider, Steffi Sutter and Stefan Tuffner.

Sascha Kurz

Declaration

This is to certify that I wrote this thesis on my own and that the references include all the sources of information I have utilized. This thesis is freely available for study purposes.

Bayreuth, 17. October 2003 ....

1 Introduction

A **polyomino** is a connected interior-disjoint union of axis-aligned unit squares joined edge-to-edge, in other words, an edge-connected union of cells in the planar square lattice. There are at least three ways to define two polyominoes as equivalent, namely factoring out just translations (fixed polyominoes), rotations and translations (chiral polyominoes), or reflections, rotations and translations (free polyominoes). In the literature polyominoes are sometimes named **animals** or one speaks of the **cell-growth problem** [74, 98]. For the origin of polyominoes we quote Klarner [75]: "Polyominoes have a long history, going back to the start of the 20th century, but they were popularized in the present era initially by Solomon Golumb [56, 57, 58, 59, 60, 61, 62, 55], then by Martin Gardner in his *Scientific American* columns." To give an illustration of polyominoes Figure 1.1 depicts the free polyominoes consisting of at most 5 unit squares.

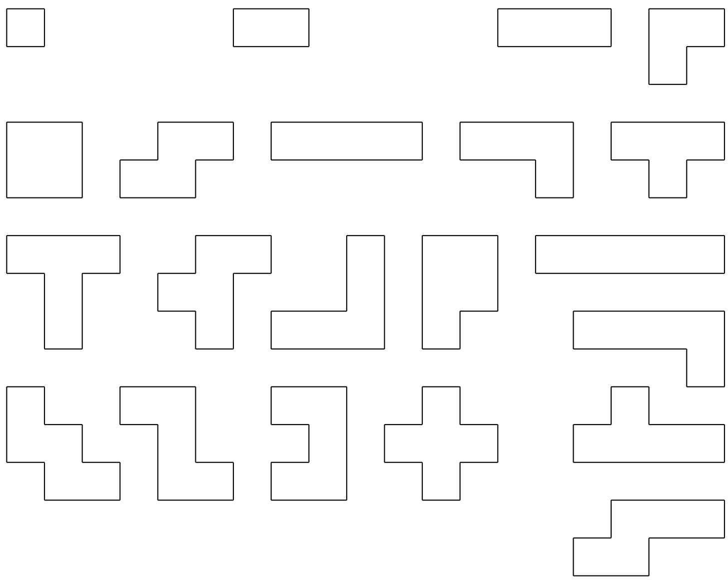

Figure 1.1. Polyominoes with at most 5 squares.

There are several generalizations of polyominoes i.e. polyiamonds (edge-to-edge unions of unit equilateral triangles) [8, 53, 66, 86, 106], polyhexes (edge-to-edge unions of unit regular hexagons) [6, 52, 53, 86], polyabolos (edge-to-edge unions of unit right isosceles triangles) [52], polycubes (face-to-face unions of unit cubes) [3, 87], etc. One can also define polyominoes as connected systems of cells on archimedean tesselations [9]. In this thesis we regard a d-dimensional polyomino as a facet-to-facet connected system of d-dimensional unit hypercubes. If nothing else is mentioned the term polyominoes is used for free polyominoes.

Before we introduce the theorems of this thesis we would like to mention a few applications and problems for polyominoes. The term cell-growth problem certainly suggests applications in medicine and biology. Polyominoes are useful for the *Ising Model* [24] modelling neural networks, flocking birds, beating heart cells, atoms, protein folds, biological membrane, social behavior, etc. Further applications of polyominoes lie in the fields of chemistry and physics. As problems concerning polyominoes one might mention counting polyominoes [1,2,4,5,10-23,25-50,54,64,67,69-72,76,78-97,99-101,105,107,108,110,111, 113], generating polyominoes [109, 112], achievement games [6, 7, 8, 9, 51, 66] and extremal animals [65, 68, 73, 77]. In Appendix A we give tables for the exact number of some types of polyominoes for small numbers of cells.

This thesis is about polyominoes with maximum convex hull. At the end of this introduction we would like to mention the proven theorems.

In [73] it is proved that the area of the convex hull of any facet-to-facet connected system of n unit squares is at most $n + \frac{1}{2}\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$. We will prove their conjuncture for the d-dimensional case.

Theorem 1. The d-dimensional volume of the convex hull of any facet-to-facet connected system of n unit hypercubes is at most

$$\sum_{I \subset \{1,\ldots,d\}} \frac{1}{|I|!} \prod_{i \in I} \left\lfloor \frac{n-2+i}{d} \right\rfloor .$$

The authors of [73] also asked for the number of different polyominoes with

n cells and maximum area of the convex hull. We enumerated them for the $\mathbf{R^d}$.

Theorem 2. The number $c_2(n)$ of polyominoes in $\mathbf{R^2}$ with maximum area of the convex hull is given by

$$n \equiv 0 \;\; mod \; 4 : c_2(n) = \frac{n^3 - 2n^2 + 4n}{16},$$

$$n \equiv 1 \; mod \; 4 : c_2(n) = \frac{n^3 - 2n^2 + 13n + 20}{32},$$

$$n \equiv 2 \; mod \; 4 : c_2(n) = \frac{n^3 - 2n^2 + 4n + 8}{16},$$

$$n \equiv 3 \;\; mod \; 4 : c_2(n) = \frac{n^3 - 2n^2 + 5n + 8}{32}.$$

Theorem 3. The number $c_d(n)$ of polyominoes in $\mathbf{R^d}$ with maximum volume of the convex hull is given by

$$n \equiv a \; mod \; d \; : \; c_d(n) = \binom{\lceil \frac{n-a}{2d} \rceil + d - a}{d - a + 1} \binom{\lceil \frac{n+d-a}{2d} \rceil + a - 2}{a - 1}$$

with $0 < a \leq d$ and $d \geq 3$.

Knowing the maximum area of the convex hull, one can also asked for which numbers a there is a polyomino with n cells and an area a of the convex hull. For the 2-dimensional case the situation is fully described by the following statement.

Theorem 4. The existence of a 2-dimensional polyomino consisting of n cells with area a of the convex hull is equivalent to $a \in A_n$ with

$$A_n = \left\{ n + \frac{m}{2} \middle| m \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor, m \in \mathbb{N}_0 \right\} - \left\{ n + \frac{1}{2} \middle| if \, n + 1 \, is \, prime \right\} .$$

2 Proof of Theorem 1

Will first prove the theorem for $d = 2, 3$ before we prove it in any dimension.

Definition 2.1.

$$f_2(l_1, l_2, v_1, v_2) := 1 + (l_1 - 1) + (l_2 - 1) + \tfrac{(l_1-1)(l_2-1)}{2} + v_1 + v_2 + \tfrac{v_1(l_2-1)}{2} + \tfrac{v_2(l_1-1)}{2},$$

$$f_3(l_1, l_2, l_3, v_1, v_2, v_3) := 1 + (l_1 - 1) + (l_2 - 1) + (l_3 - 1) + \tfrac{(l_1-1)(l_2-1)}{2} + \tfrac{(l_1-1)(l_3-1)}{2} + \tfrac{(l_2-1)(l_3-1)}{2} + \tfrac{(l_1-1)(l_2-1)(l_3-1)}{6} + \tfrac{v_1(l_2-1)}{2} + \tfrac{v_1(l_3-1)}{2} + \tfrac{v_2(l_1-1)}{2} + \tfrac{v_2(l_3-1)}{2} + \tfrac{v_3(l_1-1)}{2} + \tfrac{v_3(l_2-1)}{2} + \tfrac{v_1(l_2-1)(l_3-1)}{6} + \tfrac{v_2(l_1-1)(l_3-1)}{6} + \tfrac{v_3(l_1-1)(l_2-1)}{6} + \tfrac{v_1v_2(l_3-1)}{6} + \tfrac{v_1v_3(l_2-1)}{6} + \tfrac{v_2v_3(l_1-1)}{6} + v_1 + v_2 + v_3 + \tfrac{v_1v_2}{2} + \tfrac{v_1v_3}{2} + \tfrac{v_2v_3}{2} + \tfrac{v_1v_2v_3}{6}.$$

We number the standard coordinate axes of $\mathbf{R^d}$ by $1, \ldots, d$. Every d-dimensional polyomino has a smallest surrounding box with side length $l_1, \ldots, l_d$, where l_i is the length in direction i. If we build up a polyomino cell by cell then one of the l_i increases by 1 or none of the l_i increases. In the second case we increase v_i by one, where the new hypercube has a facet-neighbor in direction of axis i. If N is the set of axis-directions of facet-neighbors of the new hypercube, then v_i is increased by one for only one $i \in N$. Since at this position there is the possibility to choose, we must live with the fact that there might be different tuples $(l_1, \ldots, l_d, v_1, \ldots, v_d)$ for one polyomino. We define $v_1 = \ldots = v_d = 0$ for the polyomino consisting of a single hypercube. This definition of l_i and v_i leads to the following equation

$$n = 1 + \sum_{i=1}^{d} (l_i - 1) + \sum_{i=1}^{d} v_i\,. \qquad (\dagger)$$

Lemma 2.2. The area of the convex hull of a 2-dimensional polyomino is bounded by $f_2(l_1, l_2, v_1, v_2)$.

Proof. We prove the statement by induction on n, using equation †. For $n = 1$ only $l_1 = l_2 = 1$, $v_1 = v_2 = 0$ is possible. Since $f_2(1, 1, 0, 0) = 1$ is clearly the area of the convex hull of a unit square, the induction start is

made. Now we may assume that the lemma is proved for all possible tuples (l_1, l_2, v_1, v_2) with $1 + \sum_{i=1}^{2}(l_i - 1) + \sum_{i=1}^{2} v_i = n - 1$.
We consider the growth of the l_i, v_i and the area a of the convex hull by adding the n-th square.

(i) l_1 is increased by 1:

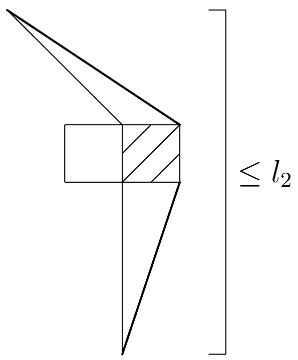

Figure 2.1. Increasing l_1.

We depict (see Figure 2.1) the new square by 3 diagonal lines. Since l_1 is increased the new square must have a left or a right neighbor, without loss of generality it has a left neighbor. The new square contributes at most 2 (thick) lines to the convex hull of the polyomino. As we draw lines from the neighbor square to the endpoints of the new lines we see that the growth is at most $1 + \frac{l_2-1}{2}$. One for the new square and the rest for the triangles. Since $f_2(l_1 + 1, l_2, v_1, v_2) - f_2(l_1, l_2, v_1, v_2) = 1 + \frac{l_2-1}{2}$ the induction step follows.

(ii) v_1 is increased by one:

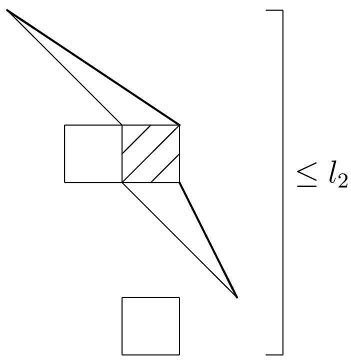

Figure 2.2. Increasing v_1.

Again we depict (Figure 2.2) the new square by 3 diagonal lines. Without loss of generality the new square has a left neighbor, and square contributes at most 2 (thick) lines to the convex hull of the polyomino. As l_1 is not increased there must be a square in the same column as the new square. By the same argument as in (i) the growth is less than $1 + \frac{l_2-1}{2}$. With $f_2(l_1, l_2, v_1 + 1, v_2) - f_2(l_1, l_2, v_1, v_2) = 1 + \frac{l_2-1}{2}$ the induction step follows.

(iii) l_2 or v_2 is increased by one:
Due to symmetry this case is analogue to (i) or (ii).□

Lemma 2.3. The area of the convex hull of a polyomino with n unit squares is at most $n + \frac{1}{2}\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor$.

Proof.

(i) If $v_1 > 0$ we decrease v_1 by one and increase l_1 by one, and consider $f_2(l_1+1, l_2, v_1-1, v_2) - f_2(l_1, l_2, v_1, v_2) = \frac{v_2}{2}$. Since due to the proof of Lemma 2.2 the maximum $f_2(l_1, l_2, v_1, v_2)$ cannot be achieved for $v_1 \neq 0$ this substitution increases the area. So we conclude with an symmetry argument $v_1 = v_2 = 0$.

(ii) Since we can renumber the coordinate axes we can assume that $l_1 \leq l_2$ holds. If $l_2 - l_1 \geq 2$ we increase l_1 and decrease l_2 by one. Considering $f_2(l_1+1, l_2-1, 0, 0) - f_2(l_1, l_2, 0, 0) = \frac{l_2-l_1}{2} > 0$ shows that the area of the convex hull increases by this substitution. So we conclude $l_2 - l_1 \leq 1$.

(iii) Using equation † we get $l_1 = \left\lfloor\frac{n+1}{2}\right\rfloor$, $l_2 = \left\lfloor\frac{n+2}{2}\right\rfloor$. Inserting in Lemma 2.2 yields $f_2(l_1, l_2, v_1, v_2) \leq n + \frac{1}{2}\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor$.

(iv) The proposed value is attained for the example in Figure 2.3.

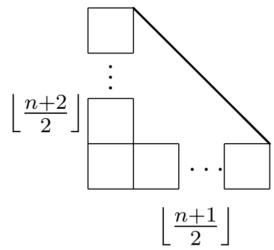

Figure 2.3. 2-dimensional polyomino with maximum convex hull.

□

Lemma 2.4. The volume of the convex hull of a 3-dimensional polyomino is bounded by $f_3(l_1, l_2, l_3, v_1, v_2, v_3)$.

Proof. We prove the statement by induction on n, using equation †. For $n = 1$ only $l_1 = l_2 = l_3 = 1$, $v_1 = v_2 = v_3 = 0$ is possible. Since $f_3(1,1,1,0,0,0) = 1$ is clearly the volume of the convex hull of a single unit cube, the induction start is made. Now we may assume that the lemma is proved for all possible tuples $(l_1, l_2, l_3, v_1, v_2, v_3)$ with $1 + \sum_{i=1}^{3}(l_i - 1) + \sum_{i=1}^{3} v_i = n - 1$.
We consider the growth of the l_i, v_i and the volume a of the convex hull by adding the n-th cube.

(i) l_1 is increased by 1:

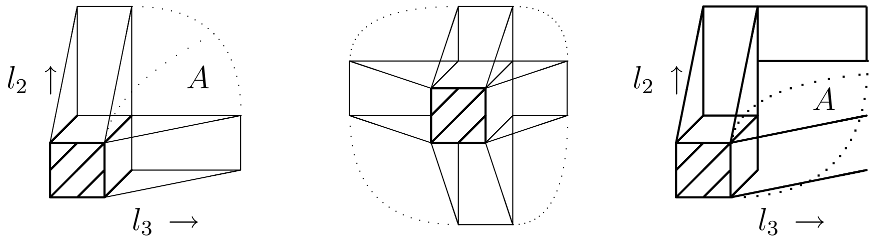

Figure 2.4. Increasing l_1 in the 3-dimensional case.

Consider the left picture in Figure 2.4. The new cube (depicted in Figure 2.4 by 3 diagonal thick lines) contributes itself a volume of 1. As in the proof of Lemma 2.2 we draw the lines of the convex hull of the new cube, and the cube below. If we look at the facets in direction of coordinate axis 3 we have a contribution of at most $\frac{1\times 1\times(l_3-1)}{2}$. Analogue in direction 2. Beside from this there is an area A which contributes $\frac{1\times A}{3}$. By a look at the proof of Lemma 2.2 we get $A \le \frac{(l_2-1)(l_3-1)}{2} + \frac{v_2(l_3-1)}{2} + \frac{v_3(l_2-1)}{2} + v_2 + v_3$ (the two summands $l_2 - 1$ and $l_3 - 1$ are already considered before). In total we get a maximal contribution of $1+\frac{l_2-1}{2}+\frac{l_3-1}{2}+\frac{(l_2-1)(l_3-1)}{6}+\frac{v_2(l_3-1)}{6}+\frac{v_3(l_2-1)}{6}+\frac{v_2}{3}+\frac{v_3}{3}$ by adding the new cube. Now consider $f_3(l_1+1, l_2, l_3, v_1, v_2, v_3) - f_3(l_1, l_2, l_3, v_1, v_2, v_3)$ $= 1 + \frac{l_2-1}{2} + \frac{l_3-1}{2} + \frac{(l_2-1)(l_3-1)}{6} + \frac{v_2(l_3-1)}{6} + \frac{v_3(l_2-1)}{6} + \frac{v_2}{2} + \frac{v_3}{2} + \frac{v_2 v_3}{6}$. Since this difference is not less than the maximal contribution of the new cube we have the induction step.

The two pictures on the right in Figure 2.4 show, that the estimations also hold, when the new cube does not lie as special as in the picture on the left. In the picture on the left in Figure 2.4 we distinguish the terms $(l_2-1)+(l_3-1)$ and the area A. The first term is multiplied by a factor of $\frac{1}{2}$ and the second term is multiplied by a factor of $\frac{1}{3}$. The rightmost picture in Figure 2.4 shows that this distinction must not be the right one in general. But because $\frac{1}{2} > \frac{1}{3}$ we are on the safer side. One should spend a little thought on the non-deterministic definition of the v_i. We need the v_i for estimating the area A. In the case where a cube, with neighbors in 2 or 3 directions and increasing a v_i, is added there is a choose which v_i is increased by one. If the 2 directions are direction 2 and 3, then we have seen in the proof of Lemma 2.2, that the estimation for A is correct in any case of choose. If there is a neighbor in direction 1 the area in the 2×3-plane is not increased. For our purpose it doesn't matter to get the best possible estimation, we only need one that holds. So we have seen that the estimations hold for any choose of the v_i.

(ii) v_1 is increased by one:
Similar to the case (ii) in the proof of Lemma 2.2, we use the same consideration as in case (i). Since on one side, in direction 2 or 3, there is already a cube on the level of the new cube the growth is less than $1 + \frac{l_2-1}{2} + \frac{l_3-1}{2} + \frac{(l_2-1)(l_3-1)}{6} + \frac{v_2(l_3-1)}{6} + \frac{v_3(l_2-1)}{6} + \frac{v_2}{3} + \frac{v_3}{3}$. Now we consider $f_3(l_1, l_2, l_3, v_1+1, v_2, v_3) - f_3(l_1, l_2, l_3, v_1, v_2, v_3) = 1 + \frac{l_2-1}{2} + \frac{l_3-1}{2} + \frac{(l_2-1)(l_3-1)}{6} + \frac{v_2(l_3-1)}{6} + \frac{(l_2-1)v_3}{6} + \frac{v_2}{2} + \frac{v_3}{2} + \frac{v_2 v_3}{6}$. Since the last difference is not less than the maximum growth by the new cube we have the induction step.

(iii) l_2, l_3, v_2 or v_3 is increased by one:
Due to symmetry this is analogue to case (i) or (ii). □

Lemma 2.5. The volume of the convex hull of a polyomino consisting of n unit cubes is at most

$$1 + \left\lfloor \frac{n-1}{3} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor + \frac{\left\lfloor \frac{n-1}{3} \right\rfloor \left\lfloor \frac{n}{3} \right\rfloor}{2} + \frac{\left\lfloor \frac{n-1}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor}{2} + \frac{\left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor}{2} + \frac{\left\lfloor \frac{n-1}{3} \right\rfloor \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor}{6}$$

Proof.
(i) If $v_1 > 0$ we decrease v_1 and increase l_1 by one. Now we consider the difference of $l_3(l_1+1, l_2, l_3, v_1-1, v_2, v_3)$ and $l_3(l_1, l_2, l_3, v_1, v_2, v_3)$, which is 0. Since due to the proof of Lemma 2.4 the maximum $f_3(l_1, l_2, l_3, v_1, v_2, v_3)$ cannot be achieved for $v_1 \neq 0$ this substitution increases the maximum possible volume. So we conclude by an symmetry argument $v_1 = v_2 = v_3 = 0$.

(ii) Since we can renumber the coordinate axes we can assume that $l_1 \leq l_2 \leq l_3$ holds. If $l_3 - l_1 \geq 2$ we increase l_1 and decrease l_3 by one. Considering $f_3(l_1+1, l_2, l_3-1, 0, 0, 0) - f_3(l_1, l_2, l_3, 0, 0, 0) = \frac{(l_3-l_1-1)(l_2+2)}{6} > 0$ shows that the area of the convex hull increases by this substitution. So we conclude $|l_i - l_j| \leq 1$.

(iii) Using equation † we get $l_i = \left\lfloor \frac{n+1+i}{3} \right\rfloor$. Inserting in Lemma 2.4 yields the proposed formula.

(iv) An extremal configuration consist of 3 pairwise orthogonal linear arms of $\left\lfloor \frac{n-2+i}{3} \right\rfloor$ cubes ($i = 1 \dots 3$) joined at a central cube. □

Now we use the same structure of the lemmas for the 2- and 3-dimensional case for the d-dimensional case.

Definition 2.6.

$$f_d(l_1, \dots, l_d, v_1, \dots, v_d) := \sum_{I \subset \{1,\dots,d\}} \frac{1}{|I|! 2^{d-|I|}} \sum_{b=0}^{2^d-1} \prod_{i \in I} q_{b,i}$$

with $b = \sum_{j=1}^{d} b_j 2^{j-1}$, $b_j \in \{0, 1\}$, $q_{b,i} = \begin{cases} l_i - 1 & for \quad b_i = 0\,, \\ v_i & for \quad b_i = 1\,. \end{cases}$

Lemma 2.7. The d-dimensional volume of the convex hull of a polyomino consisting of n unit hypercubes is bounded by $f_d(l_1, \dots, l_d, v_1, \dots, v_d)$.
Proof. We prove the statement by double induction on d and n, using equation †. The cases $d = 2, 3$ are already treated, so we can first induct on d. For $n = 1$ only $l_i = 1$, $v_i = 0$ $i \in \{1, \dots, d\}$ is possible. Since $f_d(1, \dots, 1, 0, \dots, 0) = 1$ is clearly the volume of the convex hull of a single unit hypercube, the induction step is made. Now we may assume that the lemma is proved for all possible tuples $(l_1, \dots, l_d, v_1, \dots, v_d)$ with $1 + \sum_{i=1}^{d} (l_i - 1) + \sum_{i=1}^{d} v_i = n - 1$.
We consider the growth of the l_i, v_i and the volume a of the convex hull by adding the n-th hypercube.

(i) l_1 is increased by one:
The new hypercube has itself a volume of 1. Similar to the proof of Lemma 2.4 there is a hypervolume $H \le f_{d-1}(l_2, \dots, l_d, v_2, \dots, v_d) - 1$ which grows in direction 1. As in Lemma 2.4 we must split H in different parts which are multiplied by $\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{d}$ respectively plus the volume 1 of the new cube to get the maximum growth of the volume by adding the new cube. We choose the parts in such a way that the parts with the higher factors are as big

as theoretical possible. For every $0 \leq r \leq d-2$ we can consider the sets $\{i_1, i_2, \ldots, i_r\}$ with $1 \neq i_a \neq i_b$ for $a \neq b$. Let Y be such a set. The axis directions $1, i_1, \ldots, i_r$ span a hyperplane of the $\mathbb{R}^d$. Define $\overline{Y} = \{j_1, \ldots, j_{d-r-1}\}$ by $Y \cap \overline{Y} = \{\}$ and $1 \cup Y \cup \overline{Y} = \{1, \ldots, d\}$. So the spanned hyperplanes of Y and of $\overline{Y}$ are orthogonal. The maximum volume in the hyperplane spanned by $\overline{Y}$ is at most $f_{d-r-1}(l_{j_1}, \ldots, l_{j_{d-r-1}}, v_{j_1}, \ldots, v_{j_{d-r-1}}) - 1$ due to induction. Now look through a facet of the new cube which is parallel to Y. This would yield a contribution of $\frac{1}{d-r}(f_{d-r-1}(l_{j_1}, \ldots, l_{j_{d-r-1}}, v_{j_1}, \ldots, v_{j_{d-r-1}}) - 1)$ to the volume of the convex hull. In terms of Definition 2.6 this is

$$\frac{1}{d-r} \sum_{\{\} \neq I \subset \{j_1, \ldots, j_{r-d-1}\}} \frac{1}{|I|! 2^{d-r-1-|I|}} \sum_{b=0}^{2^{d-r-1}-1} \prod_{i \in I} q_{b,i} \, .$$

To avoid double counting we now only sum over subsets I of cardinality $r-d-1$. The subsets of less cardinality will be recognized in the consideration of the subspaces of the hyperplane spanned by $\overline{Y}$. So that we finally get for Y a maximum contribution of

$$\frac{1}{d-r} \frac{1}{|d-r-1|!} \sum_{b=0}^{2^{d-r-1}-1} \prod_{i \in \overline{Y}} q_{b,i} \, .$$

If we do so for all possible sets Y we have assigned a factor between $\frac{1}{2}$ and $\frac{1}{d}$ to every summand of $f_{d-1}(l_2, \ldots, l_d, v_2, \ldots, v_d) - 1$. To get the induction step we now must only remark that the described sum above with its factors is exactly the difference between $f_d(l_1+1, \ldots, l_d, v_1, \ldots, v_d)$ and $f_d(l_1, \ldots, l_d, v_1, \ldots, v_d)$.

(ii) v_1 is increased by one:
Due to the symmetry of l_i and v_i in Definition 2.6 this is analogue to (i) with the addition that the maximum cannot be achieved in this case because there is already a cube on this level and a part of the contribution of the new cube to the volume of the convex hull is double counted.

(iii) $l_2, \ldots, l_d, v_2, \ldots, v_d$ is increased by one:
Due to symmetry this is analogue to (i) or (ii).□

Theorem 1. The d-dimensional volume of the convex hull of any facet-to-facet connected system of n unit hypercubes is at most

$$\sum_{I \subset \{1,\ldots,d\}} \frac{1}{|I|!} \prod_{i \in I} \left\lfloor \frac{n-2+i}{d} \right\rfloor .$$

Proof.

(i) If $v_1 > 0$ we decrease v_1 and increase l_1 by one. Now we consider the difference of $f_d(l_1+1, l_2, \ldots, l_d, v_1-1, v_2, \ldots, v_d)$ and $f_d(l_1, l_2, \ldots, l_d, v_1, v_2, \ldots, v_d)$ which is 0. Since due to the proof of Lemma 2.7 the maximum volume $f_d(l_1, \ldots, l_d, v_1, \ldots, v_d)$ cannot be achieved if $v_1 \neq 0$ this substitution increases the volume. So we conclude by an symmetry argument $v_1 = \ldots = v_d = 0$.

(ii) Since we can renumber the coordinate axes we can assume that $l_1 \leq l_2 \leq \ldots \leq l_d$ holds. If $l_d - l_1 \geq 2$ we increase l_1 and decrease l_d by one. Now consider $f_d(l_1+1, l_2, \ldots, l_{d-1}, l_d-1, 0, 0, \ldots, 0) - f_d(l_1, l_2, \ldots, l_d, 0, 0, \ldots, 0)$. If a summand of $f_d(\ldots)$ contains only the term l_1 and not l_d than there is a corresponding term with l_1 replaced by l_d, so those terms raise each other in the above difference. Clearly the summands containing none of the terms l_1 or l_d raise each other in the difference. So there only the summands with both terms l_1 and l_d left. Since $(l_1+1-1)(l_d-1-1)-(l_1-1)(l_d-1) = l_d-l_1-1 > 0$ the above difference is > 0, so this substitution increases the volume, and we conclude $|l_i - l_j| \leq 1$.

(iii) Using equation † we get $l_i = \left\lfloor \frac{n-2+i+d}{d} \right\rfloor$. Inserting in Lemma 2.7 yields the proposed formula.

(iv) An extremal configuration consist of d pairwise orthogonal linear arms of $\left\lfloor \frac{n-2+i}{d} \right\rfloor$ cubes ($i = 1 \ldots d$) joined at a central cube. $\square$

3 Proof of Theorem 2

Lemma 3.1. Every polyomino with the maximum area of the convex hull must be a part of the two shapes depicted below, up to symmetry. Additionally this shapes all have the maximum area given by $f_2(a, b, 0, 0)$.

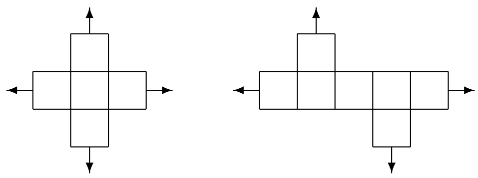

Figure 3.1. The 2 shapes of polyominoes with maximum convex hull.

Proof.

We denote the length of the middle strip by a, and the height of the total polyomino by b. By h_1 and h_2 we denote the heights of the two vertical strips with $h_1 + 1 + h_2 = b$. So we compute the area of the convex hull to $\frac{(a-1)h_1}{2} + \frac{(a-1)h_2}{2} + h_1 + l_1 + h_2 = \frac{(a-1)(b-1)}{2} + n$. Figure 3.2 shows that an extremal polyomino cannot grow in another way than depicted in Figure 3.1, because the area would not be optimal. □

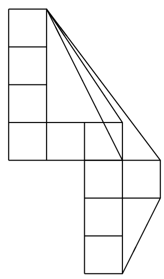

Figure 3.2.

Now we start to count the polyominoes with the shape of Lemma 3.1. Therefore we distinguish whether $a = b$ or not and the two cases for the shapes.

(i) $a = b$, shape 1.
We can describe each polyomino of this form by a tuple (r, t), where r should give the position of the horizontal strip, and t the position of the vertical strip. Reflecting at the horizontal and vertical symmetry-axis yields $1 \leq r \leq \lceil \frac{a}{2} \rceil$, $1 \leq t \leq \lceil \frac{a}{2} \rceil$. If we reflect at the diagonal we get $r \leq t$. As there are no more symmetries we get

$$c_{2,1}(a, b) = \sum_{r=1}^{\left\lceil \frac{a}{2} \right\rceil} \sum_{t=1}^{r} 1 = \sum_{r=1}^{\left\lceil \frac{a}{2} \right\rceil} r = \frac{\left\lceil \frac{a}{2} \right\rceil \left\lceil \frac{a+2}{2} \right\rceil}{2}.$$

(ii) $a = b$, shape 2.
By mirroring at the diagonal we can assume that the 'broken' strip is vertically and not horizontally. For the middle strip only the rows 1 to $\lceil \frac{a}{2} \rceil$ are possible. Row 1 leads to (i). If the horizontal line lies exactly on the center then the horizontal symmetry-axis exists. We handle this case extra. So there are $\lfloor \frac{a-2}{2} \rfloor$ cases. We fist must only consider the vertical symmetry-axis. As there are two vertical stripes we describe their position by t_1 and t_2. If t_1 does not lie in the middle we have the conditions $2 \le r \le \lfloor \frac{a}{2} \rfloor$, $t_1 \ne t_2$, $1 \le t_1 \le \lfloor \frac{a}{2} \rfloor$ (vertical symmetry-axis) and we get

$$c_{2,2}(a,b) = \left\lfloor \frac{a-2}{2} \right\rfloor \left\lfloor \frac{a}{2} \right\rfloor (a-1)\,.$$

If t_1 lies in the middle which is only possible for $a \equiv 1 \ mod\ 2$ we have the conditions $2 \le r \le \lfloor \frac{a}{2} \rfloor$, $t_1 = \frac{a+1}{2}$, $t_2 < \frac{a+1}{2}$ and we get an additional

$$c_{2,3}(a,b) = \left\lfloor \frac{a-2}{2} \right\rfloor \left\lfloor \frac{a}{2} \right\rfloor .$$

Now consider the case that the horizontal strip lies exactly in the middle. This is only possible for $a \equiv 1 \ mod\ 2$. We describe the polyominoes by triples (r, t_1, t_2). By mirroring at the vertical symmetry-axis we can achieve that t_1 lies nearer at the border: $\min(t_1, a+1-t_1) \le \min(t_2, a+1-t_2)$. Mirroring at the horizontal symmetry-axis yields $1 \le t_1 \le \lfloor \frac{a}{2} \rfloor$, so we get

$$c_{2,4}(a,b) = \sum_{t_1=1}^{\lfloor \frac{a}{2} \rfloor} \sum_{t_2=t_1+1}^{a+1-t_1} 1 = \sum_{t_1=1}^{\lfloor \frac{a}{2} \rfloor} a+1-2t_1 = (a+1)\left\lfloor \frac{a}{2} \right\rfloor - \left\lfloor \frac{a}{2} \right\rfloor \left\lfloor \frac{a+2}{2} \right\rfloor = \left\lfloor \frac{a}{2} \right\rfloor \left\lceil \frac{a}{2} \right\rceil .$$

(iii) $a \ne b$, shape 1.
Because $a \ne b$ we cannot mirror at the diagonal. Again we describe the polyominoes by (r, t). Mirroring at the horizontal and vertical symmetry-axis yields $1 \le r \le \left\lceil \frac{a}{2} \right\rceil$, $1 \le t \le \left\lceil \frac{b}{2} \right\rceil$, so we get

$$c_{2,5}(a,b) = \left\lceil \frac{a}{2} \right\rceil \left\lceil \frac{b}{2} \right\rceil .$$

(iv) $a \neq b$, shape 2.
Analogue to (ii) we get

$$c_{2,6}(a,b) = \left\lfloor \frac{b-2}{2} \right\rfloor \left\lfloor \frac{a}{2} \right\rfloor (a-1)$$

plus for $a \equiv 1 \ mod \ 2$ an extra

$$c_{2,7}(a,b) = \left\lfloor \frac{b-2}{2} \right\rfloor \left\lfloor \frac{a}{2} \right\rfloor$$

and for $b \equiv 1 \ mod \ 2$ an extra

$$c_{2,8}(a,b) = \left\lfloor \frac{a}{2} \right\rfloor \left\lceil \frac{a}{2} \right\rceil .$$

The exchange of a and b is obvious.

We summarize the results

$a = b$:

$a \equiv 0 \ mod \ 2$: $c_2(a,b) = \frac{2a^3-5a^2+6a}{8}$ for $a \geq 0$,
$a \equiv 1 \ mod \ 2$: $c_2(a,b) = \frac{2a^3-5a^2+10a+1}{8}$ for $a \geq 1$.

$a \neq b$:

$a \equiv 0, b \equiv 0 \ mod \ 2$: $c_2(a,b) = \frac{ab(a+b-1)-2a^2-2b^2+2a+2b}{4}$ for $a, b \geq 2$,

$$a \equiv 1, b \equiv 0 \ mod \ 2\text{: } c_2(a,b) = \begin{cases} \frac{ab(a+b-1)-2a^2-2b^2+4b+2a}{4} & \text{for} \quad a \geq 3, b \geq 2\,, \\ 1 & \phantom{\text{for}} \quad a = 1, b \geq 2\,, \end{cases}$$

$$a \equiv 0, b \equiv 1 \ mod \ 2\text{: } c_2(a,b) = \begin{cases} \frac{ab(a+b-1)-2a^2-2b^2+4a+2b}{4} & \text{for} \quad a \geq 2, b \geq 3\,, \\ 1 & \phantom{\text{for}} \quad a \geq 2, b = 1\,, \end{cases}$$

$$a \equiv 1, b \equiv 1 \ mod \ 2\text{: } c_2(a,b) = \begin{cases} \frac{ab(a+b-1)-2a^2-2b^2+4a+4b-2}{4} & \text{for} \quad a, b \geq 3\,, \\ 1 & \phantom{\text{for}} \quad (a-1)(b-1) = 0\,. \end{cases}$$

Theorem 2. The number $c_2(n)$ of polyominoes in $\mathbf{R^2}$ with maximum area of the convex hull is given by

$$n \equiv 0 \ \ mod\ 4 : c_2(n) = \frac{n^3 - 2n^2 + 4n}{16},$$
$$n \equiv 1 \ mod\ 4 : c_2(n) = \frac{n^3 - 2n^2 + 13n + 20}{32},$$
$$n \equiv 2 \ mod\ 4 : c_2(n) = \frac{n^3 - 2n^2 + 4n + 8}{16},$$
$$n \equiv 3 \ \ mod\ 4 : c_2(n) = \frac{n^3 - 2n^2 + 5n + 8}{32}.$$

Proof. The polyominoes with maximal area of the convex hull are given by $|a - b| \leq 1$ so Theorem 2 follows from the above. □

Conclusion 3.2. The generating function for the number $c_2(n)$ of polyominoes in $\mathbf{R^2}$ with maximum area of the convex hull is given by

$$\frac{1 + x - x^2 - x^3 + 2x^5 + 8x^6 + 2x^7 + 4x^8 + 2x^9 - x^{10} + x^{12}}{(1 - x^2)^2(1 - x^4)^2}.$$

4 Proof of Theorem 3

Lemma 3.1. A polyomino in $\mathbf{R}^d$, $d \geq 3$ with the maximum volume of the convex hull consists of a central cube with linear arms at his faces.
Proof. We will prove by double induction on d and n. Therefore we consider the process of building up a polyomino cube by cube. If the final polyomino has the maximum volume of the convex hull in every step of the building process the volume of the convex hull must be given by $f_d(l_1, \ldots, l_d, 0, \ldots, 0)$. If $l_i = 1$ for an i we can delete axis i and project the polyomino in the $\mathbf{R}^{d-1}$. Now we assume $d \geq 3$. In Lemma 2.7 we have seen, that when l_i increases by one then the volume grows by the term

$$\frac{\sum_{j=1}^{d}(l_j - 1)\ - (l_i - 1)}{2}$$

considering the faces of the new cube.

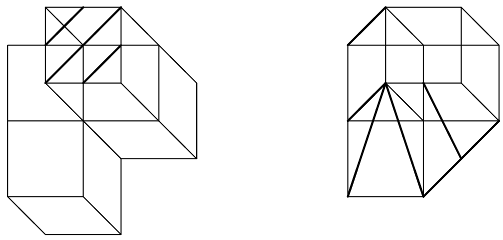

Figure 4.1. Adding a new cube.

The new cube in Figure 4.1 is depicted by 3 diagonals on the left picture. The right picture shows the convex hull. The new cube contributes a volume of 1 to the convex hull. The two faces of the cube contribute $\frac{1}{2}+\frac{1}{4}$, and the corner of the new cube contributes $\frac{1}{3}$. By a look at Figure 2.4 and the proof of Lemma 2.4 we see that here the area A is $\frac{1}{2}$ bigger than estimated and that the strip in direction 2 is $\frac{1}{2}$ fewer than estimated. As the first term is multiplied by $\frac{1}{3}$ and the second by $\frac{1}{2}$ we loose $\frac{1}{12}$. This holds in general, because the contribution of a face of dimension k is multiplied by $\frac{1}{d-k+1}$, so the contribution of a face of dimension k must be as big as estimated in Lemma 2.7. We conclude, that the polyominoes with maximum volume of the convex hull must have the proposed shape for $n \geq 4$ (because we use $d \geq 3$), It is easy to see that the polyominoes with maximum volume of the convex hull consisting of $n < 4$ cubes also have the proposed shape. $\square$

Theorem 3. The number $c_d(n)$ of polyominoes in $\mathbf{R^d}$ with maximum volume of the convex hull is given by

$$n \equiv a \; mod \; d \; : \; c_d(n) = \binom{\lceil \frac{n-a}{2d} \rceil + d - a}{d-a+1} \binom{\lceil \frac{n+d-a}{2d} \rceil + a - 2}{a-1}$$

with $0 < a \leq d$ and $d \geq 3$.

Proof. From the proof of Theorem 1 we know the length of the linear arms to be $\lfloor \frac{n-2+i}{d} \rfloor$. For $n \equiv a \; mod \; d$, $0 < a \leq d$ this is $(d-a+1)$ times the length $\frac{n-a}{d}$ and $a-1$ times the length $\frac{n+d-a}{d}$. We can describe every polyomino in the shape of Lemma 4.1 as a set of the sets $\{a_i, b_i\}$ with $a_i + b_i = \lfloor \frac{n-2+i}{d} \rfloor$, $a_i, b_i \in \mathbb{N}_0$, since we can reflect on the hyperplanes and renumber the coordinate axes by rotation. Using the fact that every symmetry of a hypercube can be composed by the rotations and the reflections on hyperplanes the theorem follows by a simple combinatorial formulae. □

5 Proof of Theorem 4

Theorem 4. The existence of a 2-dimensional polyomino consisting of n cells with area a of the convex hull is equivalent to $a \in A_n$ with

$$A_n = \left\{ n + \frac{m}{2} \middle| m \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor, m \in \mathbb{N}_0 \right\} - \left\{ n + \frac{1}{2} \middle| if\, n+1\, is\, prime \right\} .$$

Proof. $\subseteq$:
Since the corner points of a polyomino lie on a unit square grid the area of the convex hull must be a natural multiple of $\frac{1}{2}$. With Lemma 2.3 and the fact that the area of n unit squares is n we got that the set of possible areas of the convex hull of polyominoes with n cells must be a subset of

$$\left\{ n + \frac{m}{2} \middle| m \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor, m \in \mathbb{N}_0 \right\}.$$

A polyomino consisting of n cells with area n of the convex hull must be convex. If the area of the convex hull is $n + \frac{1}{2}$ there must be a triangle of area $\frac{1}{2}$. If we extend the triangle to a rectangle we get a polyomino consisting of $n+1$ cells.

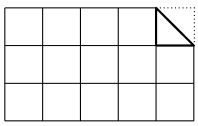

Figure 5.1. Polyominoes with n squares and area $n + \frac{1}{2}$ of the convex hull.

Thus we can construct all polyominoes consisting of n unit squares with area $n + \frac{1}{2}$ of the convex hull by deleting a square at the corner of a convex polyomino consisting of $n+1$ cells. The convex polyominoes are exactly the $a \times b$ rectangles. Therefore for $n+1$ prime there is only the $1 \times (n+1)$

rectangle. Deleting a square yields area n of the convex hull.

$\supseteq$:
For $m = 0$ we have the $a \times b$ rectangles with $ab = n$ as examples. The above consideration for area $n + \frac{1}{2}$ of the convex hull yields a construction for $n+1$ composite. Now we give 6 constructions to handle the other values for m.

(i)

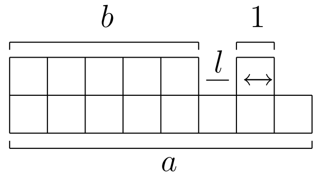

Figure 5.2. Construction 1.

We let a run from $\lceil \frac{n}{2} \rceil$ to $n-2$ and l run from 0 to $a-b-1$. With $a+b+1 = n$ we get $a \geq b+1$ so that the construction depicted in Figure 5.2 is possible. For $n \equiv 0 \ mod\ 2$ the attained values for m are $(0), (2, \ldots, 4), (4, \ldots, 8), \ldots, (a - b - 1, \ldots, 2a - 2b - 2), \ldots, (n - 4, \ldots, 2n - 8) = 0, 2, 3, \ldots, 2n - 8$. For $n \equiv 1 \ mod\ 2$ the attained values for m are $(1, 2), (3, \ldots, 6), (5, \ldots, 10), \ldots, (a - b - 1, \ldots, 2a - 2b - 2), \ldots, (n - 4, \ldots, 2n - 8) = 0, 2, 3, \ldots, 2n - 8$.
So we can handle $2 \leq m \leq 2n - 8$.

(ii)

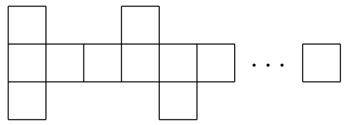
Figure 5.3. Construction 2.

For $n \geq 9$ Construction 2 yields $m = 2n - 7$. The remaining cases $5 \leq n \leq 8$ for $m = 2n - 7$ are treated in Figure 5.4.

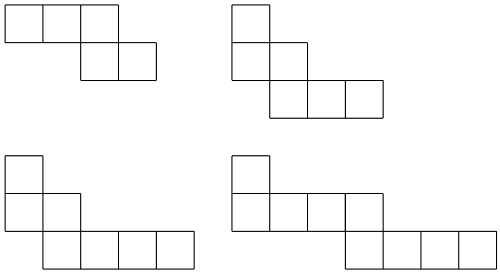

Figure 5.4. $m = 2n - 7$ for $5 \leq n \leq 8$.

(iii)

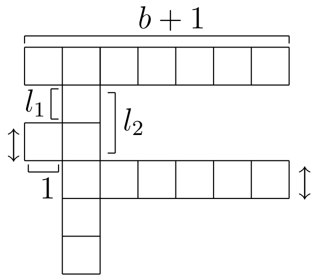

Figure 5.5. Construction 3.

The conditions for a possible construction of Figure 5.5 are $0 \leq l_1, l_2 \leq n - 2b - 2$ and $2b + 2 \leq n$. We demand $n - 2b - 2 \geq b$ which is equivalent to $b \leq \frac{n}{3}$. With given l_1, l_2, b, n it holds $m = bn - 2b^2 - 2b + l_1 + l_2(b-1)$. Because we have demanded $n - 2b - 2 \geq b - 1$ we can vary l_1 at least between 0 and $b-2$ and so we can get by changing l_1 and l_2 all values between $b(n-2b-2)$ and $2b(n - 2b - 2)$. Now we want to combine those intervals for successive values for b. The assumption that the intervals do not intersect is equivalent to

$$\begin{aligned} & 2(b-1)(n-2(b-1)-2) < b(n-2b-2) \\ \Leftrightarrow \quad & bn - 2n - 2b^2 + 6b < 0 \\ \Leftrightarrow \quad & n(b-2) < 2b(b-3) \\ \Leftrightarrow \quad & n < 2b\tfrac{b-3}{b-2} < 2b \end{aligned}$$

As we already have to fulfill $b \leq \frac{n}{3}$ the intervals intersect. We choose $2 \leq b \leq \left\lfloor \frac{n}{4} \right\rfloor$ and get constructions for m the interval $2n-6, 2n-5 \ldots, \left\lceil \frac{n^2-4n}{4} \right\rceil$.

(iv)

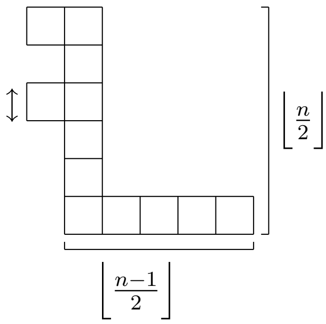

Figure 5.6. Construction 4.

For $n \geq 4$ Construction 4 is possible and for $n \equiv 0 \mod 2$ we can get m from $\frac{n^2-4n}{4}$ to $\frac{n^2-2n-8}{4}$. For $n \equiv 1 \mod 2$ we can get m from $\frac{n^2-4n-1}{4}$ to $\frac{n^2-2n-11}{4}$.

(v)

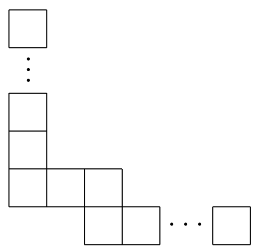

Figure 5.7. Construction 5.

The height and the width of Figure 5.7 is be given by $\left\lfloor \frac{n+1}{2} \right\rfloor$ and $\left\lfloor \frac{n+2}{2} \right\rfloor$ For $n \geq 7$ Construction 5 is possible. We need it only for odd n to obtain $m = \frac{n^2-2n-7}{4}$. For $n \leq 5$ we remark $2n - 7 \geq \frac{n^2-2n-7}{4}$.

(vi)

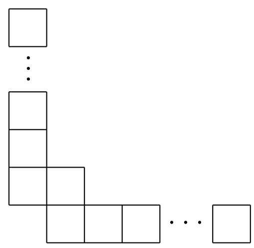

Figure 5.8. Construction 6.

The height and the width of Figure 5.8 is as is Figure 5.7. For even n we get $m = \frac{n^2-2n-4}{4}$ and for odd n we get $m = \frac{n^2-2n-3}{4}$.

The proof is completed by Figure 2.3.$\square$

In higher dimensions the situation analogue to Theorem 4 is more complicates as we will see in the next Lemma.

Lemma 5.1. For every integer t there is a 3-dimensional polyomino consisting of n cubes so that the denominator of its volume of the convex hull exceeds t.
Proof. The volume of the convex hull of the polyomino in Figure 5.9 is given by $2n - 3 + \frac{1}{6}(n - 1 + \frac{1}{2(n-3)})$. $\square$

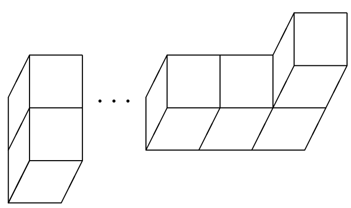

Figure 5.9. Polyomino with a big denominator for the volume of the convex hull.

We enumerated the 2-dimensional polyominoes with maximum area of the convex hull in Theorem 2. For the minimum area n of the convex hull we enumerate them in Lemma 5.2 and for area $n+\frac{1}{2}$ of the convex hull we enumerate them in Lemma 5.3. Therefore we denote the number of divisors of an integer n by $\tau(n)$.

Lemma 5.2. The number of polyominoes consisting of n unit squares with minimum area n of the convex hull is given by $\left\lceil \frac{\tau(n)}{2} \right\rceil$.
Proof. Those polyominoes are convex (in the sense of geometry) and so they are all rectangle polyominoes. Considering the symmetries yields the division by two and the ceiling. □

Lemma 5.3. The number of polyominoes consisting of n unit squares with area $n+\frac{1}{2}$ of the convex hull is given by $\left\lceil \frac{\tau(n+1)}{2} \right\rceil - 1$.
Proof. See Figure 5.1 and the proof of Theorem 4 for a description of those polyominoes. □

Lemma 5.4. The number of d-dimensional polyominoes consisting of n unit hypercubes with minimum volume n is the number of ways to decompose n into sets of d factors.
Proof. Those polyominoes are convex (in the sense of geometry) and so they are hyper rectangle polyominoes.

6 Outlook

In the last sections we enumerated the d-dimensional polyominoes with maximum and those with minimum volume of the convex hull. Here we do not treat the equivalent problem for polyiamonds, polyhexes or other kinds of polyominoes. For polyiamonds consisting of n unit equilateral triangles the minimum area of the convex hull is n since there are convex (in the sense of geometry) polyiamonds for every integer n. It is not difficult to describe the shape of those extremal animals but to find an elegant way to enumerate their number is another thing. Polyhexes with more then one cell cannot be convex (in the sense of geometry). We propose that the set of polyhexes with the minimum are of the convex hull is equivalent to the set of polyhexes with minimum perimeter. The later set is not enumerated yet, at least to the authors knowledge.

Another class of problems which is also related to our topic is the question for the maximum area of the convex hull of all edge-to-edge packings of n regular k-gons in the plane.

Conjuncture 6.1. The maximum area of the convex hull of n regular k-gons of area 1 is at most

$$1 + \frac{(n-1)^2}{\pi\sqrt{3}} + \frac{2\sqrt{3}(n-1)}{\pi}$$

with equality only for $k = \infty$, more precisely circles, and $n \equiv 1 \; mod \; 3$.

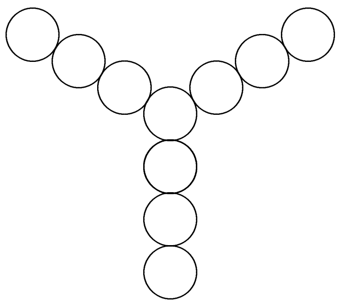

Figure 6.1. An example of circles with big area of the convex hull.

References

[1] F. Abn-Majdoub-Hassani. *Combinatorics of Polyominoes and Oscillating Shifted Tableaux.* PhD thesis, Université Paris-Sud, 1996.

[2] A. A. Ali. Enumeration of directed site animals on the decorated square lattices. *Phys. A*, 202:520–528, 1994.

[3] Laurent Alonso and Raphaël Cerf. The three-dimensional polyominoes of minimal area. *Electron. J. Combin.*, 3(1):39 pp., 1996. Symbolic computation in combinatorics Δ_1 (Ithaca, NY, 1993).

[4] J. Bétréma and J.-G. Penaud. Animaux et abres guingois. *Theoret. Comp. Sci.*, 117:67–89, 1993.

[5] V.K. Bhat et al. Enumeration of directed compact site animals in two dimensions. *J. Phys. A*, 19:3261–3265, 1968.

[6] Jens-P. Bode and Heiko Harborth. Hexagonal polyomino achievement. *Discrete Math.*, 212(1-2):5–18, 2000. Graph theory (Dörnfeld, 1997).

[7] Jens-P. Bode and Heiko Harborth. Triangular mosaic polyomino achievement. *Congr. Numer.*, 144:143–152, 2000. Proceedings of the Thirty-first Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 2000).

[8] Jens-P. Bode and Heiko Harborth. Triangle polyomino set achievement. *Congr. Numer.*, 148:97–101, 2001. Proceedings of the Thirty-second Southeastern International Conference on Combinatorics, Graph Theory and Computing (Baton Rouge, LA, 2001).

[9] Jens-P. Bode and Heiko Harborth. Achievement games for polyominoes on archimedean tesselations. In Tor Helleseth Jong-Seon No, Hong-Yeop Song and P. Vijay Kumar, editors, *Mathematical Properties Of Sequences And Other Combinatorial Structures*. Kluwer, 2003.

[10] Mireille Bousquet-Mélou. *q-Énumération de polyomios convexes*. PhD thesis, Université of Bordeaux I, 1991.

[11] Mireille Bousquet-Mélou. Convex polyominoes and algebraic languages. *J. Phys. A*, 25(7):1935–1944, 1992.

[12] Mireille Bousquet-Mélou. Convex polyominoes and heaps of segments. *J. Phys. A*, 25(7):1925–1934, 1992.

[13] Mireille Bousquet-Mélou. Une bijection entre les polyominos convexes dirigés et les mots de dyck bilatères. (french) [a bijection between convex directed polyominoes and bilateral dyck words]. *RAIRO Inform. Thor. Appl.*, 26(3):205–219, 1992.

[14] Mireille Bousquet-Mélou. q-énumération de polyominos convexes. (french) [q-enumeration of convex polyominoes]. *J. Combin. Theory Ser. A*, 64(2):265–288, 1993.

[15] Mireille Bousquet-Mélou. Codage des polyominos convexes et équations pour l'énumération suivant l'aire. (french) [coding of convex polyominos and equations for enumeration according to the area]. *Discrete Appl. Math.*, 48(1):21–43, 1994.

[16] Mireille Bousquet-Mélou. A method for the enumeration of various classes of column-convex polygons. *Discrete Math.*, 154:1–25, 1996.

[17] Mireille Bousquet-Mélou. New enumerative results on two-dimensional directed animals. *Discrete Math.*, 180:73–106, 1998.

[18] Mireille Bousquet-Mélou and Jean-Marc Fédou. The generating function of convex polyominoes: the resolution of a q-differential system. *Discrete Math.*, 137(1-3):53–75, 1995.

[19] Mireille Bousquet-Mélou and A. J. Guttmann. Enumeration of three-dimensional convex polygons. *Annal Comb.*, 1:27–53, 1997.

[20] Mireille Bousquet-Mélou and A. Rechnitzer. Lattice animals and heaps of dimers. *Discrete Math.*, 258:235–274, 2002.

[21] Mireille Bousquet-Mélou and Xavier Gérard Viennot. Empilements de segments et q-énumération de polyominos convexes dirigés. (french) [stacks of segments and q-enumeration of directed convex polyominoes]. *J. Combin. Theory Ser. A*, 60(2):196–224, 1992.

[22] R. Brak and A. J. Guttmann. Exact solution of the staircase and row-convex polygon perimeter and area generating function. *J. Phys. A: Math. Gen.*, 23:4581–4588, 1990.

[23] V. Chvátal C. Berge, C. C. Chen and C. S. Seow. Combinatorial properties of polyominoes. *Combinatorica*, 1(3):217–224, 1981.

[24] B. A. Cibra. An introduction to the ising model. *Amer. Math. Monthly*, 94:937–957, 1987.

[25] A. R. Conway and A. J. Guttmann. On two-dimensional percolation. *J. Phys. A: Math. Gen.*, 28:891–904, 1995.

[26] M. A. Prassad D. P. Bhatia and D. Arora. Asymptotic results for the number of multidimensional partitions of an integer and directed compact lattice animals. *J. Phys. A: Math. Gen.*, 30:2281–2285, 1997.

[27] M. Delest and S. Dulucq. Enumeration of directed column-convex animals with given perimeter and area. *Croat. Chem. Acta*, 66:59–80, 1993.

[28] Marie-Pierre Delest. *Utilisation des l'anguges algeébraiques et du Calcul Formel pour le Codage et l'Enumeration des Polyominoes.* PhD thesis, Université Bordeaux I, May 1987.

[29] Marie-Pierre Delest. Generating functions for column-convex polyominoes. *J. Comb. Theory Ser. A.*, 48(1):12–31, 1988. Symbolic computation in combinatorics Δ_1 (Ithaca, NY, 1993).

[30] Marie-Pierre Delest. Enumeration of polyominoes using macsyma. *Theoret. Comput. Sci.*, 79(1):209–226, 1991. Algebraic and computing treatment of noncommutative power series (Lille, 1988).

[31] Marie-Pierre Delest. Polyominoes and animals: some recent results. *J. Math. Chem.*, 8(1-3):3–18, 1991. Mathematical chemistry and computation (Dubrovnik, 1990).

[32] Marie-Pierre Delest and J.-M. Fédou. Counting polyominoes using attribute grammars. In *Lecture Notes in Comput. Sci.*, volume 461, pages 46–60. Springer, 1990. Attribute grammars and their applications (Paris, 1990).

[33] Marie-Pierre Delest and Gérard Viennot. Algebraic languages and polyominoes enumeration. *Theoret. Comput. Sci.*, 34(1-2):169–206, 1984.

[34] D. Dhar. Exact solution of a directed-site animals-enumeration problem in three dimensions. *Phys. Rev. Lett.*, 51:853–856, 1983.

[35] D. Dhar et al. Enumeration of directed site animals. *J. Phys. A*, 15:L279–L284, 1982.

[36] Virgil Domocoş. A combinatorial method for the enumeration of column-convex polyominoes. *Discrete Math.*, 152(1-3):1155–123, 1996. Twelfth British Combinatorial Conference (Norwich, 1989).

[37] Kim Dongsu. The number of convex polyominos with given perimeter. *Discrete Math.*, 70(1):47–51, 1988.

[38] Philippe Duchon. q-grammars and wall polyominoes. *Ann. Comb.*, 3(2-4):311–321, 1999. On combinatorics and statistical mechanics.

[39] J. M. Fédou E. Barcucci, A. Del Lungo and R. Pinzani. Steep polyominoes, q-motzkin numbers and q-bessel functions. *Discrete Math.*, 189(1-3):21–42, 1998.

[40] Renzo Pinzani Elena Barcucci and Renzo Sprugnoli. Directed column-convex polyominoes by recurrence relations. In *Lecture Notes in Comput. Sci.*, volume 668, pages 282–298. Springer, 1993. TAPSOFT '93: theory and practice of software development (Orsay, 1993).

[41] Svetlan Feretić Emeric Deutsch and Marc Noy. Diagonally convex directed polyominoes and even trees: a bijection and related issues. *Discrete Math.*, 256(3):645–654, 2002. LaCIM 2000 Conference on Combinatorics, Computer Science and Applications (Montreal, QC).

[42] Svetlan Feretić. A new coding for column-convex directed animals. *Croat. Chem. Acta*, 66:81–90, 1993.

[43] Svetlan Feretić. The column-convex polyominoes perimeter generationg function for everybody. *Croat. Chem. Acta*, 69:741–756, 1995.

[44] Svetlan Feretić. A bijective perimeter enumeration of directed convex polyominoes. *J. Statist. Plann. Inference*, 101(1-2):81–94, 2002. Special issue on lattice path combinatorics and applications (Vienna, 1998).

[45] Svetlan Feretić. A q-enumeration of directed diagonally convex polyominoes. *Discrete Math.*, 246(1-3):99–109, 2002. Formal power series and algebraic combinatorics (Barcelona, 1999).

[46] Svjetlan Feretić. A new way of counting the column-convex polyominoes by perimeter. *Discrete Math.*, 180(1-3):173–184, 1998. Proceedings of the 7th Conference on Formal Power Series and Algebraic Combinatorics (Noisy-le-Grand, 1995).

[47] Svjetlan Feretić. An alternative method for q-counting directed column-convex polyominoes. *Discrete Math.*, 210(1-3):55–70, 2000.

Formal power series and algebraic combinatorics (Minneapolis, MN, 1996).

[48] Svjetlan Feretić and Dragutin Svrtan. The area generating function for the column-convex polyominoes on the checkerboard lattice. *Croat. Chem. Acta*, 68(1):75–90, 1995. Proceedings of the Ninth Dubrovnik International Course and Conference MATH/CHEM/COMP (Dubrovnik, 1994).

[49] Svjetlan Feretić and Dragutin Svrtan. Combinatorics of diagonally convex directed polyominoes. *Discrete Math.*, 157(1-3):147–168, 1996. Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994).

[50] G. Forgacs and V. Privman. Directed compact lattice animals: exact results. *J. Statist. Phys.*, 49:1165–1180, 1987.

[51] Heiko Harborth Frank Harary and Markus Seemann. Handicap achievement for polyominoes. *Congr. Numer.*, 145:65–80, 2000. Proceedings of the Thirty-first Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 2000.

[52] Martin Gardner. *Mathematical Magic Show*, chapter 11. Polyhexes and Polyaboloes, pages 175–187. Vintage, 1978.

[53] Martin Gardner. *Time Travel and Other Mathematical Bewilderments*, chapter 14. Tiling with Polyominoes,Polyiamonds and Polyhexes, pages 175–187. W.H. Freeman, 1988.

[54] Ira-M. Gessel. On the number of convex polyominoes. *Ann. Sci. Math. Québec*, 24(1):63–66, 2000.

[55] S. W. Golomb. *Polyominoes.* Charles Scribner's Sons, 1965.

[56] Solomon W. Golomb. Checker boards and polyominoes. *Amer. Math. Monthly*, 61:675–682, 1954.

[57] Solomon W. Golomb. Tiling with polyominoes. *J. Combinatorial Theory*, 1:280–296, 1966.

[58] Solomon W. Golomb. Tiling with sets of polyominoes. *J. Combinatorial Theory*, 9:60–71, 1970.

[59] Solomon W. Golomb. Polyominoes which tile rectangles. *J. Combin. Theory Ser. A*, 51(1):117–124, 1989.

[60] Solomon W. Golomb. *Puzzles, patterns, problems, and packings. With diagrams by Warren Lushbaugh.* Princeton University Press, second edition, 1994.

[61] Solomon W. Golomb. Tiling rectangles with polyominoes. *Math. Intelligencer*, 18(2):38–47, 1996.

[62] Solomon W. Golomb and Lloyd R. Welch. Perfect codes in the lee metric and the packing of polyominoes. *SIAM J. Appl. Math.*, 18:302–317, 1970.

[63] D. Gouyou-Beauchamps and Gérard Viennot. Equivalence of the two-dimensional directed animal problem to a one-dimensional path problem. *Adv. in Appl. Math.*, 9(3):334–357, 1988.

[64] A. J. Guttmann. The number of convex polygons on the squares and honeycomb lattices. *J. Phys. A*, 21:467–474, 1988.

[65] F. Harary and Heiko Harborth. Extremal animals. *Journal of Combinatorics, Information & System Sciences*, 1(1):1–8, 1976.

[66] F. Harary and Heiko Harborth. Achievement and avoidance games with triangular animals. *J. Recreational Math.*, 18:110–115, 1985.

[67] F. Harary and R.C. Read. The enumeration of tree-like polyhexes. *Proc. Edinb. Math. Soc. (2)*, 17:1–13, 1970.

[68] Heiko Harborth and Christian Thürmann. Limited snakes of polyominoes. *Congr. Numer.*, 133:211–218, 1998. Proceedings of the Twenty-ninth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1998).

[69] Dean Hickerson. Counting horizontally convex polyominoes. *J. Integer Seq.*, 2, 1999. Article 99.1.8, 1 HTML document (electronic).

[70] B. Derrida J. P. Nadal and J. Vannimenus. Directed lattice animals in 2 dimensions: numerical and exact resultats. *J. Physique*, 43:1561–1574, 1982.

[71] I. Jensen. Enumeration of lattice animals and trees. *J. Statistical Physics*, 102:865–881, 2001.

[72] G. S. Joyce and A. J. Guttmann. Exact results for the generating function of directed column-convex animals on the square lattice. *J. Phys. A: Math. Gen.*, 27:4359–4367, 1994.

[73] Peter BraßKároly Bezdek and Heiko Harborth. Maximum convex hulls of connected systems of segments and of polyominoes. *Beiträge Algebra Geom.*, 35(1):37–43, 1994. Festschrift on the occasion of the 65th birthday of Otto Krötenheerdt.

[74] David A. Klarner. Cell growth problems. *Canad. J. Math.*, 19:851–863, 1967.

[75] David A. Klarner. Polyominoes. In Jacob E. Goodman and Joseph O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 12, pages 225–242. CRC Press LLC, 1997.

[76] J. V. Knop et al. On the total number of polyhexes. *Match*, 16:851–863, 1984.

[77] Sascha Kurz. Counting polyominoes with minimum perimeter. in preparation, 2003.

[78] Jaques Labelle. Self-avoiding walks and polyominoes in strips. *Bull. Inst. Combin. Appl.*, 23:88–98, 1998.

[79] Pierre Leroux and Ėtienne Rassart. Enumeration of symmetry classes of parallelogram polyominoes. *Ann. Sci. Math. Québec*, 25(1):71–90, 2001.

[80] K. Y. Lin. Perimeter generating function for row-convex polygons on the rectangular lattice. *J. Phys. A: Math. Gen.*, 23:4703–4705, 1990.

[81] K. Y. Lin. Exact solution of the convex polygon perimeter and area generating function. *J. Phys. A: Math. Gen.*, 24:2411–2417, 1991.

[82] K. Y. Lin and S. J. Chang. Rigorous results for the number of convex polygons on the square and honeycomb lattices. *J. Phys. A*, 21:2635–2642, 1988.

[83] K. Y. Lin and W. J. Tzeng. Perimeter and area generating functions of the staircase and row-convex polygons on the rectangular lattice. *Internat. J. Modern Phys. B*, 5:1913–1925, 1991.

[84] K. Y. Lin and F. Y. Wu. Unidirectional-convex polygons on the honeycomb lattice. *J. Phys. A: Math. Gen.*, 23:5003–5010, 1990.

[85] W. F. Lunnon. Counting polyominoes. In A.O.L. Atkin and B.J.Birch, editors, *Computers in Number Theory*, pages 347–372. Academic Press, 1971.

[86] W. F. Lunnon. Counting hexagonal and triangular polyominoes. In R. C. Read, editor, *Graph Theory and Computing*, pages 87–100. Academic Press, 1972.

[87] W. F. Lunnon. Counting multidimensional polyominoes. *Comput. J.*, 18(4):366–367, 1975.

[88] D. Gouyou-Beauchamps Marie-Pierre Delest and B. Vauquelin. Enumeration of parallelogram polyominoes with given bond and site perimeter. *Graphs Combin.*, 3(4):325–339, 1987.

[89] J. P. Dubernard Marie-Pierre Delest and I. Dutour. Parallelogram polyominoes and corners. *J. Symbolic Comput.*, 20(5-6):503–515, 1995. Symbolic computation in combinatorics Δ_1 (Ithaca, NY, 1993).

[90] S. Mertens. Lattice animals: a fast enumeration algorithm and new perimeter polyominals. *J. of Statistical Physics*, 58(5-6):1095–1108, 1990.

[91] Stephan Mertens and Markus E. Lautenbacher. Counting lattice animals: a parallel attack. *J. Statis. Phys.*, 66(1-2):669–678, 1992.

[92] W.P. Orrick Mireille Bousquet-Mélou, A.J. Guttmann and A. Rechnitzer. Inversion relations, reciprocity and polyominoes. *Ann. Comb.*, 3(2-4):223–249, 1999. On combinatorics and statistical mechanics.

[93] J.-G. Penaud. *Arbres et Animaux.* PhD thesis, Université Bordeaux I, May 1990.

[94] Ėtienne Rassart Pierre Leroux and A. Robitaille. Enumeration of symmetry classes of convex polyominoes in the square lattice. *Adv. in Appl. Math.*, 21(3):343–380, 1998.

[95] G. Pólya. On the number of certain lattice polygons. *J. Comb. Th.*, 6:102–105, 1969.

[96] V. Privman and N. M. Švrakić. Exact generating function for fully directed compact lattice animals. *Phys. Rev. Lett.*, 60:1107–1109, 1988.

[97] A. J. Guttmann R. Brak and I. G. Enting. Exact solution of the row-convex perimeter generation function. *J. Phys. A*, 23:2319–2326, 1990. Symbolic computation in combinatorics Δ_1 (Ithaca, NY, 1993).

[98] R. C. Read. Contributions to the cell growth problem. *Canad. J. Math.*, 14:1–20, 1962.

[99] R. C. Read. Some applications of computers in graph theory. In L. W. Beineke and R. J. Wilson, editors, *Selected Topics in Graph Theory*, pages 417–444. Academic Press, 1978.

[100] A. D. Rechnitzer. *Some problems in the counting of lattice animals, polyominoes, polygons and walks.* PhD thesis, University of Melbourne, August 2000.

[101] D. Hugh Redelmeier. Counting polyominoes: yet another attack. *Discrete Math.*, 36(2):191–203, 1981.

[102] N. J. A. Sloane. The on-line encyclopedia of integer sequences. published electronically at http://www.research.att.com/ njas/sequences/, 2001.

[103] N. J. A. Sloane. The on-line encyclopedia of integer sequences. *Notices of the AMS*, 50(8):912–915, 2003.

[104] Ian Stewart. *Die Reise nach Pentagonien - 16 mathematische Kurzgeschichten.* dtv, 1995.

[105] Robert A. Sulanke. Three recurrences for parallelogram polyominoes. *J. Differ. Equations Appl.*, 5(2):155–176, 1999.

[106] P. J. Torbijn. Polyiamonds. *J. Rec. Math.*, 2:216–227, 1969.

[107] L. Turban. Lattice animals on a staircase and fibonacci numbers. *J. Phys. A*, 33:2587–2595, 2000.

[108] W. J. Tzeng and K. Y. Lin. Exact solution of the row-convex polygon generating function for the checkerboard lattice. *Internat. J. Modern Phys. B*, 5:2551–2562, 1991.

[109] M. Loebl W. Hochstaettler and C. Moll. Generating convex polyominoes at random. Technical Report No. 92.112, Angewandte Mathematik und Informatik Universitaet zu Koeln, 1992.

[110] S. G. Whittington and C. E. Soteros. Lattice animals: rigorous results and wild guesses. In G. R. Grimmett and D. J. A. Welsh, editors, *Disorder in physical systems: a volume in honour of J. M. Hammersley.* Clarendon Press, 1990.

[111] Martin Loebl Winfried Hochstättler and Christoph Moll. Ėnumération de polyominos convexes dirigés. (french) [enumeration of directed convex polyominoes]. *Discrete Math.*, 157(1-3):70–90, 1996. Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994).

[112] Martin Loebl Winfried Hochstättler and Christoph Moll. Generating convex polyominoes at random. *Discrete Math.*, 153(1-3):165–176, 1996. Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993).

[113] Doron Zeilberger. The umbral transfer-matrix method iii, counting animals. *New York J. of Mathematics*, 7:223–231, 2001.

A Exact numbers of different types of polyominoes

Since I am generally interested in enumerating of polyominoes the bibliography, with most entries concerning enumeration of polyominoes, will be followed by the, at least to me, known exact numbers of some kinds of polyominoes.

In this context I would like to mention N.J.A. Sloane's marvellous **Online Encyclopedia of Integer Sequences** [102, 103]. This archive contains over 80.000 integer sequences and numerous references. Suppose your are working on a topic were an integer sequence is involved. Calculating the first few terms and using the Look-Up interface of the Online Encyclopedia of Integer Sequences might give you the next terms, the name, references, generating functions,... . Since the numbers of polyominoes given in the next subsections will not be up to date for a long time we cite the corresponding sequences of [102] to give the reader the chance of getting the newest numbers.

Before we give the numbers we would like to encourage all mathematical authors to support this archive by contributing the integer sequences from their mathematical work. The author itself had submitted and extended over 500 sequences for this archive.

A.1 Number of square polyominoes

n	A0001055(n)	n	A0001055(n)	n	A0001055(n)
1	1	11	4655	21	2870671950
2	1	12	17073	22	11123060678
3	1	13	63600	23	43191857688
4	2	14	238591	24	168047007728
5	5	15	901971	25	654999700403
6	12	16	3426576	26	2557227044764
7	35	17	13079255	27	9999088822075
8	108	18	50107909	28	39153010938487
9	369	19	192622052	29	153511100594603
10	1285	20	742624232		

Table A.1. A0001055 Polyominoes or square animals.

n	A001168(n)	n	A001168(n)
1	1	24	5239988770268
2	2	25	20457802016011
3	6	26	79992676367108
4	19	27	313224032098244
5	63	28	1228088671826973
6	216	29	4820975409710116
7	760	30	18946775782611174
8	2725	31	74541651404935148
9	9910	32	293560133910477776
10	36446	33	1157186142148293638
11	135268	34	4565553929115769162
12	505861	35	18027932215016128134
13	1903890	36	71242712815411950635
14	7204874	37	281746550485032531911
15	27394666	38	1115021869572604692100
16	104592937	39	4415695134978868448596
17	400795844	40	17498111172838312982542
18	1540820542	41	69381900728932743048483
19	5940738676	42	275265412856343074274146
20	22964779660	43	1092687308874612006972082
21	88983512783	44	4339784013643393384603906
22	345532572678	45	17244800728846724289191074
23	1344372335524	46	68557762666345165410168738

Table A.2. A001168 Fixed polyominoes with n cells.

A.2 Number of polyiamonds

n	A000577(n)	n	A000577(n)	n	A000577(n)
1	1	11	1186	21	41835738
2	1	12	3334	22	121419260
3	1	13	9235	23	353045291
4	3	14	26166	24	1028452717
5	4	15	73983	25	3000800627
6	12	16	211297	26	8769216722
7	24	17	604107	27	25661961260
8	66	18	1736328	28	75195166667
9	160	19	5000593		
10	448	20	14448984		

Table A.3. A000577 Triangular polyominoes (or polyiamonds) with n cells (turning over is allowed, holes are allowed, must be connected along edges).

n	AA001420(n)	n	A001420(n)	n	A001420(n)
1	2	11	14016	21	501994070
2	3	12	39169	22	1456891547
3	6	13	110194	23	4236446214
4	14	14	311751	24	12341035217
5	36	15	886160	25	36009329450
6	94	16	2529260	26	105229462401
7	250	17	7244862	27	307942754342
8	675	18	20818498	28	902338712971
9	1838	19	59994514		
10	5053	20	173338962		

Table A.4. A001420 Fixed 2-dimensional triangular-celled animals with n cells.

A.3 Number of polyhexes

n	A000228(n)	n	A000228(n)	n	A000228(n)
1	1	8	1448	15	76581875
2	1	9	6572	16	372868101
3	3	10	30490	17	1822236628
4	7	11	143552	18	8934910362
5	22	12	683101	19	43939164263
6	82	13	3274826	20	216651036012
7	333	14	15796897		

Table A.5. A000228 Hexagonal polyominoes.

n	A001207(n)	n	A001207(n)	n	A001207(n)
1	1	9	77359	17	21866153748
2	3	10	362671	18	107217298977
3	11	11	1716033	19	527266673134
4	44	12	8182213	20	2599804551168
5	186	13	39267086	21	12849503756579
6	814	14	189492795	22	63646233127758
7	3652	15	918837374		
8	16689	16	4474080844		

Table A.6. A001207 Fixed hexagonal polyominoes with n cells.

A.4 Number of 3-dimensional polyominoes

n	A000162(n)	n	A000162(n)	n	A000162(n)
1	1	6	166	11	2522522
2	1	7	1023	12	18598427
3	2	8	6922	13	138462649
4	8	9	48311		
5	29	10	346543		

Table A.7. A000162 3-dimensional polyominoes (or polycubes) with n cells.

n	A001931(n)	n	A001931(n)	n	A001931(n)
1	1	7	23502	13	3322769321
2	3	8	162913	14	24946773111
3	15	9	1152870	15	188625900446
4	86	10	8294738	16	1435074454755
5	534	11	60494549	17	10977812452428
6	3481	12	446205905		

Table A.8. A001931 Fixed 3-dimensional polyominoes with n cells; lattice animals in the simple cubic lattice (6 nearest neighbors), face-connected cubes.

A.5 Number of polyominoes on archimedean tessellations

The numbers given in this section are all taken from [9]. The eight Archimedean tessellations are depicted in Figure A.4 to A.11 where the cyclic sequences $(p_1, p_2, \ldots, p_q)$ represent the lists of the numbers of sides of all polygons surrounding any vertex in this order. In the tables n denotes the number of cells and $a(n)$ denotes the number of free polyominoes with n cells on the given tessellation.

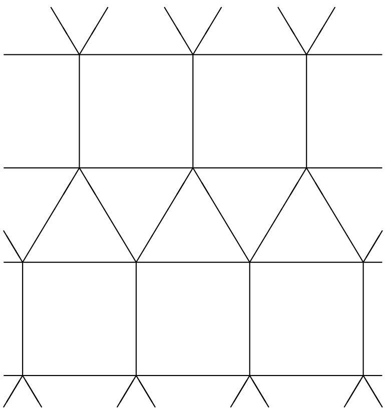

Figure A.1. (3,3,3,4,4).

n	a(n)	n	a(n)
1	2	9	2822
2	3	10	9207
3	5	11	30117
4	13	12	99708
5	32	13	331219
6	96	14	1106870
7	281	15	3710728
8	891	16	

Table A.9. (3,3,3,4,4).

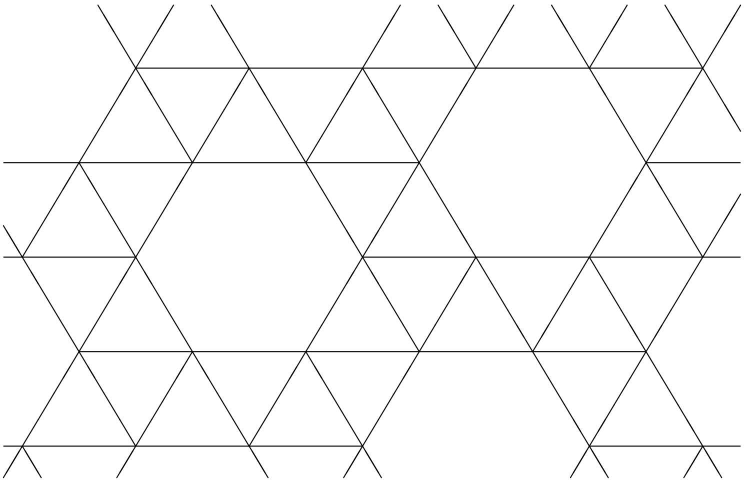

Figure A.2. (3,3,3,3,6).

n	a(n)	n	a(n)	n	a(n)	n	a(n)
1	3	5	69	9	8943	13	1345840
2	3	6	228	10	31164	14	4758782
3	7	7	762	11	108840		
4	23	8	2594	12	382063		

Table A.10. (3,3,3,3,6).

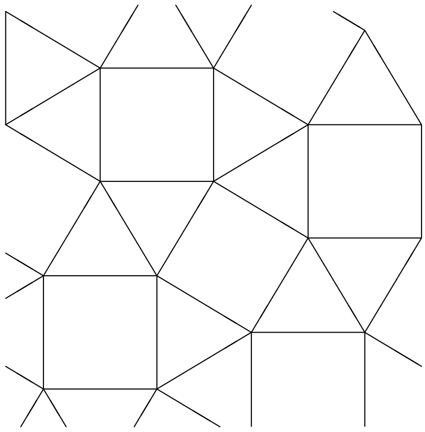

Figure A.3. (3,3,4,3,4).

n	a(n)	n	a(n)
1	2	9	2161
2	2	10	6690
3	4	11	20881
4	10	12	65593
5	28	13	207171
6	79	14	657301
7	233	15	2093785
8	705	16	

Table A.11. (3,3,4,3,4).

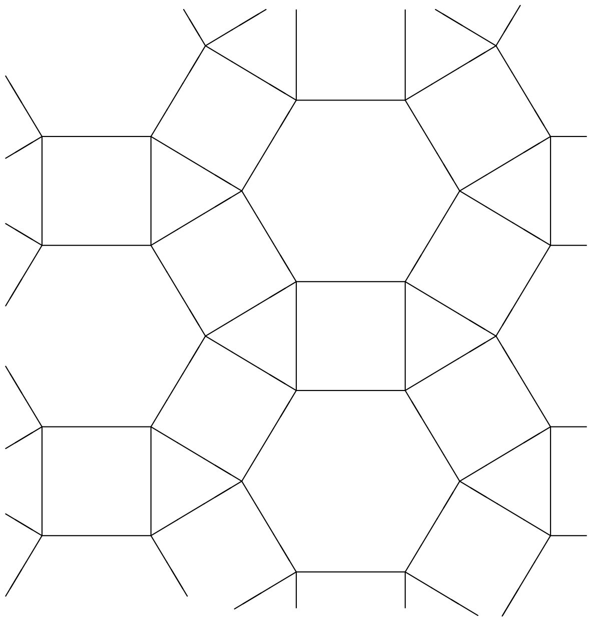

Figure A.4. (3,4,6,4).

n	a(n)
1	3
2	2
3	7
4	16
5	59
6	194
7	790
8	3116
9	13091
10	55021
11	235754
12	1015101
13	4408515

Table A.12. (3,4,6,4).

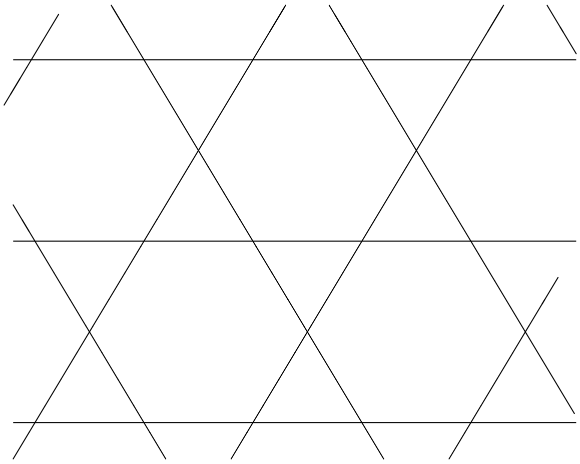

Figure A.5. (3,6,3,6).

n	a(n)
1	2
2	1
3	4
4	9
5	29
6	90
7	330
8	1167
9	4393
10	16552
11	63618
12	245732
13	957443
14	3745541

Table A.13. (3,6,3,6).

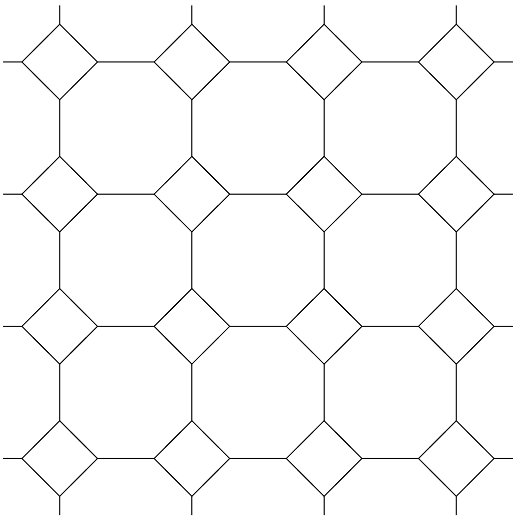

Figure A.6. (4,8,8).

n	a(n)	n	a(n)
1	2	7	1914
2	2	8	9645
3	7	9	50447
4	21	10	266992
5	90	11	1432165
6	388		

Table A.14. (4,8,8).

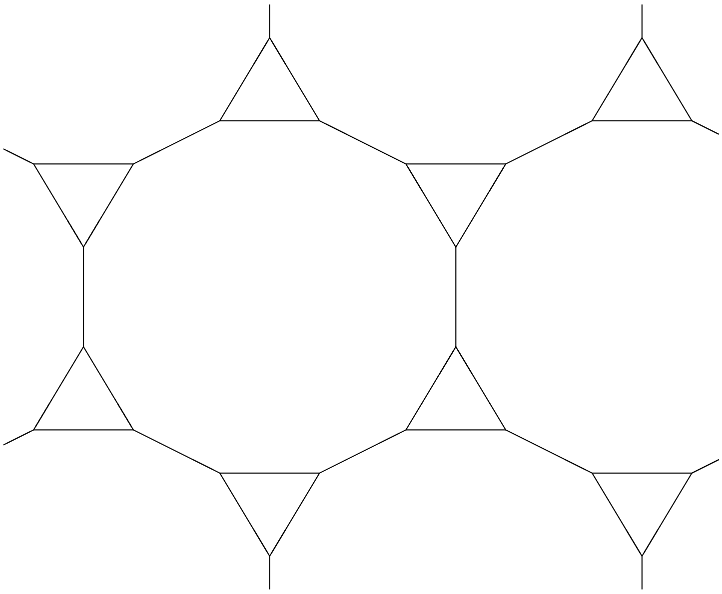

Figure A.7. (3,12,12).

n	a(n)	n	a(n)	n	a(n)	n	a(n)
1	2	4	35	7	5949	10	1541542
2	2	5	173	8	37198		
3	8	6	983	9	237762		

Table A.15. (3,12,12).

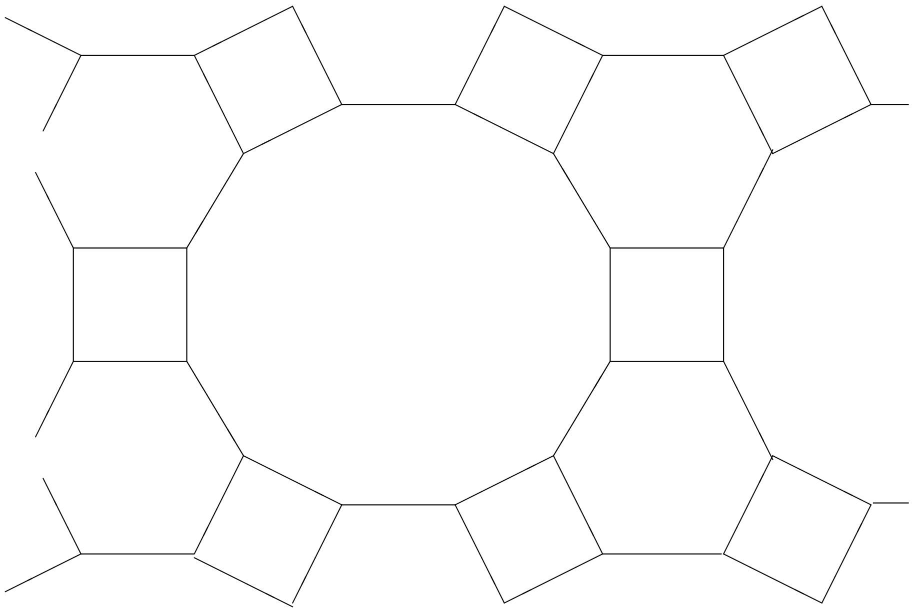

Figure A.8. (4,6,12).

n	a(n)	n	a(n)	n	a(n)	n	a(n)
1	3	4	49	7	7796	10	1697278
2	3	5	255	8	45876	11	10472378
3	14	6	1327	9	278002		

Table A.16. (4,6,12).

B Deutsche Zusammenfassung

Ein **Polyomino** ist eine über Kanten verbundene Vereiningung von Zellen im ebenen Quadratgitter. Es gibt mindestens 3 Möglichkeiten zu definieren wann 2 Polyominoes als äquivalent betrachtet werden sollen. Man bezeichnet dies als fixe Polyominoes wenn äquivalente Polyominoes durch Verschiebungen auseinander hervorgehen, als chirale Polyominoes falls wenn äquivalente Polyominoes durch Verschiebungen und Drehungen auseinander hervorgehen und man bezeichnet sie als freie Polyominoes wenn äquivalente Polyominoes durch Verschiebungen, Drehungen und Spiegelungen auseinander hervorgehen. In der Literatur werden sie auch manchmal **animals** genannt, oder man spricht vom **Zellwachstums-Problem** [74, 98]. Für den Ursprung von Polyominoes zitiere ich in freier Übersetzung Klarner [75]: "Polyominoes haben eine lange Geschichte, die bis zum Anfang des 20. Jahrhunderts zurück geht. Aber einer breitern Öffentlichkeit bekannt gemacht wurden sie zunächst von Solomon Golomb [56, 57, 58, 59, 60, 61, 62, 55] und Martin Gardner in seinen Kolumnen des *Scientific American*." Um die abstrakt definierten Polyominoes zu veranschaulichen sind in der Graphik 1.1 auf Seite 1 die Polyominoes aus höchstens 5 Quadraten dargestellt.

Es gibt mehrere Verallgemeinerungen von Polyominoes, z.B. Polyiamonds (kantenbenachbarte Vereinigungen von gleichseitigen Einheitsdreiecken) [8, 53, 66, 86, 106], Polyhexes (kantenbenachbarte Vereinigungen von regulären Einheitssechsecken) [6, 52, 53, 86], Polyabolos (kantenbenachbarte Vereinigungen von rechtwinkligen gleichschenkligen Einheitsdreiecken) [52], Polycubes (flächenbenachbarte Vereinigungen von Einheitswürfeln) [3, 87], usw. Desweiteren kann man Polyominoes als Vereinigung von Zellen auf den Archimedischen Parkettierungen [9] definieren. In dieser Arbeit betrachten wir d-dimensionale Polyominoes als flächenbenachbarte Vereinigung von d-dimensionalen Einheitswürfeln. Falls nicht anders erwähnt sind mit dem Terminus Polyominoes die freien Polyominoes gemeint.

Bevor die Sätze dieser Arbeit aufgelistet werden sollen noch ein paar Anwendungen und Probleme von Polyominoes genannt werden. Der Ausdruck Zellwachstums-Problem suggeriert Anwendungen in Medizin und Biologie. Polyominoes sind nützlich für das *Ising Model* [24] mit dem man z.B. Nervennetzwerke, Vogelschwärme, schlagende Herzzellen, Atome, Proteinfaltungen, Membrane, soziales Verhalten, usw. modellieren kann. Weitere Anwendungen liegen auf dem Gebiet der Chemie und der Physik. Als Probleme mit Polyominoes seien das Abzählen von Polyominoes [1,2,4,5,10-23,25-50,54,64,67,69-72,76,78-97,99-101,105,107,108,110,111,113], das Erzeugen von Polyominoes [109, 112], Vollendungsspiele [6, 7, 8, 9, 51, 66] und extremale Polyominoes [65, 68, 73, 77] erwähnt. Im Anhang A werden exakte Anzahlen für einige Arten von Polyominoes aufgelistet.

Der Hauptteil dieser Arbeit handelt von Polyominoes mit maximalem Flächeninhalt der konvexen Hülle. In [73] wurde gezeigt, daßder maximale Flächeninhalt der konvexen Hülle eines Polyominoes aus n Einheitsquadraten $n + \frac{1}{2}\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor$ beträgt. In dieser Arbeit wird die Vermutung aus [73] für den d-dimensionalen Fall bewiesen.

Satz 1. Das d-dimensionale Volumen der konvexen Hülle einer flächenbenachbarten Vereinigung von n Einheitshyperwürfeln is höchstens

$$\sum_{I\subset\{1,\ldots,d\}} \frac{1}{|I|!}\prod_{i\in I}\left\lfloor\frac{n-2+i}{d}\right\rfloor .$$

Die Autoren von [73] fragten nach der Anzahl von verschiedenen Polyominoes aus n Quadraten mit maximalem Flächeninhalt der konvexen Hülle.

Theorem 2. Die Anzahl $c_2(n)$ von Polyominoes im $\mathbf{R^2}$ maximalem Flächeninhalt der konvexen Hülle bestehend aus n Einheitsquadraten ist gegeben durch

$$n \equiv 0 \;\; mod\ 4 : c_2(n) = \frac{n^3-2n^2+4n}{16},$$
$$n \equiv 1 \; mod\ 4 : c_2(n) = \frac{n^3-2n^2+13n+20}{32},$$
$$n \equiv 2 \; mod\ 4 : c_2(n) = \frac{n^3-2n^2+4n+8}{16},$$
$$n \equiv 3 \;\; mod\ 4 : c_2(n) = \frac{n^3-2n^2+5n+8}{32}.$$

Satz 3. Die Anzahl $c_d(n)$ von Polyominoes im $\mathbf{R^d}$ mit maximalem Flächeninhalt der konvexen Hülle bestehend aus n Einheitshyperwürfeln ist gegeben durch

$$n \equiv a \; mod \; d \; : \; c_d(n) = \binom{\lceil \frac{n-a}{2d} \rceil + d - a}{d - a + 1} \binom{\lceil \frac{n+d-a}{2d} \rceil + a - 2}{a - 1}$$

mit $0 < a \leq d$ und $d \geq 3$.

Wenn man den maximalen Flächeninhalt der konvexen Hülle kennt kann man fragen welche Flächeninhalte möglich sind. Für den zweidimensionalen Fall wird die Situation vollständig durch den nästen Satz beschrieben.

Satz 4. Die Existenz eines zweidimensionalen Polyomino bestehend aus n Zellen mit einer Fläche a der konvexen Hülle ist äquivalent zu $a \in A_n$ mit

$$A_n = \left\{ n + \frac{m}{2} \middle| m \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor, m \in \mathbb{N}_0 \right\} - \left\{ n + \frac{1}{2} \middle| if \; n+1 \; is \, prime \right\} .$$

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