Enumerating (0,1) Matrices Uniquely Reconstructable From Their Row and Column Sum Vectors

Chad Brewbaker
Departments of Computer Science and Mathematics
Iowa State University, Ames Iowa, USA
crb002@iastate.edu

Submitted: October 13, 2003;
MR Subject Classifications: 05A15, 11B68, 11B73, 15A36

Abstract

We examine the set of (0,1) matrices that can be uniquely reconstructed from their row and column sum vectors. Their cardinality is shown to be the Poly-Bernoulli numbers of negative index. The graphs generated by adjacency matrices of this type are then surveyed.

1 Introduction

In the literature many of the results dealing with the row and column sums of (0-1) matrices are linked to a paper by Ryser[9], and further work done by Brualdi[10]. A good reference on the subject is chapter two of [11]. Discrete Tomography, the study of constructing (0,1) matrices from their row and column sums, is another branch which is surveyed in [13]. For this paper we will be dealing with (0,1) matrices that can be uniquely reconstructed from their row and column sums. We will need the following theorem.

Theorem 1 (Ryser-Fulkerson 1950's) *Any (0,1) matrix with row sum vector R and column sum vector S can be transformed into any other (0,1) matrix with row sum vector R and column sum vector S via* ***interchange operations****.*

An ***interchange operation*** is one of the following: replacing the sub-matrix $\binom{01}{10}$ with $\binom{10}{01}$, or replacing the sub-matrix $\binom{10}{01}$ with $\binom{01}{10}$. Note that row/column sums stay unchanged after interchange operations are preformed.

Any set of (0,1) matrices with the same row sum vector, and the same column sum vector, are said to be in the same ***Ryser Class***. For example, the Ryser Class of $\binom{011}{100}$ is $\{\binom{011}{100}, \binom{101}{010}, \binom{110}{001}\}$

By the Ryser-Fulkerson Theorem, the Ryser classes can be represented as graphs where each vertex is a matrix in the Ryser class, and each edge is an interchange operation.

For this paper we will be interested in Ryser classes of size one. By the Ryser-Fulkerson theorem we can deduce that a (0,1) matrix can be uniuqely determined by its row and column sums IFF no interchange operation can be preformed on the matrix. We shall refer to the sub-matrices $\binom{01}{10}$ or $\binom{10}{01}$ as ***Forbiden Minors***, and the class of (0,1) matrices uniquely determined by their row and column sums as ***Lonesum Matrices***.

2 Enumeration of Lonesum Matrices

The ***Poly-Bernoulli Numbers*** are defined as:
$B_n^{(k)} = \sum_{m=0}^{n}(-1)^{n+m}m!\left\{{n \atop m}\right\}(m+1)^{-k}$[6]
Note that $\left\{{n \atop m}\right\}$ are Stirling Numbers of the Second Kind.

Theorem 2 (Main Result) *The number of distinct $n \times k$ Lonesum Matrices is $B_n^{(-k)}$.*

Proof:
We shall view this as a language counting problem. The symbols in our alphabet will be {0,1} columns. Forbidden words will have two symbols(columns) that form one of the forbidden minors : $\binom{01}{10}$ or $\binom{10}{01}$. For example, the three symbol word $\binom{010}{110}$ is allowed while the three symbol word $\binom{011}{110}$ is forbidden because it contains the forbidden minor $\binom{01}{10}$.

Lemma 1 (Symbol Weight) *: No two symbols of the same* weight *occur in the same word of our language, unless they are the same symbol.*

Proof: By weight we mean the number of 1's in a given symbol. Let S_1 and S_2 be symbols in our alphabet, $S_1 \neq S_2$, and weight(S_1) = weight(S_2). Since $S_1 \neq S_2$ there must be a row where they have different values. Scanning both symbols from top to bottom find the first row where their weights differ and call it X. At this point the running total of one symbol weight is bigger than the other, so scan down to the row where their weights are equal again and call that row Y. Thus, we have $\binom{X(0,1)}{Y(1,0)}$ or $\binom{X(1,0)}{Y(0,1)}$. Since both are forbidden minors the lemma holds. We can't have two symbols of the same weight occur in the same word of our language unless they are the same symbol.□

Corollary 1 (Row Weight) *By a similar argument no two rows have the same weight unless they are identical.*

Lemma 2 (Swap) *Permuting rows/columns does not change membership in the class of Lonesum matrices.*

Proof:
Note that the forbidden minors are mirror images. Swapping either the rows or columns of one forbidden minor will give you the other. Thus, any permutation of rows or columns will still yield the same number of forbidden minors. □

A $(i+1)$ by (i) matrix who's rows are $1^{(i-j)}0^j$ for $(0 \leq j \leq i)$ shall be refered to as a ***Upper 1 Matrix***. Let G_i be an onto mapping from the rows of an i symbol word to the rows of an Upper 1 Matrix:

Example: $\begin{pmatrix} row(1) \\ row(2) \\ \vdots \\ row(n) \end{pmatrix} ONTO- > \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Lemma 3 (Mapping) *The words formed by the onto function G_i when combined with the all 1's symbol and the all 0's symbol form every possible Lonesum Matrix.*

Proof:

Proposition 1 (Uniqueness)

This will be proved by a contradiction. Assume that a Lonesum Matrix exists whose symbols don't arise from the onto mapping G_i along with the all 1's symbol and the all 0's symbol. Erase all but one copy of every duplicate symbol in the matrix, and any all-1's or all-0's symbols. By the weight lemma all remaining symbols have a different weight, so sort them in decreasing order of their weight. Since 1*0* rows are in the onto mapping we want a row of the form $*0*1*$, thus two columns of the form A(...0...) B(...1...). Column A has a higher weight than column B, so two copies of the row (1 0) (1 0) must exist. We have a forbidden minor $\binom{01}{10}$ or$\binom{10}{01}$, and thus a contradiction.□

Proposition 2 (Existance)

Again we will use contradiction. Assume that the mapping G_i created a non-Lonesum matrix. Note that the row (...0...1...) must exist to create a forbidden minor. This row does not exist in the mapping, and thus we have a contradiction.□
By the two propositions we have proved the Mapping Lemma. □

By the Mapping Lemma we can count the number of legal words on m symbols. If each symbol is n digits tall, we get the onto functions from an n set to an m set. This is $m!\left\{{n \atop m}\right\}$.[2] We can now construct a counting formula using inclusion-exclusion on these words.

Lemma 4 (Counting Formula) *The number of $n \times m$ Lonesum Matrices is* $\sum_{m=1}^{n}(-1)^{n+m}m!\left\{{n \atop m}\right\}(m+1)^k$

By inclusion-exclusion we know that our counting formula is:
$\sum_{m=1}^{n}(-1)^{n+m}A*B$, were A is the number symbol sets of size m-1, and B is the number of strings length k on an m-1 symbol set along with the all-1 and all-0 symbols. By the mapping lemma $A = m!\left\{{n \atop m}\right\}$. Trivially $B = (m+1)^k$. Thus, the formula is:
$\sum_{m=1}^{n}(-1)^{n+m}m!\left\{{n \atop m}\right\}(m+1)^k$. □

By the counting lemma the number of Lonesum matrices is equal to the Poly-Bernoulli numbers of negative index. Thus, we have proved our main theorem.$\square$

We can now use transformations on sets of Lonesum matrices to simplify proofs about the Poly-Bernoulli Numbers of Negative Index. For instance, take this identity proved by Kaneko[6].

Corollary 2 (Inversion) *For any $n, k \geq 0$ we have $B_n^{(-k)} = B_k^{(-n)}$*

Lonesum Matrix Proof:
Transpose the set of $n \times k$ Lonesum Matrices. Since $\binom{01}{10}$ and $\binom{10}{01}$ are duals under this transformation no forbidden matrices are added or deleted from our set. $\square$

3 Survey of n by n Lonesum Matrices

The following is indended to be a small survey. Almost, if not all of the results presented here can be found in the literature. First we will look at n by n Lonesum Matrices with no restriction.

Corollary 3 (Labeled Directed Graphs) *The number of n by n labeled directed graphs uniquely reconstructable from their in and out degrees is B_n^{-n}*

Proof:
By viewing the square (0,1) matrix as an adjacency matrix the row and column sum vectors correspond to in and out degrees for each vertex. Thus, n by n Lonesum Matrices are equivalent to labeled directed graphs uniquely reconstructable from their in and out degree vectors. From above we know that the number of these matrices is B_n^{-n}.$\square$
How about labeled directed graphs without loops? In this case we restrict ourselves to square Lonesum matrices that have 0 down the main diagonal; $\forall_i a_{i,i} = 0$. Instead of enumerating them outright we will show that they are equivalent to labeled interval orders on n elements A079144[3].

Definition 1 (Interval Order) *a, b, c, d are distinct elements in a set I with two properties:*
(i)Irreflexive: $a < a$ is not valid.
(ii)Relativity:$a < b$,$c < d$ implies at least one of $a < d$,$c < b$

Observation 1 (Labeled Interval Orders on n Elements) *Square Lonesum Matrices with 0 down the main diagonal are the set of labeled interval orders on n elements*

Proof:
View the matrix as $a_{i,j} = 1 \Leftrightarrow i < j$.
(i) is satisfied because $a_{i,i} = 0$, thus we never have $a < a$
(ii)is satisfied because we have no forbidden minors.

Take the intervals a, b, e, f. The forbidden minor:

$$\begin{array}{cc} < & \begin{matrix} e & f \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

shows that $a < e$, $b < f$. It violates condition (ii) because $a < f$ or $b < e$ must exist.$\square$

The set of Lonesum matrices that correspond the adjacency matrices of simple graphs is pretty dull. If $a_{i,j} = 1$ then $a_{j,i} = 1$ by symmetry, and we get the forbidden minor:

$$\begin{array}{cc} \spadesuit & \begin{matrix} a & b \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

Thus, the only matrix in this class is the all zero n by n matrix; the empty graph.

Suppose we allow every vertex to have a loop. This is the set of Lonesum n by n matrices with $a_{i,j} = a_{j,i}$.

Let's list the forbidden minors and their implications. Assume a, b, c, d are distinct vertexes:

$$\begin{array}{cc} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

(1) Every edge is connected to a looped vertex. If the edge ab exists, but a loop on a or b does not, then we get the above forbidden minor.

$$\begin{array}{cc} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

(2) Every two loops are connected by an edge. Two loops not connected by an edge forms the above forbidden minor.

$$\begin{array}{cc} & \begin{matrix} a & c \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

(3)Every loop is distance ≤ 1 from every edge. From the above forbidden minor, given the loops a and c we must also include at least one of the edges ac or ab. Either edge would make the distance between a and bc at most one.

$$\begin{array}{cc} & \begin{matrix} a & c \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

(4)Any two edges sharing a non-loop vertex form a triangle. If a is a non-loop vertex, and ac, bc are edges in the graph, then edge bc must be added to avoid the above forbidden minor.

$$\begin{array}{cc} & \begin{matrix} c & d \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

(5)Every two non-adjacent edges are connected by an edge. Given the edges ac, bd we must insert either edge bc or edge ad to avoid the above forbidden minor.

The resulting graph consists of a completely connected and looped center, surrounded by a set of non-adjacent and unlooped vertexes.

Theorem 3 (Labeled Lonesum Graphs) *The number of labeled Lonesum graphs on n vertexes is:*
$G(n) = \sum_{k=0}^{n}[\binom{n}{k}2^{k*(n-k)}]$

Proof:
Since the looped vertexes are totally connected make a choice for each set of k looped vertexes. Next we must decide how to connect the remaining $n-k$ vertexes to the looped vertexes. Each vertex has k possible edges to a looped vertex,or 2^kedge configurations. There are $n-k$ vertexes that must make this choice, thus $2^{k*(n-k)}$ total edge configurations for every k set of looped vertexes.□

4 Acknowledgments

Thanks to Dan Ashlock for posing the question of enumerating Lonesum Matrices to me, and for our discussions.

References

[1] J.H. van Lint & R.M.Wilson,*A Course in Combinatorics 2nd Ed.*, 2001, ISBN 0521006015

[2] Miklos Bona,*A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory*, 2002, ISBN 9810249004

[3] Neil Slone,*The On-Line Encyclopedia of Integer Sequences*,http://www.research.att.com/' 'njas/sequences

[4] Dan Ashlock,*On Sets of 0-1 Matrices with Equal Marginal Sums*, Spring 2003 offering of Math552:Enumerative Combinatorics, Iowa State University

[5] Martin J. Erickson, *Introduction to Combinatorics*, pg. 82, 1996, ISBN 0471154083

[6] Masanobu Kaneko,*Poly-Bernoulli Numbers*, Journal de Théorie des Nombres de Bordeaux 9, 1997, 221-228

[7] Tsuneo Arakawa and Masanobu Kaneko, *On Poly-Bernoulli Numbers*, July 12, 1999

[8] Arnold Adelberg, *Kummer Congruences for Universal Bernoulli Numbers and Related Congruences for Poly-Bernoulli Numbers*,International mathematical journal 1 (2002), no. 1, 53-63.

[9] H.J. Ryser, *Combinatorial Properties of Matrices of Zeros and Ones*, Canadian Journal of Mathematics (9) 371-377, 1957

[10] Richard A. Brualdi, *Matrices of Zeros and Ones with Fixed Row and Column Sums*,Linear Algebra and Its Applications (33) 159-231, 1980

[11] V.N. Sachkov and V.E.Tarakanov,*Combinatorial Properties of Nonnegative Matrices*, Chapter 2: Ryser Classes 45-82, 2002, ISBN 082182788X

[12] Thomas Lockman Greenough, *Representation and Enumeration of Interval Orders and Semiorders*, PhD thesis, Dartmouth College, June 1976

[13] *Special Issue on Discrete Tomography*, Linear Algebra and Its Applications, Volume 339, 2001