Admissible partitions and the square of the Vandermonde determinant

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Abstract. The expansion of the second power of the Vandermonde determinant as a finite sum of Schur functions is considered.

1. Introduction

Laughlin[1] has described the fractional quantum Hall effect in terms of a wavefunction

$$\Psi^m_{Laughlin}(z_1, \ldots, z_N) = \prod_{i<j}^{N}(z_i - z_j)^{2m+1} \exp\left(-\frac{1}{2}\sum_{i=1}^{N}|z_i|^2\right) \tag{1}$$

The Vandermonde alternating function in N variables is defined as

$$V(z_1, \ldots, z_N) = \prod_{i<j}^{N}(z_i - z_j) \tag{2}$$

$$\frac{\Psi_{Laughlin}}{V} = V^{2m} = \sum_{\lambda\vdash n} c^\lambda s_\lambda \tag{3}$$

where $n = mN(N-1)$ and the s_λ are Schur functions. The coefficients c_λ are signed integers.

Dunne[2] and Di Francesco *et al*[3] have discussed properties of the expansions while Scharf *et al*[4] have given specific algorithms for computing the expansions for $m = 1$ with N from 2 to 9. The author has extended these results to $N = 10$ leading to a number of new conjectures.

1.1. Expansion of the Laughlin wavefunction

Henceforth we consider the case where $m = 1$. The partitions, (λ), indexing the Schur functions are of weight $N(N-1)$. For a given N the partitions are bounded by a highest partition $(2N-2, 2N-4, \ldots, 0)$ and a lowest partition $((N-1)^{N-1})$ with the partitions being of length N and $N-1$.

Let

$$n_k = \sum_{i=0}^{k} \lambda_{N-i} - k(k+1) k = 0, 1, \ldots, N-1 \tag{4}$$

Di Francesco *et al*[3] define *admissible partitions* as satisfying Eq(4) with *all* $n_k \geq 0$. They computed the number of admissible partitions A_N for $N \leq 29$ and conjectured that A_N was the number of distinct partitions arising in the expansion, Eq(3), *provided none of the coefficients vanished*.

The conjecture has been shown[4] to fail for $N \geq 8$. We find the number of admissible partitions associated with vanishing coefficients as

$$(N=8) \quad 8,\ (N=9) \quad 66,\ (N=10) \quad 389$$

The coefficients of s_λ and s_{λ_r} are equal if[2]

$$(\lambda_r) = (2(N-1) - \lambda_N, \ldots, 2(N-1) - \lambda_1) \tag{5}$$

We list the 8 partitions for $N = 8$ as reverse pairs

$$\begin{array}{lll} \{13\ 11\ 985^2 41\} & \{13\ 10\ 9^2 6531\} & (Q1) \\ \{13\ 11\ 9854^2 2\} & \{13\ 10\ 987531\} & (Q2) \\ \{13\ 11\ 976541\} & \{12\ 10^2 96531\} & (Q3) \\ \{12\ 11\ 97^2 4^2 1\} & \{12\ 10^2 7^2 532\} & (Q4) \end{array}$$

1.2. The $q-$discriminant

Let $q\mathbf{x} = (qx_1, qx_2, \ldots, qx_N)$ and the $q-$discriminant of $\mathbf{x}$ be

$$D_N(q; \mathbf{x}) = \prod_{1 \leq i \neq j \leq N} (x_i - qx_j) \tag{6}$$

and

$$R_N(q; \mathbf{x}) = \prod_{1 \leq i \neq j \leq N} (x_i - qx_j)(qx_i - x_j) \qquad = \sum_\lambda c^\lambda(q) s_\lambda(\mathbf{x}) \tag{7}$$

So that

$$V_N^2(\mathbf{x}) = \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 = R_N(1; \mathbf{x}) \tag{8}$$

Introduce $q-$polynomials such that

$$R_N(q; \mathbf{x}) = \sum_\lambda c^\lambda(q) s_\lambda(\mathbf{x}) \tag{9}$$

$$R_N(q; \mathbf{x}) = \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \sum_{\nu \subseteq (N-1)^N} ((-q)^{|\nu|)} + (-q)^{N^2 - |\nu|}) \\ \times s_{(N-1)^N/\nu}(\mathbf{x}) s_{\nu'}(\mathbf{x})$$

Such expansions have been evaluated as polynomials in q for all admissible partitions for $N = 2...6$ with many examples for $N = 7, 8, 9$.

N=2	[1]	q	$\{2\}$
	[-3]	$-(q^2+q+1)$	$\{1^2\}$
N=3	[1]	q^3	$\{42\}$
	[-3]	$-q^2(q^2+q+1)$	$\{41^2\} + \{3^2\}$
	[6]	$+q(q^2+q+1)(q^2+1)$	$\{321\}$
	[-15]	$-(q^2+q+1)(q^4+q^2+q+1)$	$\{2^3\}$
N=4	[1]	q^6	$\{642\}$
	[-3]	$-q^5(q^2+q+1)$	$\{641^2\} + \{63^2\} + \{5^2 2\}$
	[6]	$+q^4(q^2+q+1)(q^2+1)$	$\{6321\} + \{543\}$
	..	.....	....

The $q-$polynomials for the four pairs of partitions designated earlier as $Q(1)...Q(4)$ are

$$
\begin{aligned}
&Q(1) - q^{17}(q^2-q+1)^2(q^2+1)^2(q^2+q+1)^5(1-q)^4 \\
&Q(2) + q^{16}(q^2-q+1)^2(q^2+1)(q^2+q+1)^6(1-q)^4 \\
&Q(3) + q^{16}(q^2-q+1)^2(q^2+1)^3(q^2+q+1)^5(1-q)^4 \\
&Q(4) + q^{14}(q^2-q+1)^2(q^2+q+1)^5(1-q)^4 \\
&\qquad \times (q^{10}+q^9+3q^8+4q^6+q^5+4q^4+3q^2+q+1)
\end{aligned}
$$

Note the factor $(q-1)^4$ which vanishes for $q=1$.

1.3. A conjecture

The following conjecture has been verified to hold for $N \leq 10$

If a $q-$*polynomial is of the form* $(-1)^\phi q^p Q(q)$ *then under* $N \to N+1$

$$\phi \to \phi,\ p \to p+N,\ Q(q) \to Q(q),\ \{\lambda\} \to \{2N-2, \lambda\}$$

Define

$$QS(N) = \sum_\lambda c_\lambda(q)$$

then

$$QS(N) = \prod_{x=0}^{[N/2]} (-3x+1) \prod_{x=0}^{[(N-1)/2]} (6x+1)$$

Di Francesco *etal*[3] establish the remarkable result that the sum of the squares of the coefficients of the second power of the Vandermonde with $q=1$ is

$$\frac{(3N)!}{N!(3!)^N}$$

What is the corresponding result for the $q-$polynomials? For $N=4$ one finds

$$
\begin{aligned}
&q^{24} + 6q^{23} + 22q^{22} + 58q^{21} + 128q^{20} + 242q^{19} \\
&+ 418q^{18} + 646q^{17} + 929q^{16} + 1210q^{15} + 1490q^{14} \\
&+ 1670q^{13} + 1760q^{12} + 1670q^{11} + 1490q^{10} + 1210q^9 \\
&+ 646q^8 + 418q^6 + 242q^5 + 128q^4 + 58q^3 + 22q^2 + 6q + 1
\end{aligned}
$$

Note the polynomial is symmetrical and unimodal! Can the general result be found?

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References

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