Enumeration of planar two-face maps ⋆

Michel Bousquet, Gilbert Labelle, Pierre Leroux

Lacim, Département de Mathématiques, Université du Québec à Montréal.

Abstract

We enumerate unrooted planar maps (up to orientation preserving homeomorphism) having two faces, according to the number of vertices and to their vertex and face degree distributions, both in the (vertex) labelled and unlabelled cases. We first consider plane maps, i.e., maps which are embedded in the plane, and then deduce the case of planar (or sphere) maps, embedded on the sphere. A crucial step is the enumeration of two-face plane maps having an antipodal symmetry and use is made of Liskovets' method in the process. The motivation for this research comes from the topological classification of Belyi functions.

Résumé

Nous dénombrons les cartes planaires (à homéomorphisme préservant l'orientation près) non pointées à deux faces, selon le nombre de sommets et selon la distribution des degrés des sommets et des faces, étiquetées (aux sommets) ou non. Nous abordons d'abord les cartes planes, c'est-à-dire plongées dans le plan, et déduisons ensuite le cas des cartes planaires (ou sphériques), plongées sur la sphère. Une étape cruciale est le dénombrement des cartes planes à deux faces admettant une symétrie antipodale et la méthode de Liskovets est utilisée pour cela. La motivation de cette recherche provient de la classification topologique des fonctions de Belyi.

Key words: Planar maps, unrooted maps, plane maps, sphere maps, degree distributions, species, Belyi functions,

1 Introduction.

The interest of studying maps is now well established. Not only are they interesting on their own, but the combinatorics of maps is also closely related to other topics, such as Galois theory, algebraic number theory or the theory of Riemann surfaces and algebraic combinatorics (see Arnold [1], Goulden

⋆ Work was partially supported by NSERC (Canada) and FCAR (Québec)

and Jackson [11] and Shabat and Zvonkin [21]). The enumeration of maps is a difficult problem. One way to approach this problem is to consider *rooted* maps, that is, maps with a distinguished and directed edge. The fact that rooted maps have only the trivial automorphism facilitates their enumeration. For papers on the enumeration of rooted planar maps, see Tutte ([24],[26]), Cori [8], Arquès [2], Bender and Wormald [5].

This paper deals with the enumeration of *unrooted planar maps having two faces*. Our main objective is to enumerate these maps according to their vertex and face degree distributions. This problem is motivated by the classification of Belyi functions, which are in correspondance with planar (hyper)maps; see Magot [18], Magot and Zvonkin [19], and Shabat and Zvonkin [21]. The case of only one face reduces to plane trees and has been completely solved; see Harrary, Prins and Tutte [12] and Tutte [25] for rooted trees, and Walkup [27] and Labelle and Leroux [14] for unrooted trees.

For other work on the enumeration of unrooted maps, see Liskovets [15]–[16], Liskovets and Walsh [17], Tutte [23] and Wormald [28], [29]. Note also that Magot [18] has given an algorithm for the generation of non rooted planar two-face maps, according to their face degree distribution.

A *planar map* m is a cellular embedding of a connected graph (multiple edges and loops permitted) into the 2-sphere S^2. This defines a partition of S^2 into vertices (points), edges (open arcs whose endpoints are vertices) and faces (regions of S^2 obtained by deletion of the vertices and edges, which are homeomorphic to open discs). Two planar maps are called *equivalent* if there exists an orientation preserving homeomorphism of S^2 which sends one into the other.

By contrast, a *plane* map, or graph, is a proper embedding of a connected graph into the plane. It can be seen as a planar map with a distinguished (exterior) face. Although not traditional, the more precise terminology of *sphere maps*, for planar maps, seems appropriate here to distinguish them from plane maps. This terminology will be used in the rest of this paper.

We will consider sphere and plane maps (up to equivalence) as structures on the set of labelled vertices. Let m and m' be two sphere maps (resp. plane maps) with vertex sets $U = \mathcal{V}(\mathsf{m})$ and $U' = \mathcal{V}(\mathsf{m}')$ respectivly. Then an *isomorphism* of maps $\mathsf{m}\tilde{\rightarrow}\mathsf{m}'$ is a bijection of the vertices $\sigma : U\tilde{\rightarrow}U'$ which is induced by an orientation preserving (possibly trivial) homeomorphism of the sphere (resp. of the plane) sending the map m into m'. In this manner, *unlabelled maps*, that is isomorphism classes, correspond exactly to the topological equivalence classes of maps.

In order to enumerate two-face maps, we first express the species of two-face *plane* maps in terms of circular permutations and of planted plane trees (see section 2). This yields the enumeration of both labelled and unlabelled two-face plane maps with n vertices, using Lagrange inversion. Moreover, the above expression can be refined, using appropriate weights, to incorporate the vertex degree and the face degree distributions.

In a second stage, two-face *sphere* maps are considered as orbits of two-face plane maps, under the antipodal transformation which exchanges the interior and the exterior faces. A crucial step then is to enumerate plane maps having an antipodal symmetry. In the labelled case, this is easily done since only the one-vertex and two-vertex cycles have this symmetry. In the unlabelled case one can use a direct bijective approach or compute the cycle index polynomial of a particular action of the dihedral group; see Bousquet [6]. Here, we rather adopt a hybrid but simpler approach which makes use of Liskovets' method [15,16], for the enumeration of sphere maps: unlabelled two-face sphere maps on $n \geq 3$ vertices can be considered as orbits of the symmetric group acting on labelled sphere maps. One difference with [15] is that the symmetric group acts on the vertices here instead of the half-edges or bits (or "brins").

An important use is made of the following fact:

Lemma 1 (See [3]). *Any periodic orientation preserving homeomorphism of the 2-sphere is conjugate by an orientation preserving homeomorphism to a rotation around a certain axis.*□

It follows that a non trivial automorphism of a sphere map leaves exactly two cells (vertex, edge, or face) fixed and that for $n \geq 3$, the representation of map automorphisms by vertex permutations is faithful. For two-face sphere maps, we can classify all possible automorphisms and enumerate their fixed points, using the concept of quotient maps as in [15,16]. This approach is easily adapted to include the vertex and face degree distributions and gives the desired results. See section 3.

We would like to thank A. Zvonkin for suggesting and motivating this work, R. Cori and G. Schaeffer for useful discussions, and the referees for helpful suggestions.

2 Two-face plane maps.

Our analysis of two-face plane maps will involve the species A of *planted plane trees*, that is, of rooted plane trees with a half edge attached to the root, which contributes one unit to the root degree and prevents the other incident edges from fully rotating around the root (see Figure 1).

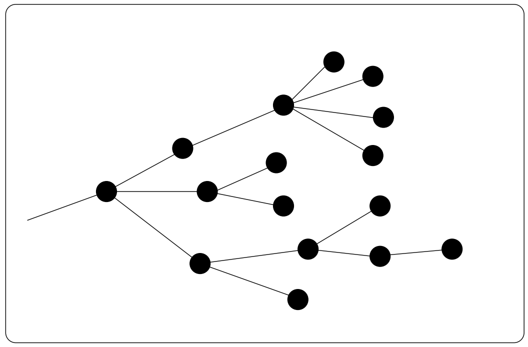

Fig. 1. A planted plane tree.

A planted plane tree is therefore an asymmetric structure. If the sets of labelled and unlabelled planted plane trees with n vertices are respectively denoted by A_n and $\widetilde{A}_n$, then their cardinalities satisfy the relation

$$|A_n| = n!|\widetilde{A}_n|$$

and the corresponding generating series

$$A(x) = \sum_{n\geq 1} |A_n|\frac{x^n}{n!} \quad \text{and} \quad \widetilde{A}(x) = \sum_{n\geq 1} |\widetilde{A}_n|x^n$$

of labelled (exponential series) and of unlabelled planted plane trees, are equal: $A(x) = \widetilde{A}(x)$. The species A of planted plane trees satisfies the combinatorial identity

$$A = XL(A), \tag{1}$$

where X is the species of singletons, and L, that of total orders (lists). This implies the following well known relation (see Tutte [25]) on the generating series :

$$A(x) = \frac{x}{1 - A(x)},$$

which can be solved algebraically to obtain

$$A(x) = \sum_{n\geq 1} \frac{1}{n}\binom{2n-2}{n-1}x^n. \tag{2}$$

More generally, by Lagrange inversion, for any integer $\alpha \geq 0$, we have

$$A^{\alpha}(x) = \sum_{n\geq \alpha} \frac{\alpha}{2n-\alpha}\binom{2n-\alpha}{n}x^n. \tag{3}$$

To keep track of the vertex degree distribution in a planted plane tree, we introduce a sequence $\boldsymbol{r} = (r_1, r_2, r_3, \ldots)$ of formal variables and a weight function w which assigns to each planted plane tree a, the weight

$$w(a) = r_1^{d_1} r_2^{d_2} r_3^{d_3} \cdots, \tag{4}$$

where d_i is the number of vertices of degree i in a. The vertex degree distribution is thus described by a vector $\boldsymbol{d} = (d_1, d_2, \ldots)$ and the following notations are used throughout this paper:

$$|\boldsymbol{d}| = \sum_i d_i \quad \text{and} \quad ||\boldsymbol{d}|| = \sum_i i d_i \tag{5}$$

corresponding respectively to the number of vertices and the total degree. The corresponding weighted species, denoted by $A_{\boldsymbol{r}}$, satisfies the combinatorial identity

$$A_{\boldsymbol{r}} = X L_{\boldsymbol{r}}(A_{\boldsymbol{r}}), \tag{6}$$

where

$$L_{\boldsymbol{r}} = 1_{r_1} + X_{r_2} + X_{r_3}^2 + \cdots$$

is the weighted species of lists where a list of length i has the weight r_{i+1}. We then have $A_{\boldsymbol{r}}(x) = x \sum_{j \geq 0} r_{j+1} A_{\boldsymbol{r}}^j(x)$, $\tilde{A}_{\boldsymbol{r}}(x) = A_{\boldsymbol{r}}(x)$, and it follows from Lagrange inversion (see Tutte [25]) that

$$A_{\boldsymbol{r}}^{\alpha}(x) = \sum_{\beta, \boldsymbol{h}} \frac{\alpha}{\beta} \binom{\beta}{\boldsymbol{h}} \boldsymbol{r}^{\boldsymbol{h}} x^{\beta}, \tag{7}$$

where

$$\binom{\beta}{\boldsymbol{h}} = \binom{\beta}{h_1, h_2, h_3, \ldots} \quad \text{and} \quad \boldsymbol{r}^{\boldsymbol{h}} = r_1^{h_1} r_2^{h_2} r_3^{h_3} \cdots,$$

the sum being taken over all integers $\beta \geq \alpha$, and vectors $\boldsymbol{h}$ such that $|\boldsymbol{h}| = \beta$ and $||\boldsymbol{h}|| = 2\beta - \alpha$.

Let C denotes the species of oriented cycles, for which

$$C(x) = \sum_{\gamma \geq 1} \frac{x^{\gamma}}{\gamma} = \log \frac{1}{1-x}, \quad \tilde{C}(x) = \frac{1}{1-x}, \tag{8}$$

and the cycle index series Z_C is given by (see [4], [13])

$$Z_C(x_1, x_2, x_3, \dots) = \sum_{m\geq 1} \frac{\phi(m)}{m} \log \frac{1}{1-x_m}, \tag{9}$$

where ϕ is the Euler phi function.

Recall that a two-face plane map is a two-face sphere map with a distinguished face. See Figure 2 for an example where the exterior (infinite) face is the distinguished one. We see that any two-face plane map can be decomposed as an oriented cycle of $XL^2(A)$-structures, where an $XL^2(A)$-structure is interpreted as a vertex to which is attached an ordered pair of lists of planted plane trees (Figure 3). In conclusion, we have the following structure theorem for the species of two-face plane maps, denoted by M.

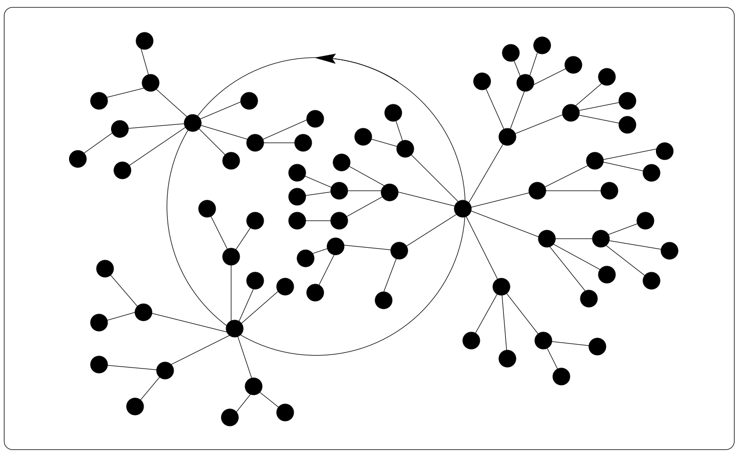

Fig. 2. A two-face plane map.

Theorem 2 *The species* M *of two-face plane maps satisfies the following combinatorial identity:*

$$\mathsf{M} = C(XL^2(A)). \tag{10}$$

□

Note that since $A = XL(A)$, we have

$$(XL^2(A))(x) = \frac{A^2(x)}{x}. \tag{11}$$

Let M_n be the set of labelled two-face plane maps over the vertex set $[n] = \{1, 2, \dots, n\}$ and $\widetilde{\mathsf{M}}_n$ the corresponding set of unlabelled maps. We have

$$|\mathsf{M}_n| = n![x^n]\mathsf{M}(x) \ \text{ and } \ |\widetilde{\mathsf{M}}_n| = [x^n]\widetilde{\mathsf{M}}(x). \tag{12}$$

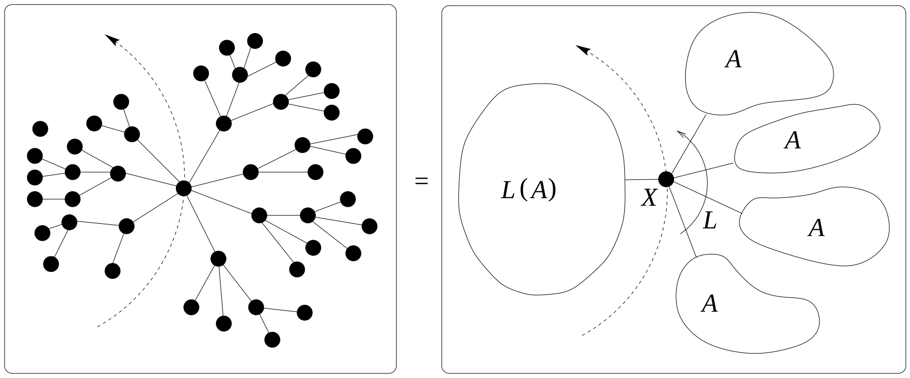

Fig. 3. An $XL^2(A)$-structure.

By using (10) and (11), we have

$$\mathsf{M}(x) = \sum_{\gamma\geq 1} \frac{A^{2\gamma}(x)}{\gamma x^{\gamma}}. \tag{13}$$

Using (3), we deduce that

$$\begin{aligned} |\mathsf{M}_n| &= (n-1)! \sum_{\gamma=1}^{n} \binom{2n}{n+\gamma} \\ &= \frac{(n-1)!}{2}\left(2^{2n} - \binom{2n}{n}\right). \end{aligned}$$

It follows from Theorem 2 and (9) and from general principles (see Theorem 1.4.2 of [4]) that

$$\begin{aligned} \widetilde{\mathsf{M}}(x) &= Z_C\left(\left(XL^2(A_L)\right)^{\sim}(x^m)\right)_{m\geq 1} \\ &= \sum_{m\geq 1} \frac{\phi(m)}{m} \log\left(1 - \frac{A^2(x^m)}{x^m}\right)^{-1} \end{aligned}$$

from which we deduce the value (15) of $|\widetilde{\mathsf{M}}_n|$ below. Hence, we have:

Theorem 3 *The numbers $|\mathsf{M}_n|$ and $|\widetilde{\mathsf{M}}_n|$ of labelled and unlabelled two-face plane maps on n vertices are respectively given by*

$$|\mathsf{M}_n| = \frac{(n-1)!}{2}\left(2^{2n} - \binom{2n}{n}\right). \tag{14}$$

and

$$|\widetilde{\mathsf{M}}_n| = \frac{1}{2n}\sum_{d|n}\phi(\frac{n}{d})\left(2^{2d} - \binom{2d}{d}\right). \tag{15}$$

□

Remark 4 *Let t_n be the number of (unlabelled) rooted sphere maps having two faces and n vertices (or n edges). It is easy to see that $n|\mathsf{M}_n| = n!t_n$ so that $t_n = \frac{1}{(n-1)!}|\mathsf{M}_n|$ and formula* (14) *is equivalent to*

$$t_n = \frac{1}{2}\left(2^{2n} - \binom{2n}{n}\right) = 2^{2n-1} - \binom{2n-1}{n-1}. \tag{16}$$

The sequence $\{t_n\}$, whose first terms are $1, 5, 22, 93, 386, 1586, \ldots$ appears in Tutte rm [26] and is presented in Sloane-Plouffe's Encyclopedia of integer sequences [22] *under* #M3920. *Similarly formulas* (28), (37) *and* (44) *below could be reformulated in terms of rooted sphere maps.*

Vertex degree distribution.

To enumerate two-face plane maps according to their vertex degree distribution, we define the weight function w_v on the species M: given a two-face map m, we set

$$w_v(\mathsf{m}) = r_1^{d_1} r_2^{d_2} r_3^{d_3} \cdots, \tag{17}$$

where d_k is the number of vertices of m of degree k. For example, the map in Figure 2 has the weight $r_1^{46}r_2^2r_3^{13}r_4^7r_8^2r_9$. It is well known (see J.W. Moon [20]) that there exists a tree having $\boldsymbol{d} = (d_1, d_2, \ldots)$ as vertex degree distribution if and only if $||\boldsymbol{d}|| = 2|\boldsymbol{d}| - 2$. It easily follows that a two-face map with vertex degree distribution $\boldsymbol{d}$ exists if and only if

$$||\boldsymbol{d}|| = 2|\boldsymbol{d}|. \tag{18}$$

Theorem 2 can be generalized to express the species M_{w_v} of two-face plane maps weighted by vertex degree in terms of the species $A_{\boldsymbol{r}}$ of planted plane trees weighted by vertex degree, defined by (6):

$$\begin{aligned}\mathsf{M}_{w_v} &= C(\sum_{m,k\geq 0} X_{r_{m+k+2}} A_{\boldsymbol{r}}^{m+k}) \\ &= C(\sum_{\lambda\geq 1} \lambda X_{r_{\lambda+1}} A_{\boldsymbol{r}}^{\lambda-1}).\end{aligned} \tag{19}$$

where X_{r_i} denotes the species of singletons, with weight r_i.

Let $\mathsf{M}_{\boldsymbol{d}}$ denote the set of labelled two-face plane maps over the set $[|\boldsymbol{d}|]$ and having $\boldsymbol{d}$ as vertex degree distribution. From (19), we deduce that

$$|\mathsf{M}_{\boldsymbol{d}}| = |\boldsymbol{d}|![\boldsymbol{r}^{\boldsymbol{d}} x^{|\boldsymbol{d}|}]\mathsf{M}_{w_v}(x), \tag{20}$$

where

$$\begin{aligned}\mathsf{M}_{w_v}(x) &= \sum_{\gamma\geq 1} \frac{x^\gamma}{\gamma} \left(r_2 + 2r_3 A_{\boldsymbol{r}}(x) + 3r_4 A_{\boldsymbol{r}}^2(x) + 4r_5 A_{\boldsymbol{r}}^3(x) + \cdots\right)^\gamma \\ &= \sum_{\substack{\gamma\geq 1\\ g_2+g_3+\cdots=\gamma}} \frac{x^\gamma}{\gamma}\binom{\gamma}{g_2, g_3, \ldots} r_2^{g_2}(2r_3)^{g_3}(3r_4)^{g_4}\cdots(A_{\boldsymbol{r}}(x))^{g_3+2g_4+\cdots}. \end{aligned}\tag{21}$$

In this sum, g_i corresponds to the number of vertices of degree i on the cycle. Note that g_1 does not appear, which is consistant with the fact that there cannot be any vertices of degree one on the cycle. We also have $|\boldsymbol{g}| = \gamma$ and $g_3 + 2g_4 + 3g_5 + \cdots = ||\boldsymbol{g}|| - 2|\boldsymbol{g}| = ||\boldsymbol{g}|| - 2\gamma$, so we can write (21) as

$$\mathsf{M}_{w_v}(x) = \sum_{\gamma\geq 1} \frac{x^\gamma}{\gamma} \sum_{\alpha\geq 0} \sum_{\substack{|\boldsymbol{g}|=\gamma\\ ||\boldsymbol{g}||=\alpha+2\gamma}} \binom{\gamma}{\boldsymbol{g}} 2^{g_3}3^{g_4}\cdots \boldsymbol{r}^{\boldsymbol{g}} A_{\boldsymbol{r}}^\alpha(x). \tag{22}$$

In this sum, α represents the number of planted plane trees which lie around the cycle. If $\alpha = 0$, all the vertices are on the cycle. Using (7), we can rewrite (22) as

$$\mathsf{M}_{w_v}(x) = \sum_{n\geq 1} \frac{x^n}{n} r_2^n + \sum \frac{\alpha}{\gamma\beta}\binom{\gamma}{\boldsymbol{g}}\binom{\beta}{\boldsymbol{h}} 2^{g_3}3^{g_4}\cdots \boldsymbol{r}^{\boldsymbol{g}+\boldsymbol{h}} x^{\gamma+\beta}, \tag{23}$$

the second sum being taken over all integers $\gamma, \alpha, \beta \geq 1$ and all vectors $\boldsymbol{g} = (g_1, g_2, \ldots)$ and $\boldsymbol{h} = (h_1, h_2, \ldots)$ such that $|\boldsymbol{g}| = \gamma, ||\boldsymbol{g}|| = \alpha + 2\gamma, g_1 = 0, |\boldsymbol{h}| = \beta$, and $||\boldsymbol{h}|| = 2\beta - \alpha$. One can write α, β and γ in terms of $\boldsymbol{g}$ and $\boldsymbol{h}$, that is

$$\alpha = ||\boldsymbol{g}|| - 2|\boldsymbol{g}|, \quad \beta = |\boldsymbol{h}| \text{ and } \gamma = |\boldsymbol{g}|. \tag{24}$$

A pure coefficient extraction, in the case $\alpha \geq 1$, gives

$$\begin{aligned} H(\boldsymbol{d}) &:= [\boldsymbol{r}^{\boldsymbol{d}} x^{|\boldsymbol{d}|}]\mathsf{M}_{w_v}(x) \\ &= \sum_{\boldsymbol{g},\boldsymbol{h}} \frac{||\boldsymbol{g}|| - 2|\boldsymbol{g}|}{|\boldsymbol{g}|\,|\boldsymbol{h}|} \binom{|\boldsymbol{g}|}{\boldsymbol{g}}\binom{|\boldsymbol{h}|}{\boldsymbol{h}} 2^{g_3}3^{g_4}\cdots, \end{aligned}\tag{25}$$

the sum being taken over all pairs of non-zero vectors $(\boldsymbol{g}, \boldsymbol{h})$ such that $\boldsymbol{g}+\boldsymbol{h} = \boldsymbol{d}$ and $g_1 = 0$.

For unlabelled two-face plane maps having $\boldsymbol{d}$ as vertex degree distribution, we deduce from (19) that

$$|\widetilde{\mathsf{M}}_{\boldsymbol{d}}| = [\boldsymbol{r}^{\boldsymbol{d}} x^{|\boldsymbol{d}|}]\widetilde{\mathsf{M}}_{w_v}(x), \tag{26}$$

with, by the composition theorem for weighted species (see [4], section 4.3),

$$\widetilde{\mathsf{M}}_{w_v}(x) = Z_C\left(\sum_{\lambda\geq 1} \lambda r_{\lambda+1}^m x^m A_{\boldsymbol{r}^m}^{\lambda-1}(x^m)\right)_{m\geq 1}, \tag{27}$$

where $A_{\boldsymbol{r}^m}$ is the weighted species of planted plane trees in which the weight of each structure, as defined in (4), is raised to the m-th power. After expanding and extracting coefficients we obtain the following result.

Theorem 5 *Let $\boldsymbol{d}$ be a vector satisfying $||\boldsymbol{d}|| = 2|\boldsymbol{d}|$. Then the number $|\mathsf{M}_{\boldsymbol{d}}|$ of labelled two-face plane maps having $\boldsymbol{d}$ as vertex degree distribution is given by $(|\boldsymbol{d}|-1)!$ if $|\boldsymbol{d}| = d_2$, and otherwise, by*

$$|\mathsf{M}_{\boldsymbol{d}}| = |\boldsymbol{d}|!H(\boldsymbol{d}), \tag{28}$$

where $H(\boldsymbol{d})$ is given by (25). Also the number $|\widetilde{\mathsf{M}}_{\boldsymbol{d}}|$ of unlabelled two-face plane maps having $\boldsymbol{d}$ as vertex degree distribution is given by $|\widetilde{\mathsf{M}}_{\boldsymbol{d}}| = 1$ if $|\boldsymbol{d}| = d_2$, and otherwise by

$$|\widetilde{\mathsf{M}}_{\boldsymbol{d}}| = \sum_{m|\boldsymbol{d}} \frac{\phi(m)}{m} H(\boldsymbol{d}/m), \tag{29}$$

the sum being taken over common divisors m of all components of $\boldsymbol{d}$, with $\boldsymbol{d}/m = (d_1/m, d_2/m, \ldots)$. □

Face degree distribution.

In order to enumerate two-face plane maps according to their face degree distribution, we introduce a new weight function w_f defined, for a two-face map m, by

$$w_f(\mathsf{m}) = s^{\gamma} t^m u^k, \tag{30}$$

where s, t and u are formal variables and γ, m and k respectively denote the number of vertices lying *on*, *outside* and *inside* the cycle. For example, the map appearing on Figure 2 has the weight $s^3 t^{43} u^{25}$.

Let α denote the degree of the outer face and β, the degree of the inner face. The triplet (γ, m, k) is sufficient to determine this degree distribution. Indeed, we have

$$\alpha = \gamma + 2m \quad \text{and} \quad \beta = \gamma + 2k, \tag{31}$$

and $\alpha + \beta = 2(\gamma + k + m) = 2n$, where n is the number of vertices of the map. We then deduce that α and β must have the same parity. One can easily verify that this condition is also sufficient for the existence of a two-face sphere map having face degree distribution (α, β).

The species M_{w_f} of two-face plane maps, weighted by w_f, can then be expressed as

$$\mathsf{M}_{w_f} = C\left(X_s \cdot L(A(X_t)) \cdot L(A(X_u))\right), \tag{32}$$

where X_s is the species of singletons weighted by s and similarly for X_t and X_u. Let $\alpha > 0$ and $\beta > 0$ have the same parity and set $n = (\alpha + \beta)/2$. Let $\mathsf{M}_{(\alpha,\beta)}$ denote the set of all two-face plane maps on $[n]$ having (α, β) as face degree distribution. We have

$$|\mathsf{M}_{(\alpha,\beta)}| = n! \sum_{\substack{1 \le \gamma \le min(\alpha,\beta) \\ 2|\gamma+\alpha}} [s^\gamma t^{(\alpha-\gamma)/2} u^{(\beta-\gamma)/2} x^n] \mathsf{M}_{w_f}(x). \tag{33}$$

Note that

$$A(X_t) = X_t L(A(X_t)), \tag{34}$$

so that at the level of generating series,

$$L(A(X_t))(x) = \frac{A(xt)}{xt}, \tag{35}$$

and similarly for $A(X_u)$. Therefore, using (32) and (3), we have

$$\mathsf{M}_{w_f}(x) = \sum_{\gamma \ge 1} \frac{s^\gamma}{\gamma (tux)^\gamma} A^\gamma(xt) A^\gamma(xu)$$

$$= \sum_{\gamma,i,j} \frac{1}{\gamma} \frac{\gamma}{2i-\gamma} \binom{2i-\gamma}{i} \frac{\gamma}{2j-\gamma} \binom{2j-\gamma}{j} s^{\gamma} t^{i-\gamma} u^{j-\gamma} x^{i+j-\gamma}$$

$$= \sum_{\gamma,m,k} \frac{\gamma}{(2m+\gamma)(2k+\gamma)} \binom{2m+\gamma}{m+\gamma} \binom{2k+\gamma}{k+\gamma} s^{\gamma} t^{m} u^{k} x^{\gamma+m+k}$$

$$= \sum_{\gamma,\alpha,\beta} \frac{\gamma}{\alpha\beta} \binom{\alpha}{\frac{\alpha+\gamma}{2}} \binom{\beta}{\frac{\beta+\gamma}{2}} s^{\gamma} t^{(\alpha-\gamma)/2} u^{(\beta-\gamma)/2} x^{(\alpha+\beta)/2}, \qquad (36)$$

the last sum being taken over all triplets of integers γ, α, β such that $\gamma \geq 1, \alpha, \beta \geq \gamma, 2|\alpha-\gamma, 2|\beta-\gamma$. The next result follows, using the identity, for $\alpha \equiv \beta \pmod 2$,

$$\sum_{\substack{\gamma=1 \\ 2|\alpha+\gamma}}^{\min(\alpha,\beta)} \gamma \binom{\alpha}{\frac{1}{2}(\alpha+\gamma)} \binom{\beta}{\frac{1}{2}(\beta+\gamma)} = \frac{\alpha\beta}{\frac{1}{2}(\alpha+\beta)} \binom{\alpha-1}{\lfloor \alpha/2 \rfloor} \binom{\beta-1}{\lfloor \beta/2 \rfloor}$$

which can be deduced from a formula due to Knuth (see [10], eq. 3.152), with similar computations in the unlabelled case.

Theorem 6 *Let α and β be two strictly positive integers having the same parity. The number $|\mathsf{M}_{(\alpha,\beta)}|$ of labelled two-face plane maps having (α, β) as face degree distribution is given by*

$$|\mathsf{M}_{(\alpha,\beta)}| = (n-1)! \binom{\alpha-1}{\lfloor \alpha/2 \rfloor} \binom{\beta-1}{\lfloor \beta/2 \rfloor}, \qquad (37)$$

where $n = (\alpha+\beta)/2$ is the number of vertices. Moreover, the corresponding number $|\widetilde{\mathsf{M}}_{(\alpha,\beta)}|$ of unlabelled 2-face plane maps is given by

$$|\widetilde{\mathsf{M}}_{(\alpha,\beta)}| = \frac{1}{n} \sum_{\ell|(\alpha,\beta)} \phi(\ell) \binom{\frac{\alpha}{\ell}-1}{\lfloor \frac{\alpha}{2\ell} \rfloor} \binom{\frac{\beta}{\ell}-1}{\lfloor \frac{\beta}{2\ell} \rfloor}. \qquad (38)$$

□

Joint vertex and face degree distributions.

Consider the plane map shown in Figure 2. The vertex and face degree distributions are respectively given by

$$\boldsymbol{d} = (46, 2, 13, 7, 0, 0, 0, 2, 1, 0, \ldots) \text{ and } (\alpha, \beta) = (89, 53). \qquad (39)$$

The vector $\boldsymbol{d}$ decomposes as the sum of the three vectors

$$\boldsymbol{d} = \boldsymbol{g} + \boldsymbol{h} + \boldsymbol{k},$$

where $\boldsymbol{g}, \boldsymbol{h}$ et $\boldsymbol{k}$ respectively denote the degree distributions of vertices that lie on, outside and inside the cycle. In our example, we have

$$\boldsymbol{g} = (0,0,0,0,0,0,0,2,1,0,\ldots),\ \boldsymbol{h} = (29,1,7,6,0,\ldots)$$
$$\text{and } \boldsymbol{k} = (17,1,6,1,0,\ldots).$$

We note that $2|\boldsymbol{h}| - ||\boldsymbol{h}|| = 10$ and $2|\boldsymbol{k}| - ||\boldsymbol{k}|| = 9$, which are respectively the number of outer and inner ordered rooted trees. The term $2|\boldsymbol{h}| - ||\boldsymbol{h}||$ is called the *residual degree* of $\boldsymbol{h}$ and is denoted by $\mathrm{res}(\boldsymbol{h})$.

Let $\boldsymbol{s} = (s_1, s_2, s_3, \ldots)$, $\boldsymbol{t} = (t_1, t_2, t_3, \ldots)$ and $\boldsymbol{u} = (u_1, u_2, u_3, \ldots)$ be three infinite sequences of formal variables and m be a two-face plane map. We consider the weight function w_{vf} defined by:

$$w_{vf}(\mathsf{m}) = \boldsymbol{s}^{\boldsymbol{g}} \boldsymbol{t}^{\boldsymbol{h}} \boldsymbol{u}^{\boldsymbol{k}},$$

where

$$\boldsymbol{s}^{\boldsymbol{g}} = s_1^{g_1} s_2^{g_2} s_3^{g_3} \cdots,\quad \boldsymbol{t}^{\boldsymbol{h}} = t_1^{h_1} t_2^{h_2} t_3^{h_3} \cdots,\quad \text{and}\quad \boldsymbol{u}^{\boldsymbol{k}} = u_1^{k_1} u_2^{k_2} u_3^{k_3} \cdots,$$

respectively describe the distributions of degrees of vertices which lie on, outside and inside the cycle. For instance, the map shown in Figure 2 has the weight $s_8^2 s_9^1 t_1^{29} t_2 t_3^7 t_4^6 u_1^{17} u_2 u_3^6 u_4$. Note that this weight is sufficient to fully describe both vertex and face degree distributions, since

$$\boldsymbol{d} = \boldsymbol{g} + \boldsymbol{h} + \boldsymbol{k},\ \alpha = 2|\boldsymbol{h}| + |\boldsymbol{g}|,\ \text{and}\ \beta = 2|\boldsymbol{k}| + |\boldsymbol{g}|.$$

The corresponding weighted species is then expressed by

$$\mathsf{M}_{w_{vf}} = C\Big(\sum_{\ell, m \geq 0} X_{s_{\ell+m+2}} A_{\boldsymbol{t}}^{\ell} A_{\boldsymbol{u}}^{m} \Big). \tag{40}$$

Let $\mathsf{M}_{\boldsymbol{d},(\alpha,\beta)}$ be the set of two-face plane maps over the set $[n]$, where $n = |\boldsymbol{d}| = (\alpha+\beta)/2$, having $\boldsymbol{d}$ and (α,β) as joint vertex and face degree distributions. Let $\mathsf{M}_{(\boldsymbol{g},\boldsymbol{h},\boldsymbol{k})}$ be the set of all two-face plane maps having $(\boldsymbol{g},\boldsymbol{h},\boldsymbol{k})$ as vertex degree distributions respectively on, outside and inside the cycle. We have

$$|\mathsf{M}_{\boldsymbol{d},(\alpha,\beta)}| = \sum_{\boldsymbol{g},\boldsymbol{h},\boldsymbol{k}} |\mathsf{M}_{(\boldsymbol{g},\boldsymbol{h},\boldsymbol{k})}|, \tag{41}$$

the sum being taken over all triplets $(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{k})$ satisfying the following conditions

$$
\begin{aligned}
&1.\ \boldsymbol{d} = \boldsymbol{g} + \boldsymbol{h} + \boldsymbol{k};\\
&2.\ \alpha = 2|\boldsymbol{h}| + |\boldsymbol{g}|,\ \ \beta = 2|\boldsymbol{k}| + |\boldsymbol{g}|;\\
&3.\ g_1 = 0,\ \ \boldsymbol{g} \neq \boldsymbol{0};\\
&4.\ \operatorname{res}(\boldsymbol{h}) \geq 0,\ \text{ and }\ \operatorname{res}(\boldsymbol{h}) = 0 \Rightarrow \boldsymbol{h} = \boldsymbol{0};\\
&5.\ \operatorname{res}(\boldsymbol{k}) \geq 0,\ \text{ and }\ \operatorname{res}(\boldsymbol{k}) = 0 \Rightarrow \boldsymbol{k} = \boldsymbol{0};
\end{aligned}
\tag{42}
$$

We find, after computations,

$$
\begin{aligned}
|\mathsf{M}_{(\boldsymbol{g},\boldsymbol{h},\boldsymbol{k})}| &= |\boldsymbol{d}|![\boldsymbol{s}^{\boldsymbol{g}}\boldsymbol{t}^{\boldsymbol{h}}\boldsymbol{u}^{\boldsymbol{k}}x^n]\mathsf{M}_{w_{vf}}(x)\\
&= \frac{|\boldsymbol{d}|!\Phi(\boldsymbol{h})\Phi(\boldsymbol{k})\Theta(\boldsymbol{g},\boldsymbol{h})}{|\boldsymbol{g}|}\binom{|\boldsymbol{g}|}{\boldsymbol{g}}\binom{|\boldsymbol{h}|}{\boldsymbol{h}}\binom{|\boldsymbol{k}|}{\boldsymbol{k}},
\end{aligned}
\tag{43}
$$

where the functions Θ and Φ are defined by

$$
\Theta(\boldsymbol{g}, \boldsymbol{h}) = [z^{\operatorname{res}(\boldsymbol{h})}](1+z)^{g_3}(1+z+z^2)^{g_4}(1+z+z^2+z^3)^{g_5}\cdots
$$

and

$$
\Phi(\boldsymbol{h}) = \begin{cases} \operatorname{res}(\boldsymbol{h})/|\boldsymbol{h}|, \text{ if } \operatorname{res}(\boldsymbol{h}) \geq 1,\\ 1, \text{ if } \boldsymbol{h} = 0,\\ 0, \text{ otherwise.} \end{cases}
$$

Similar techniques are used for the unlabelled case. We then have the following result.

Theorem 7 *Let $\boldsymbol{d}$, satisfying $||\boldsymbol{d}|| = 2|\boldsymbol{d}|$ and $\alpha, \beta > 0$, two integers having the same parity, where $|\boldsymbol{d}| = (\alpha+\beta)/2 = n$. Then the number $|\mathsf{M}_{\boldsymbol{d},(\alpha,\beta)}|$ of labelled two-face plane maps on $[n]$ having joint vertex and face degree distributions $\boldsymbol{d}$ and (α, β) is given by*

$$
|\mathsf{M}_{\boldsymbol{d},(\alpha,\beta)}| = n!H(\boldsymbol{d}, (\alpha, \beta)). \tag{44}
$$

and the corresponding number $|\widetilde{\mathsf{M}}_{\boldsymbol{d},(\alpha,\beta)}|$ of unlabelled two-face plane maps is given by

$$
|\widetilde{\mathsf{M}}_{d,(\alpha,\beta)}| = \sum_{m|(\boldsymbol{d},\alpha,\beta)} \frac{\phi(m)}{m}\, H\left(\frac{\boldsymbol{d}}{m}, (\frac{\alpha}{m}, \frac{\beta}{m})\right) \tag{45}
$$

with

$$H(\boldsymbol{d},(\alpha,\beta)) = \sum_{\boldsymbol{g},\boldsymbol{h},\boldsymbol{k}} \frac{\Phi(\boldsymbol{h})\Phi(\boldsymbol{k})\Theta(\boldsymbol{g},\boldsymbol{h})}{|\boldsymbol{g}|}\binom{|\boldsymbol{g}|}{\boldsymbol{g}}\binom{|\boldsymbol{h}|}{\boldsymbol{h}}\binom{|\boldsymbol{k}|}{\boldsymbol{k}},$$

where the sum runs over all $\boldsymbol{g},\boldsymbol{h}$ *and* $\boldsymbol{k}$ *satisfying conditions 1–5 in* (42). □

3 Sphere maps.

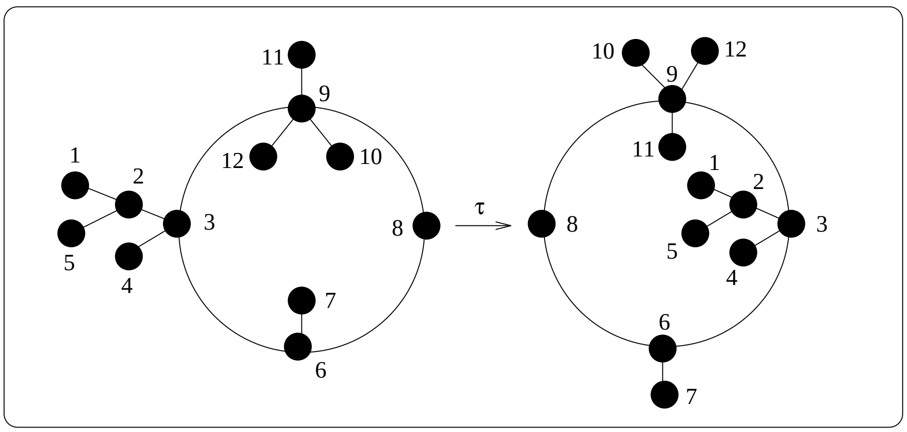

Fig. 4. Antipodal involution of a plane map.

Consider the two plane maps shown in Figure 4. Embedded in the plane, these two maps are distinct. No orientation preserving homeomorphism of the plane can send one onto the other. However, when considered embedded on the oriented sphere, both structures represent the *same* map. Imagine that the cycle lies along the equator. The left structure represents a north pole view of the map while the right structure represents a south pole view. We observe that this transformation essentially exchanges the choice of the distinguished face. Equivalently, it can be seen as a 180° rotation around an axis which passes through the equator. This transformation is clearly involutive, therefore it will be called the *antipodal involution*, and will be denoted by τ. A two-face plane map m is said to have an *antipodal symmetry* if $\tau(\mathsf{m}) = \mathsf{m}$.

Consider the group $<\tau> = \{\mathrm{Id}, \tau\}$, where Id is the identity transformation, and $\tau^2 = \mathrm{Id}$. This group *acts* on the species of two-face plane maps. More precisely, we have a familly of actions: for each finite set U, the function

$$\begin{aligned} <\tau> \times \mathsf{M}[U] &\rightarrow \mathsf{M}[U] \\ (g, \mathsf{m}) &\mapsto g \cdot \mathsf{m} \end{aligned} \qquad (46)$$

is an action of the group $<\tau>$ on the set $\mathsf{M}[U]$ of all labelled two-face plane maps over U. Also, this action commutes with any relabelling along a bijection $\sigma: U \to V$. Note that it *preserves* the vertex degree distribution and that it *reverses* the face degree distribution.

From this point of view, the two-face *sphere* maps can be seen as *orbits* of the action of $<\tau>$ on the plane maps and the species of two-face sphere maps, which will be denoted by $\mathcal{M}$, is the *quotient* of the species M of two-face plane maps by the group $<\tau>$. This is written as

$$\mathcal{M} = \mathsf{M}/<\tau> . \tag{47}$$

It follows from the Cauchy-Frobenius Theorem (alias Burnside Lemma) that for any finite class C of plane maps (labelled or unlabelled), closed under the action of τ, the cardinality of the corresponding class $\mathcal{C} = \mathsf{C}/<\tau>$ of sphere maps is given by

$$|\mathcal{C}| = |\mathsf{C}/<\tau>| = \frac{1}{2}\left(|\mathsf{C}| + |\mathrm{Fix}_{\mathsf{C}}\, \tau|\right), \tag{48}$$

where $|\mathrm{Fix}_{\mathsf{C}}\, \tau|$ is the number of maps in C having an antipodal symmetry.

3.1 Enumeration of labelled two-face sphere maps.

Let $\mathcal{M}_n$, $\mathcal{M}_{\boldsymbol{d}}$, $\mathcal{M}_{\{\alpha,\beta\}}$ and $\mathcal{M}_{\boldsymbol{d},\{\alpha,\beta\}}$ be the sets of labelled two-face *sphere* maps respectively corresponding to the sets M_n, $\mathsf{M}_{\boldsymbol{d}}$, $\mathsf{M}_{(\alpha,\beta)}$ and $\mathsf{M}_{\boldsymbol{d},(\alpha,\beta)}$ of labelled two-face plane maps. By applying equation (48) to these sets, and noting that the only labelled two-face plane maps having an antipodal symmetry are the 1-cycle (1) and the 2-cycle (12), we find:

Proposition 8 Let $\boldsymbol{d}$ satisfy $||\boldsymbol{d}|| = 2|\boldsymbol{d}|$, and $\alpha, \beta > 0$, be two integers having the same parity, and such that $n = |\boldsymbol{d}| = (\alpha+\beta)/2$ and $n \geq 3$. Then

$$|\mathcal{M}_n| = \frac{1}{2}|\mathsf{M}_n|, \tag{49}$$

$$|\mathcal{M}_{\boldsymbol{d}}| = \frac{1}{2}|\mathsf{M}_{\boldsymbol{d}}|, \tag{50}$$

$$|\mathcal{M}_{\{\alpha,\beta\}}| = \begin{cases} |\mathsf{M}_{(\alpha,\beta)}|, & \text{if } \alpha \neq \beta, \\ \frac{1}{2}|\mathsf{M}_{(\alpha,\alpha)}|, & \text{if } \alpha = \beta > 2, \end{cases} \tag{51}$$

and

$$|\mathcal{M}_{\boldsymbol{d},\{\alpha,\beta\}}| = \begin{cases} |\mathsf{M}_{\boldsymbol{d},(\alpha,\beta)}|, & \text{if } \alpha \neq \beta, \\ \frac{1}{2}|\mathsf{M}_{\boldsymbol{d},(\alpha,\alpha)}|, & \text{otherwise,} \end{cases} \tag{52}$$

where $|\mathsf{M}_n|$, $|\mathsf{M}_{\boldsymbol{d}}|$, $|M_{(\alpha,\beta)}|$ and $|\mathsf{M}_{\boldsymbol{d},(\alpha,\beta)}|$ are respectively given by equations (14), (28), (37) and (44). □

3.2 Enumeration of unlabelled two-face sphere maps.

Let $\widetilde{\mathcal{M}}_n$ denote the set of unlabelled two-face sphere maps with $n \geq 3$ vertices. Formula (48) immediately gives

$$|\widetilde{\mathcal{M}}_n| = \frac{1}{2}(|\widetilde{\mathsf{M}}_n| + |\mathrm{Fix}_{\widetilde{\mathsf{M}}_n}\tau|). \tag{53}$$

Different methods, bijective or algebraic, can be used to compute the term $|\mathrm{Fix}_{\mathcal{M}_n}\tau|$ in (53) and hence the number $|\widetilde{\mathcal{M}}_n|$. See [6], sections 3.2.2 and 3.2.3. The approach presented here uses the method of Liskovets [15,16], for the enumeration of unlabelled (and unrooted) planar (= sphere) maps: we consider unlabelled sphere maps as orbits of labelled maps under vertex relabellings, that is we write $\widetilde{\mathcal{M}}_n = \mathcal{M}_n/\mathsf{S}_n$, and invoke Burnside's Lemma, using the concept of quotient map to enumerate the fixed points. The advantage of this method is that the maps we enumerate are labelled. We have

$$|\widetilde{\mathcal{M}}_n| = \frac{1}{n!}(\mathcal{M}_n + \sum_{\sigma\in\mathcal{S}_n\backslash \mathrm{Id}} |\mathrm{Fix}_{\mathcal{M}_n}\sigma|), \tag{54}$$

where $\mathrm{Fix}_{\mathcal{M}_n}\sigma$ denotes the set of labelled two-face sphere maps for which σ is an automorphism.

It follows from Lemma 1 that any non trivial automorphism of a sphere map can be described as a rotation around an axis which intersects two of its elements. Any two-face sphere map can be drawn on the sphere in such a way that the boundary between the two faces corresponds to the equator. In this case, any non trivial automorphism is in fact a rotation around an axis of one of the four following types:

- axis intersecting the two faces: type FF;
- axis intersecting a vertex and an edge on the equator: type VE;
- axis intersecting two vertices on the equator: type VV;

- axis intersecting two edges on the equator: type EE.

Axes of type FV (face-vertex) or FE (face-edge) are obviously not allowed here since any non trivial automorphism leaving one face fixed must leave the other face fixed as well. A two-face map having an automorphism around an axis of type FF is said to have an *equatorial* symmetry, while a map having an automorphism around an axis of type VE, VV or EE is said to have an *antipodal* symmetry.

For any $\sigma \in \mathcal{S}_n$, the set $\mathrm{Fix}_{\mathcal{M}_n}\sigma$ can then be expressed as the following union

$$\mathrm{Fix}_{\mathcal{M}_n}\sigma = \bigcup_{\Gamma \in \{FF, VE, VV, EE\}} \mathrm{Fix}_{\mathcal{M}_n}(\sigma, \Gamma),$$

where $\mathrm{Fix}_{\mathcal{M}_n}(\sigma, \Gamma)$ denotes the set of maps for which σ is an automorphism of type Γ. This union is disjoint, for $n \geq 3$, and we have

$$|\widetilde{\mathcal{M}}_n| = \frac{1}{n!}(|\mathcal{M}_n| + \sum_{\substack{\sigma \in \mathcal{S}_n \setminus \mathrm{Id} \\ \Gamma \in \{FF, VE, VV, EE\}}} |\mathrm{Fix}_{\mathcal{M}_n}(\sigma, \Gamma)|\,). \tag{55}$$

In this formula, we realize that a part of the sum, namely $\sum_\sigma |\mathrm{Fix}_{\mathcal{M}_n}(\sigma, FF)|$, has essentially been computed, while enumerating two-face *plane* maps. Indeed, the analog of (54) and (55) for unlabelled plane maps is

$$\begin{aligned} |\widetilde{\mathsf{M}}_n| &= \frac{1}{n!}(\mathsf{M}_n + \sum_{\sigma \in \mathcal{S}_n \setminus \mathrm{Id}} |\mathrm{Fix}_{\mathsf{M}_n}\sigma|) \\ &= \frac{1}{n!}(\mathsf{M}_n + \sum_{\sigma \in \mathcal{S}_n \setminus \mathrm{Id}} |\mathrm{Fix}_{\mathsf{M}_n}(\sigma, FF)|) \end{aligned} \tag{56}$$

since any automorphism of a two-face plane map must leave the two faces fixed. Also, for $n \geq 3$, it is clear that

$$|\mathsf{M}_n| = 2\,|\mathcal{M}_n| \quad \text{and} \quad |\mathrm{Fix}_{\mathsf{M}_n}(\sigma, FF)| = 2\,|\mathrm{Fix}_{\mathcal{M}_n}(\sigma, FF)|$$

and we deduce from (55) that

$$|\mathcal{M}_n| = \frac{1}{2}|\widetilde{\mathsf{M}}_n| + \frac{1}{n!} \sum_{\substack{\sigma \in \mathcal{S}_n \setminus \mathrm{Id} \\ \Gamma \in \{VE, VV, EE\}}} |\mathrm{Fix}_{\mathcal{M}_n}(\sigma, \Gamma)| \tag{57}$$

and, comparing with (53), that

$$|\mathrm{Fix}_{\widetilde{\mathsf{M}}_n}\tau| = \frac{2}{n!} \sum_{\substack{\sigma\in\mathcal{S}_n\setminus\mathrm{Id} \\ \Gamma\in\{VE,VV,EE\}}} |\mathrm{Fix}_{\mathcal{M}_n}(\sigma,\Gamma)|. \tag{58}$$

Note that (58) could be proven directly using a standard result on the orbits of two commuting group actions on the same set (see [4], Exercise A.1.9), namely the groups $<\tau>$ and $\mathcal{S}_n$ acting on M_n. Another observation is that the previous reasoning remains valid if we restrict ourselves to maps having a given vertex degree distribution $\boldsymbol{d}$, with $|\boldsymbol{d}| = n \geq 3$, that is

$$|\widetilde{\mathcal{M}}_{\boldsymbol{d}}| = \frac{1}{2}(\widetilde{\mathsf{M}}_d + |\mathrm{Fix}_{\widetilde{\mathsf{M}}_{\boldsymbol{d}}}\tau|), \tag{59}$$

where

$$|\mathrm{Fix}_{\widetilde{\mathsf{M}}_{\boldsymbol{d}}}\tau| = \frac{2}{n!} \sum_{\substack{\sigma\in\mathcal{S}_n\setminus\mathrm{Id} \\ \Gamma\in\{VE,VV,EE\}}} |\mathrm{Fix}_{\mathcal{M}_{\boldsymbol{d}}}(\sigma,\Gamma)|. \tag{60}$$

There remains to compute the various terms of (58) and (60) of the form $\sum_{\sigma\in\mathcal{S}\setminus\mathrm{Id}} |\mathrm{Fix}_{\mathcal{C}}(\sigma,\Gamma)|$, for $\mathcal{C} = \mathcal{M}_n$ or $\mathcal{M}_{\boldsymbol{d}}$ and $\Gamma = VE, VV$ or EE. To do this, we will use the concept of quotient map, following Liskovets [15,16].

Computation of $\sum_{\sigma\in\mathcal{S}_n\setminus\mathrm{Id}} |\mathrm{Fix}_{\mathcal{C}}(\sigma, VE)|$.

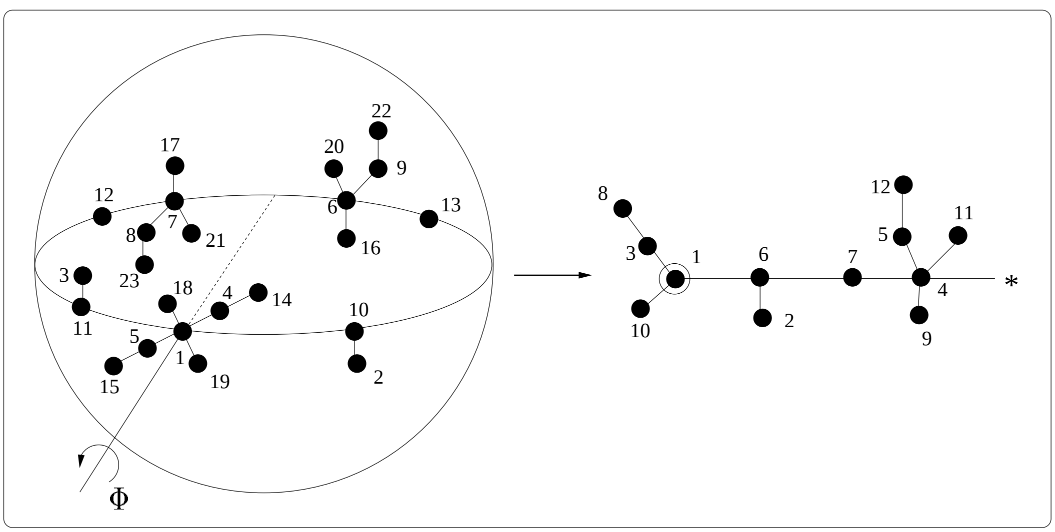

Fig. 5. A map having a symmetry of type VE and its associated quotient.

Consider a two-face sphere map m with $n \geq 3$ vertices and vertex degree distribution $\boldsymbol{d}$, having an automorphism σ of type VE. See Figure 5. In this case, σ corresponds to an antipodal rotation Φ of angle 180° around an axis intersecting one vertex and the opposite edge. This vertex is left fixed while all other vertices are exchanged pairwise. We conclude that the number n of vertices is odd and that σ is of type $\lambda(\sigma) = 1^1 2^{(n-1)/2}$. Since there are

$$\frac{n!}{2^{(n-1)/2}((n-1)/2)!}$$

permutations of cyclic type $1^1 2^{(n-1)/2}$, and $|\mathrm{Fix}_{\mathcal{C}}(\sigma, VE)|$ only depends on this cyclic type, for $\mathcal{C} = \mathcal{M}_n$ or $\mathcal{M}_{\boldsymbol{d}}$, we can write

$$\sum_{\sigma \in \mathcal{S}_n \setminus \{\mathrm{Id}\}} |\mathrm{Fix}_{\mathcal{C}}(\sigma, VE)| = \frac{n!}{2^{(n-1)/2}((n-1)/2)!} |\mathrm{Fix}_{\mathcal{C}}(\sigma_0, VE)|, \qquad (61)$$

where this time, σ_0 is the particular permutation $\sigma_0 = (1)(2,3)\cdots(n-1,n)$.

Consider the action of the subgroup $< \Phi >= \mathbb{Z}_2$ generated by the rotation Φ on the sphere S^2. The quotient space $S^2/\langle\Phi\rangle = \mathbb{Z}_2$ is obtained by identifying points on the sphere lying in the same orbit, and the induced cellular decomposition is called the *quotient map of* m by Φ. To keep track of which elements of the map were originally intersected by the rotation axis, the two corresponding elements in the quotient map are pointed. In the quotient map, the vertices are orbits (cycles) of σ_0 and they are labelled according to the increasing order of the minimum elements of the cycles.

In the present case, the quotient map $\mathsf{m}' = \mathsf{m}/\Phi$ is a labelled plane tree, having $n' = (n+1)/2$ vertices, canonically pointed at vertex 1 and planted at vertex 4 where is attached the half edge corresponding to the edge of m intersecting the rotation axis, as shown in Figure 5. The number $l(\mathsf{m}')$ of liftings of m', that is the number of different labellings of m giving rise to the same quotient is given by

$$l(\mathsf{m}') = 2^{\frac{n-1}{2}-1} = 2^{\frac{n-3}{2}} \qquad (62)$$

since after choosing the vertices $1, 2$ and 3 in a canonical way, there are two choices for each remaining cycles of σ_0. As we know from (2), there are

$$(n'-1)!\binom{2(n'-1)}{n'-1} \qquad (63)$$

labelled planted plane trees on n' vertices. If we express n' in terms of n, we get

$$|\mathrm{Fix}_{\mathcal{M}_n}(\sigma_0, VE)| = 2^{\frac{n-3}{2}} \left(\frac{n-1}{2}\right)! \binom{n-1}{(n-1)/2}. \tag{64}$$

Now, combining (61) and (64), we find, for $\mathcal{C} = \mathcal{M}_n$ and $\Gamma = VE$,

$$\sum_{\sigma \in \mathcal{S}_n \setminus \mathrm{Id}} |\mathrm{Fix}_{\mathcal{M}_n}(\sigma, VE)| = \frac{n!}{2} \binom{n-1}{(n-1)/2}. \tag{65}$$

For $\mathcal{C} = \mathcal{M}_{\boldsymbol{d}}$, it should be observed that the only fixed point of σ_0 is of even degree, say $2k$, and that the vector $\boldsymbol{d}$ has exactly one odd component, d_{2k}. Let δ_ℓ denote de vector having 1 as its ℓ^{th} component, and 0 as other components. In the quotient map m', the canonically pointed vertex number 1 has degree k and the degree distribution $\boldsymbol{d}'$ of m' is given by

$$\boldsymbol{d}' = (\boldsymbol{d} - \delta_{2k})/2 + \delta_k.$$

Using (7) with $\alpha = 1$, we know that there are $\frac{1}{n'}\binom{n'}{\boldsymbol{d}'}$ unlabelled planted plane trees having vertex degree distribution $\boldsymbol{d}'$. There are d'_k ways to select a vertex of degree k in m' and, after assigning the label 1 to it, there are $(n'-1)!$ ways to label the other vertices.

Taking into account that there are $2^{(n-3)/2}$ possible liftings, we obtain

$$|\mathrm{Fix}_{\mathcal{M}_{\boldsymbol{d}}}(\sigma_0, VE)| = 2^{((n-3)/2)} \frac{d'_k}{|n'|} \binom{n'}{\boldsymbol{d}'} (|n'|-1)! \tag{66}$$

By combining (61) and (66), and expressing $\boldsymbol{d}'$ in terms of $\boldsymbol{d}$, we obtain

$$\sum_{\sigma \in \mathcal{S}_n \setminus \mathrm{Id}} |\mathrm{Fix}_{\mathcal{M}_{\boldsymbol{d}}}(\sigma, VE)| = \frac{n!}{2} \binom{(n-1)/2}{(\boldsymbol{d} - \delta_{2k})/2}. \tag{67}$$

Computation of $\sum_{\sigma \in \mathcal{S}_n \setminus \mathrm{Id}} |\mathrm{Fix}_{\mathcal{C}}(\sigma, VV)|$.

In this case, σ corresponds to an antipodal rotation of angle $180°$ around an axis intersecting two vertices. These two vertices are left fixed while all other vertices are exchanged pairewise. Therefore the number n of vertices must be even and σ must be of type $\lambda(\sigma) = 1^2 2^{(n-2)/2}$. Since there are

$$\frac{n!}{2!2^{(n-2)/2}((n-2)/2)!}$$

permutations of cyclic type $1^2 2^{(n-1)/2}$, and $|\mathrm{Fix}_{\mathcal{C}}(\sigma, VE)|$ only depends on this cyclic type, we can write

$$\sum_{\sigma \in \mathcal{S}_n \setminus \{\mathrm{Id}\}} |\mathrm{Fix}_{\mathcal{C}}(\sigma, VV)| = \frac{n!}{2! 2^{(n-2)/2} ((n-2)/2)!} |\mathrm{Fix}_{\mathcal{C}}(\sigma_0, VV)|, \quad (68)$$

where σ_0 is the particular permutation $\sigma_0 = (1)(2)(3,4)\cdots(n-1,n)$. With this particular choice of σ_0, the quotient map is a labelled plane tree having $n' = (n+2)/2$ vertices, and canonically pointed at vertices number 1 and 2, as shown in Figure 6.

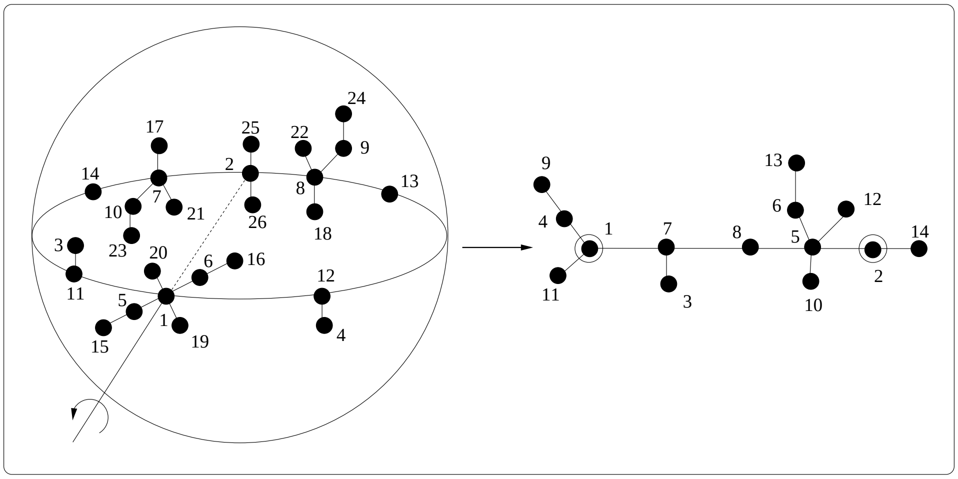

Fig. 6. A map having a symmetry of type VV and its associated quotient.

There are $\frac{(n'-2)!}{2}\binom{2(n'-1)}{n'-1}$ labelled plane trees on n' vertices (use (63) or see [4], example 3.1.17). Also note that the number of liftings, in this case, is given by $2^{(n-4)/2}$. Then, expressing n' in terms of n, we find, for $\mathcal{C} = \mathcal{M}_n$,

$$\sum_{\sigma \in \mathcal{S}_n \setminus \mathrm{Id}} |\mathrm{Fix}_{\mathcal{M}_n}(\sigma, VV)| = \frac{n!}{8}\binom{n}{n/2}. \quad (69)$$

For $\mathcal{C} = \mathcal{M}_{\boldsymbol{d}}$, note that the two fixed points of σ_0 are of even degree, say $2k$ and 2ℓ, and we may assume that $k \leq \ell$. There are two subcases to consider: either $k < \ell$ or $k = \ell$.

If $k < \ell$, the vector $\boldsymbol{d}$ has exactly two odd components, namely d_{2k} and $d_{2\ell}$. The quotient map m' is then a labelled plane tree having $\boldsymbol{d}' = (\boldsymbol{d} - \delta_{2k} - \delta_{2\ell})/2 + \delta_k + \delta_\ell$ as vertex degree distribution, and whoses vertices 1 and 2 are of degree k and ℓ, or ℓ and k. There are $(n''-2)!\binom{n'}{\boldsymbol{d}'}$ ways to select a labelled plane tree having this distribution (use (7) or see Tutte [25]). The next step consists in choosing a vertex of degree k and one of degree ℓ. There are $d'_k d'_\ell$

possibillities. This structure can then be unlabelled in $1/n'!$ ways since it is asymmetric.

Now, assign label number 1 (or 2) to the distinguished vertex of degree k. This will determine the label of the distinguished vertex of degree ℓ; there are two choices here. All other vertices are then labelled in $(n'-2)!$ possible ways. Since there are $2^{(n-4)/2}$ possible liftings, we have

$$|\mathrm{Fix}_{\mathcal{M}_{\boldsymbol{d}}}(\sigma_0, VV)| = 2^{(n-4)/2}\frac{2}{n'!}((n'-2)!)^2 d'_k d'_\ell \binom{n'}{\boldsymbol{d}'}. \tag{70}$$

Using (68) and (70), and expressing n' and $\boldsymbol{d}'$ in terms of n and $\boldsymbol{d}$, we finally find, in the case where $\boldsymbol{d}$ has exactly two odd components, d_{2k} and $d_{2\ell}$,

$$\sum_{\sigma\in\mathcal{S}_n\setminus\mathrm{Id}} |\mathrm{Fix}_{\mathcal{M}_{\boldsymbol{d}}}(\sigma, VV)| = \frac{n!}{2}\binom{(n-2)/2}{(\boldsymbol{d}-\delta_{2k}-\delta_{2\ell})/2}. \tag{71}$$

We now consider the case where $\ell = k$. This can happen only if $\boldsymbol{d}$ has no odd components. Fix $2k$ such that $d_{2k} \neq 0$, and suppose that the axis of symmetry intersects two vertices of degre $2k$. The quotient map is then a labelled plane tree having vertex degree distribution

$$\boldsymbol{d}' = \boldsymbol{d}/2 - \delta_{2k} + 2\delta_k,$$

and whose vertices number 1 and 2 are both of degree k. To construct such a map, first select one of the $(n'-2)!\binom{n'}{\boldsymbol{d}'}$ possible labelled plane trees. In this tree, select a first vertex of degree k, then a second vertex of degree k. This is possible since $d'_k \geq 2$. There are $d'_k(d'_k-1)$ possibilities. The structure obtained is now asymmetric, hence there are

$$\frac{d'_k(d'_k-1)}{n'!}(n'-2)!\binom{n'}{\boldsymbol{d}'}$$

corresponding unlabelled structures. Assign label number 1 to the first selected vertex and label 2 to the second one. The rest of the tree can be labelled in $(n'-2)!$ ways. Since there are $2^{(n-4)/2}$ possible liftings, we have, for the case where $\boldsymbol{d}$ has no odd components,

$$|\mathrm{Fix}_{\mathcal{M}_{\boldsymbol{d}}}(\sigma_0, VV)| = \sum_{\substack{k\geq 1\\ d_{2k}\neq 0}} \frac{2^{(n-4)/2}((n'-2)!)^2}{n'!} d'_k(d'_k-1)\binom{n'}{\boldsymbol{d}'} \tag{72}$$

Using (68) and (72), and expressing, n' and $\boldsymbol{d}'$ in terms of n and $\boldsymbol{d}$ we obtain in this case

$$\sum_{\sigma\in\mathcal{S}_n\setminus\mathrm{Id}} |\mathrm{Fix}_{\mathcal{M}_{\boldsymbol{d}}}(\sigma, VV)| = \frac{(n-1)!}{4}\binom{n/2}{\boldsymbol{d}/2}\sum_{k\geq 1} d_{2k}. \tag{73}$$

Computation of $\sum_{\sigma\in\mathcal{S}_n\setminus\mathrm{Id}} |\mathrm{Fix}_{\mathcal{C}}(\sigma, EE)|$.

In this case, σ corresponds to an antipodal rotation of angle 180° around an axis intersecting two edges. All vertices are exchanged pairwise. Therefore the number n of vertices must be even and σ must be of type $\lambda(\sigma) = 2^{n/2}$. Since there are $n!/(2^{n/2}(n/2)!)$ permutations of cyclic type $2^{n/2}$, and $|\mathrm{Fix}_{\mathcal{C}}(\sigma, EE)|$ only depends on this cyclic type, we can write

$$\sum_{\sigma\in\mathcal{S}_n\setminus\mathrm{Id}} |\mathrm{Fix}_{\mathcal{C}}(\sigma, EE)| = \frac{n!}{2^{n/2}(n/2)!}|\mathrm{Fix}_{\mathcal{C}}(\sigma_0, EE)|, \tag{74}$$

where σ_0 is the particular permutation of $(1,2)(3.4)\cdots(n-1,n)$. The quotient map is an (unorderly) biplanted labelled plane tree having $n' = n/2$ vertices, as shown in Figure 7. Let G denote the species of *orderly* biplanted plane trees and $|G_{n'}|$, the number of labelled G-structures on n' vertices. For $\mathcal{C} = \mathcal{M}_n$, the number of quotient structures is then given by $|G_{n'}|/2$. The species G satisfies the combinatorial identity,

$$(G+1)A = A^\bullet,$$

as shown in Figure 8, where A denotes the species of planted plane trees and $A^\bullet$, that of pointed planted plane trees. Therefore we have $G(x) = (A^\bullet(x)/A(x)) - 1$. Since

$$A(x) = \frac{1-\sqrt{1-4x}}{2} \quad\text{and}\quad A^\bullet(x) = x\frac{d}{dx}A(x) = \frac{x}{\sqrt{1-4x}},$$

we obtain

$$G(x) = \frac{1}{2}\left(\frac{1}{\sqrt{1-4x}} - 1\right).$$

After coefficient extraction, we get

$$|G_{n'}| = \frac{n'!}{2}\binom{2n'}{n'}.$$

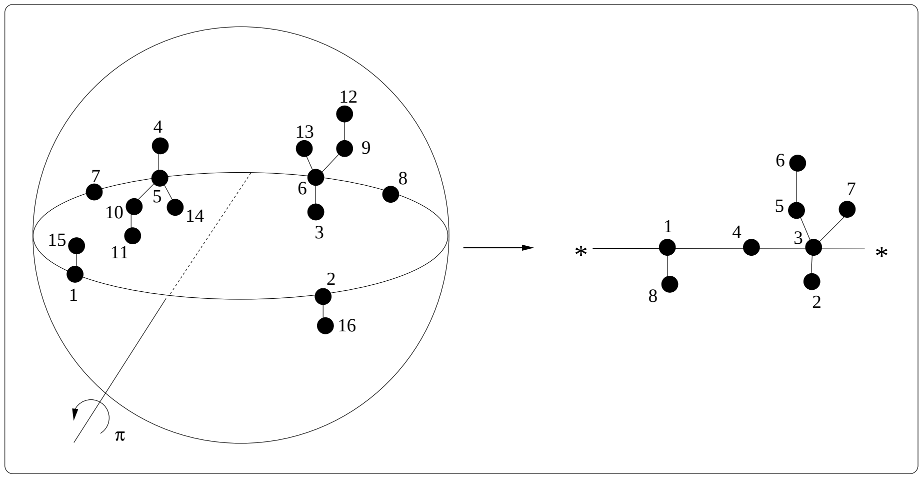

Fig. 7. A map with a symmetry of type EE and its associated quotient.

Using the fact that there are $2^{n-2/2}$ liftings and expressing n' in terms of n, we conclude that

$$\sum_{\sigma\in\mathcal{S}_n\setminus\mathrm{Id}} |\mathrm{Fix}_{\mathcal{M}_n}(\sigma, EE)| = \frac{n!}{8}\binom{n}{n/2}. \tag{75}$$

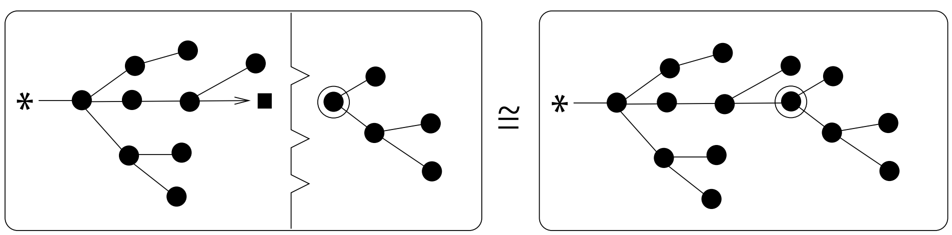

Fig. 8. $(G+1)A = A^{\bullet}$.

For $\mathcal{C} = \mathcal{M}_{\boldsymbol{d}}$, observe that the quotient map is an unorderly biplanted labelled plane tree having $\boldsymbol{d}' = \boldsymbol{d}/2$ as vertex degree distribution. To construct such a tree, first consider one of the possible $(n''-2)!\binom{n''}{\boldsymbol{d}''}$ labelled plane trees having $\boldsymbol{d}'' = \boldsymbol{d}' + 2\delta_1$ as vertex degree distribution, where $n'' = |\boldsymbol{d}''| = n'+2$. By doing so, the two star vertices in the quotient structure in Figure 7 are temporarily considered as ordinar vertices. In such a tree, select a first vertex of degree one (a leaf), and then a second vertex of degree one. There are $d_1''(d_1''-1)$ possibilities. The structure obtained has become asymmetric, hence we can divide by $n''!$ to obtain the corresponding unlabelled structures. The next step is to label all vertices except the two distinguished ones. We obtain an orderely biplanted labelled plane tree. The result has to be divided by 2 since we are aiming at unorderly biplanted plane trees. Considering the $2^{(n-2)/2}$ possible liftings, it follows that

$$|\mathrm{Fix}_{\mathcal{M}_{\boldsymbol{d}}}(\sigma_0, EE)| = \frac{1}{2}\frac{2^{(n-2)/2}((n''-2)!)^2}{n''!}\binom{n''}{\boldsymbol{d}''} d_1''(d_1''-1). \tag{76}$$

Using the two previous equations, and expressing everything in terms of $\boldsymbol{d}$ and n, we obtain

$$\sum_{\sigma\in\mathcal{S}_n\setminus\mathrm{Id}} |\mathrm{Fix}_{\mathcal{M}_{\boldsymbol{d}}}(\sigma, EE)| = \frac{n!}{4}\binom{n/2}{\boldsymbol{d}/2}. \tag{77}$$

We can now state the following results.

Theorem 9 *The number $|\widetilde{\mathcal{M}}_n|$ of unlabelled two-face sphere maps on $n \geq 3$ vertices is given by*

$$|\widetilde{\mathcal{M}}_n| = \frac{1}{4n}\sum_{s|n}\phi(\frac{n}{s})\left(2^{2s} - \binom{2s}{s}\right) + \begin{cases} \frac{1}{2}\binom{n-1}{(n-1)/2}, \text{if } n \text{ is odd,} \\ \frac{1}{4}\binom{n}{n/2}, \quad \text{otherwise.} \end{cases} \tag{78}$$

PROOF. Formula (57) states that

$$|\widetilde{\mathcal{M}}_n| = \frac{1}{2}|\widetilde{\mathsf{M}}_n| + \frac{1}{n!}\sum_{\substack{\sigma\in\mathcal{S}_n\setminus\mathrm{Id} \\ \Gamma\in\{VE,VV,EE\}}} |\mathrm{Fix}_{\mathcal{M}_n}(\sigma,\Gamma)|.$$

Replacing $|\widetilde{\mathsf{M}}_n|$ by its value, given by (15), yields the first term of (78) while summing formulas (65), where n is odd, and (69) and (75), where n is even, and dividing by $n!$, gives the second term. $\square$

Similarly, we can now use (59) and sum formulas (67), (71), (73), and (77) to obtain the following theorem. Also recall that $|\widetilde{\mathsf{M}}_d|$ is given by (29).

Theorem 10 *Let $\boldsymbol{d}$ be a vector satisfying $||\boldsymbol{d}|| = 2|\boldsymbol{d}|$, with $n = |\boldsymbol{d}| \geq 3$, and let r be the number of odd components in $\boldsymbol{d}$. Then the number $|\widetilde{\mathcal{M}}_{\boldsymbol{d}}|$ of unlabelled two-face sphere maps having $\boldsymbol{d}$ has vertex degree distribution is given by*

$$|\widetilde{\mathcal{M}}_d| = \frac{1}{2}(|\widetilde{\mathsf{M}}_{\boldsymbol{d}}| + |\mathrm{Fix}_{\widetilde{\mathsf{M}}_d}\tau|),$$

where

$$|\mathrm{Fix}_{\widetilde{\mathsf{M}}_{\boldsymbol{d}}}\tau| = \begin{cases} \frac{1}{2}\binom{n/2}{\boldsymbol{d}/2}\left(1+\frac{1}{n}\sum_{k\geq 1} d_{2k}\right), & \textit{if } r=0; \\ \binom{(n-1)/2}{(\boldsymbol{d}-\delta_{2k})/2}, & \textit{if } r=1,\ d_{2k}\ \textit{odd;} \\ \binom{(n-2)/2}{(\boldsymbol{d}-\delta_{2k}-\delta_{2\ell})/2}, & \textit{if } r=2,\ d_{2k}\ \textit{and}\ d_{2\ell}\ \textit{odd}\,; \\ 0, & \textit{if } r\geq 3. \end{cases} \tag{79}$$

□

Let $\widetilde{\mathcal{M}}_{\{\alpha,\beta\}}$ denote the set of unlabelled two-face sphere maps having $\{\alpha,\beta\}$ as face degree distribution. If $\alpha\neq\beta$, there is no antipodal symmetry, and we have

$$|\widetilde{\mathcal{M}}_{\{\alpha,\beta\}}| = |\widetilde{\mathsf{M}}_{(\alpha,\beta)}|, \tag{80}$$

since in this case, we can choose the north or inner face to be that of smallest degree. Recall that $|\widetilde{\mathsf{M}}_{(\alpha,\beta)}|$ is given by (38)

If $\alpha=\beta$, the set $\widetilde{\mathsf{M}}_{(\alpha,\alpha)}$ is closed under the action of τ and we can apply (48). We have

$$|\widetilde{\mathcal{M}}_{\{\alpha,\alpha\}}| = \frac{1}{2}\left(|\widetilde{\mathsf{M}}_{(\alpha,\alpha)}| + |\mathrm{Fix}_{\widetilde{\mathsf{M}}_{\{\alpha,\alpha\}}}\tau|\right). \tag{81}$$

Since $\alpha=\beta$, we simply have $\alpha=n$, the number of vertices. Therefore

$$|\mathrm{Fix}_{\widetilde{\mathsf{M}}_{\{\alpha,\alpha\}}}\tau| = |\mathrm{Fix}_{\widetilde{\mathsf{M}}_n}\tau|. \tag{82}$$

The term $|\mathrm{Fix}_{\widetilde{\mathsf{M}}_n}\tau|$ can be easily deduced from (48), (15) and (78), and the next result follows.

Theorem 11 *If $\alpha>0$ and $\beta>0$ have the same parity, then the number $|\widetilde{\mathcal{M}}_{\{\alpha,\beta\}}|$ of unlabelled two-face sphere maps having $\{\alpha,\beta\}$ as face degree distribution is given by*

$$|\widetilde{\mathcal{M}}_{\{\alpha,\beta\}}| = \begin{cases} |\widetilde{\mathsf{M}}_{(\alpha,\beta)}|, & \text{if} \alpha\neq\beta, \\ \frac{1}{2}|\widetilde{\mathsf{M}}_{(\alpha,\alpha)}| + \frac{1}{2}\binom{\alpha-1}{(\alpha-1)/2}, & \text{if } \alpha=\beta \text{ is odd}, \\ \frac{1}{2}|\widetilde{\mathsf{M}}_{(\alpha,\alpha)}| + \frac{1}{4}\binom{\alpha}{\alpha/2}, & \text{if } \alpha=\beta \text{ is even}, \end{cases} \tag{83}$$

□

Finally, let $\widetilde{\mathcal{M}}_{\boldsymbol{d},\{\alpha,\beta\}}$ denote the set of all unlabelled two-face sphere maps having joint vertex and face degree distribution given by $\boldsymbol{d}$ and $\{\alpha,\beta\}$. If $\alpha \neq \beta$, we have

$$|\widetilde{\mathcal{M}}_{\boldsymbol{d},\{\alpha,\beta\}}| = |\widetilde{\mathsf{M}}_{\boldsymbol{d},(\alpha,\beta)}|, \tag{84}$$

since in this case, there are no possible antipodal symmetries. Recall that $|\widetilde{\mathsf{M}}_{d,(\alpha,\beta)}|$ is given by (45).

If $\alpha = \beta$, by (48), we have

$$|\widetilde{\mathcal{M}}_{\boldsymbol{d},\{\alpha,\alpha\}}| = \frac{1}{2}|\widetilde{\mathsf{M}}_{\boldsymbol{d},(\alpha,\alpha)}| + \frac{1}{2}|\mathrm{Fix}_{\widetilde{\mathsf{M}}_{\boldsymbol{d},(\alpha,\alpha)}}\tau|, \tag{85}$$

and α is completely determined by $\boldsymbol{d}$: $\alpha = |\boldsymbol{d}|$, hence we have

$$|\mathrm{Fix}_{\widetilde{\mathsf{M}}_{d,(\alpha,\alpha)}}\tau| = |\mathrm{Fix}_{\widetilde{\mathsf{M}}_{d}}\tau|. \tag{86}$$

Theorem 12 *Let $\boldsymbol{d} \neq \boldsymbol{0}$ be a vector of nonnegative integers satisfying $||\boldsymbol{d}|| = 2|\boldsymbol{d}|$ and α, β be two positive integers having the same parity and such that $(\alpha+\beta)/2 = |\boldsymbol{d}| = n \geq 3$. Then the number of unlabelled two-face sphere maps having joint vertex and face degree distributions $\boldsymbol{d}$ and $\{\alpha,\beta\}$ is given by*

$$|\widetilde{\mathcal{M}}_{\boldsymbol{d},\{\alpha,\beta\}}| = \begin{cases} |\widetilde{\mathsf{M}}_{\boldsymbol{d},(\alpha,\beta)}|, & \text{if } \alpha \neq \beta, \\ \frac{1}{2}|\widetilde{\mathsf{M}}_{\boldsymbol{d},(\alpha,\alpha)}| + \frac{1}{2}|\mathrm{Fix}_{\widetilde{\mathsf{M}}_{\boldsymbol{d}}}\tau| & \text{if } \alpha = \beta, \end{cases} \tag{87}$$

where $|\mathrm{Fix}_{\widetilde{\mathsf{M}}_{\boldsymbol{d}}}\tau|$, is given by (79). □

References

[1] V. I. Arnold. *Topological classification of trigonometric polynomials and combinatorics of graphs with an equal number of vertices and edges*, Functional Analysis and its Applications. **30**, (1996), 1–14.

[2] D. Arquès. Une relation fonctionnelle nouvelle sur les cartes planaires pointées, J. Comb. Th. Ser. B, **39**, (1985), 27–42.

[3] L. Babai, W. Imrich et L. Lovász. *Finite homeomorphism groups of the 2-sphere*, Colloq. Math. Soc. J. Bolyai, **8**, Topics in Topology, Keszthely (Hungary), (1972), 61–75.

[4] F. BERGERON, G. LABELLE, and P. LEROUX. *Combinatorial species and tree-like structures*, Encyclopedia of Mathematics and its Applications. Vol **67**, Cambridge University Press, (1998).

[5] E. A. BENDER and N. C. WORMALD. *The number of loopless planar maps*, Discrete Mathematics, **54**, (1985), 235–237.

[6] M. BOUSQUET. *Espèces de structures et applications au dénombrement de cartes et de cactus*, Thèse de doctorat, Université du Québec à Montréal, Publications du LaCIM, Vol. 24, (1999).

[7] M. BOUSQUET, G. LABELLE, and P. LEROUX. *Dénombrement de cartes planaires à deux faces*, 10th SFCA/FPSAC Conference, Fields Institute, Toronto, (1998) 79–90.

[8] R. CORI. *Bijective census of rooted planar maps: A survey*, Proceedings of the fifth conference on Formal Power Series and Algebraic Combinatorics, A. Barlotti, M. Delest and R. Pinzani, ed., Florence (1993), 131–141.

[9] R. CORI and A. MACHI. *Maps, hypermaps and their automorphisms: a survey*, I, II, III. Expositiones Mathematicae, vol. **10**, (1992), 403–427, 429–447, 449–467.

[10] H. W. GOULD. *A standardized set of tables listing 500 binomial coefficient summations*, West Virginia University (1972).

[11] I.P. GOULDEN and D.M. JACKSON. *Maps in locally orientable surfaces, the double coset algebra, and zonal polynomials*, Canadian J. Math. **48**, (1996), 569–584.

[12] F. HARARRY, G. PRINS and W.T. TUTTE. *The number of plane trees*, Indag. Math. **26** (1964), 319-329.

[13] A. JOYAL. *Une théorie combinatoire des séries formelles*, Advances in Mathematics, **42**, (1981), 1–82.

[14] G. LABELLE and P. LEROUX. *Enumeration of (uni- or bicolored) plane trees according to their degree distribution*, Disc. Math. **157** (1996), 227–240.

[15] V.A. LISKOVETS. *A census of non-isomorphic planar maps*, Colloq. Math. Soc. J. Bolyai **25**, Algebraic Methods in Graph Theory (1981), 479–494.

[16] V.A. LISKOVETS. *Enumeration of non-isomorphic planar maps*, Selecta Math. Soviet. **4**, (1985), 303–323.

[17] V.A. LISKOVETS and T.R. WALSH. *The enumeration of non-isomorpic 2-connected planar maps*, Canad. J. Math. **35**, (1983), 417–435.

[18] N. MAGOT. *Cartes planaires et fonctions de Belyi: Aspects algorithmiques et expérimentaux*, Thèse de Doctorat, Université de Bordeaux, (1997), 150p.

[19] N. MAGOT and A. ZVONKIN. *Belyi Functions for Archimedean Solids*, Formal power series and algebraic combinatorics, 9th Conference Proceedings, Vienna, Vol. **3**, (1997), 373–389.

[20] J.W. Moon, *Counting Labelled Trees*, Canadian Mathematical Monographs **1**, Canadian Mathematical Society, 1970.

[21] G. Shabat and A. Zvonkin. *Plane trees and algebraic numbers*, Contemporary Math. **178** (1994), 233–275.

[22] Sloane and S. Plouffe. *Encyclopedia of integer sequences* Academic Press Inc. 1995.

[23] W. T. Tutte. *A census of Slicings*, Can J. Math. **14** (1962), 708–722.

[24] W. T. Tutte. *A census of Planar maps*, Can J. Math. **15** (1963), 249–271.

[25] W. T. Tutte. *The number of plane planted trees with a given partition*, Amer. Math. Monthly, **71** (1964), 272–277.

[26] W. T. Tutte. *On the enumeration of planar maps*, Bull. Amer. Math. Soc. **74** (1968), 64–74.

[27] D. Walkup, *The number of plane trees*, Mathematika **19** (1972), 200–204.

[28] N.C. Wormald. *Counting unrooted planar maps*, Discrete mathematics **36**, (1981), 205–225.

[29] N.C. Wormald. *On the number of planar maps*, Can. J. Math. **33**, No. 1, (1981), 1–11.