p^q-Catalan Numbers and Squarefree Binomial Coefficients

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Abstract

In this paper we consider the generalized Catalan numbers $F(s,n) = \frac{1}{(s-1)n+1}\binom{sn}{n}$, which we call s-Catalan numbers. For p prime, we find all positive integers n such that p^q divides $F(p^q,n)$, and also determine all distinct residues of $F(p^q,n) \pmod{p^q}$, $q \geq 1$. As a byproduct we settle a question of Hough and the late Simion on the divisibility of the 4-Catalan numbers by 4. In the second part of the paper we prove that if $p^q \leq 99999$, then $\binom{p^q n+1}{n}$ is not squarefree for $n \geq \tau_1(p^q)$ sufficiently large ($\tau_1(p^q)$ computable). Moreover, using the results of the first part, we find $n < \tau_1(p^q)$ (in base p), for which $\binom{p^q n+1}{n}$ may be squarefree. As consequences, we obtain that $\binom{4n+1}{n}$ is squarefree only for $n = 1, 3, 45$, and $\binom{9n+1}{n}$ is squarefree only for $n = 1, 4, 10$.

Keywords. Binomial Coefficients, Catalan Numbers, Congruences, Squarefree Numbers

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1 Introduction

Problems involving binomial coefficients were considered by many mathematicians for over two centuries. R.K. Guy in [6] mentions several problems on divisibility of binomial coefficients (see B31, B33). Erdös conjectured that for $n > 4$, $\binom{2n}{n}$ is never squarefree. This was proved by Sárközy in [13], for sufficiently large n, and by Granville and Ramaré in [5] for any $n > 4$ (see also [17] for another proof).

Many people (see, for instance, [1, 2, 7, 8, 9, 12, 15]) proposed and studied the following generalization of classical Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$, which we will call *s-Catalan numbers*, namely $F(s,n) = \frac{1}{(s-1)n+1}\binom{sn}{n}$. There are many interpretations of this sequence (see [2, 7, 9, 12, 15]), for instance: the number of s-ary trees with n source-nodes, the number of ways of associating n applications of a given s-ary operator, the number of ways of dividing a convex polygon into n disjoint $(s+1)$-gons with nonintersecting diagonals, and the number of s-good paths (below the line $y = sx$) from $(0,-1)$ to $(n,(s-1)n-1)$.

Naturally, some of the questions proposed by Erdös on the classical Catalan numbers, may be asked here as well, as Hough and the late Simion proposed [8]: (a) When p is prime, for what values of n is $F(p,n)$ divisible by p? (b)* For what values of n is $F(4,n)$ divisible by 4? (c)* What can you say when s takes on the other composite values? There are no answers yet known for (b) and (c). In this paper we give a simple proof to (a), and we show that $F(p^2,n)$ is divisible by p^2, unless $(p^2-1)n+1$ is an even power of p, or a p-term sum of odd powers of p, thereby answering (b), and (c) for $s = p^2$. We generalize this result describing all integers n, for which p^q divides $F(p^q,n)$, $q \geq 3$. In the second part of the paper, we show that if $p^q \leq 99999$, then $\binom{p^q n+1}{n}$ is not squarefree for $n \geq \tau_1(p^q)$ sufficiently large (computable). If $n < \tau_1(p^q)$, we employ the generalized Catalan numbers to find the set of integers n, where $\binom{p^q n+1}{n}$ might be squarefree. As consequences, we obtain that $\binom{4n+1}{n}$ is squarefree only for $n = 1, 3, 45$, and $\binom{9n+1}{n}$ is squarefree only for $n = 1, 4, 10$.

2 Preliminary Results

Let $[x]$ be the largest integer smaller than or equal to x. In this section we state a few results which will be needed later. Lucas (1878) (see [3]) found a simple method to determine $\binom{m}{n}$ (mod p).

Theorem 1 (Lucas). *If p is prime, then $\binom{m}{n} \equiv \binom{[m/p]}{[n/p]}\binom{m_0}{n_0} \pmod{p}$, where m_0, n_0 are the least non-negative residues modulo p of m, respectively n.*

In 1808 Legendre showed that the exact power of p dividing $n\,!$ is

$$[n/p] + [n/p^2] + [n/p^3] + \cdots . \tag{1}$$

We define (see [4]) the *sum of digits function* $\sigma_p(n) = n_0 + n_1 + \cdots + n_d$, if $n = n_0 + n_1 p + \cdots + n_d p^d$. Then, using σ, (1) transforms into

$$\frac{n - \sigma_p(n)}{p - 1}. \tag{2}$$

Kummer found a way to determine the power to which a prime p divides a binomial coefficient.

Theorem 2 (Kummer). *The power to which the prime p divides the binomial coefficient $\binom{m}{n}$, say $v_p(\binom{m}{n})$, is given by the number of carries when we add n and $m - n$ in base p.*

Define $n!_p$ to be the product of all integers $\leq n$, that are not divisible by p. We see that $n!_p = \dfrac{n!}{[n/p]!p^{[n/p]}}$. Granville in [4] proves the following beautiful generalization of both Lucas and Kummer's Theorems.

Theorem 3 (Granville). *Suppose that the prime power p^q and positive integers $m = n+r$ are given. Let N_j be the least positive residue of $[n/p^j] \pmod{p^q}$ for each $j \geq 0$ (that is, $N_j = n_j + n_{j+1}p + \cdots + n_{j+q-1}p^{q-1}$, where $n = n_0 + n_1 p + \cdots + n_d p^d$): also make the corresponding definitions for m_j, M_j, r_j, R_j. Let e_j be the number of indices $i \geq j$ for which $m_i < n_i$ (that is, the number of* carries*, when adding n and r in base p, on or beyond the jth digit). Then*

$$\frac{1}{p^{e_0}}\binom{m}{n} \equiv (\pm 1)^{e_{q-1}} \frac{M_0!_p}{N_0!_p\, R_0!_p} \frac{M_1!_p}{N_1!_p\, R_1!_p} \cdots \frac{M_d!_p}{N_d!_p\, R_d!_p} \pmod{p^q},$$

where $(\pm)$ is (-1) except if $p = 2$ and $q \geq 3$.

Our first result gives a complete answer to the first posed question (a), generalizing the well-known result on Catalan numbers, or equivalently, on middle binomial coefficients (see [6]), which states that $4 \mid \binom{2n}{n}$, unless $n = 2^k$, for some k. Denote by $\mathbf{N}$ the set of nonnegative integers.

Theorem 4. *Let p be a prime. Then, p divides $F(p,n)$, unless n is of the form $\frac{p^k-1}{p-1}$, $k \in \mathbf{N}$, in which case $F(p,n) \equiv 1 \pmod{p}$.*

Proof. We rewrite $F(p,n) = \frac{1}{(p-1)n+1}\binom{pn}{n} = \frac{1}{pn+1}\binom{pn+1}{n}$. Since $p \not| F(p,0)$, we assume $n > 0$. Applying Lucas' Theorem repeatedly for the base p representations $(0 \leq m_i, n_i \leq p-1)$, $m = m_0 + m_1 p + \cdots + m_d p^d$ and $n = n_0 + n_1 p + \cdots + n_d p^d$, we obtain $\binom{m}{n} \equiv \binom{m_0}{n_0}\binom{m_1}{n_1}\cdots\binom{m_d}{n_d} \pmod{p}$. For $m = pn+1 \equiv 1 \pmod{p}$, we get

$$F(p,n) \equiv \binom{pn+1}{n} \equiv \binom{1}{n_0}\binom{n_0}{n_1}\cdots\binom{n_{d-1}}{n_d} \pmod{p}, \quad n_d \neq 0.$$

If $F(p,n) \not\equiv 0 \pmod{p}$, we must have $1 \geq n_0 \geq n_1 \geq \cdots \geq n_d > 0$, therefore $n_j = 1$, $0 \leq j \leq d$. In that case, $n = 1 + p + \cdots + p^d = \frac{p^{d+1}-1}{p-1}$ and $F(p,n) \equiv 1 \pmod{p}$. $\square$

The following lemma will be extensively used throughout the paper.

Lemma 5. *We have*

$$v_p(F(p^q,n)) = \frac{\sigma_p((p^q-1)n+1)-1}{p-1}.$$

Proof. Using (2) we get that the power of p dividing $\binom{m}{n}$ is

$$v_p\left(\binom{m}{n}\right) = \frac{\sigma_p(n)+\sigma_p(m-n)-\sigma_p(m)}{p-1}. \tag{3}$$

If $m = p^q n+1$, by using the identity $F(p^q,n) = \frac{1}{p^q n+1}\binom{p^q n+1}{n}$, (3) becomes

$$v_p(F(p^q,n)) = \frac{\sigma_p(n)+\sigma_p((p^q-1)n+1)-\sigma_p(p^q n+1)}{p-1} = \frac{\sigma_p((p^q-1)n+1)-1}{p-1},$$

since $\sigma_p(p^q n+1) = \sigma_p(n)+1$. $\square$

3 Scarce squarefree p^2-Catalan numbers

Denote by $n = (ab\ldots)_p$ the base p representation of n, a being the most significant digit.

Our next result refers to the third question of Hough and Simion, if $s = p^2$.

Theorem 6. *Let p be a prime. Then, p^2 divides $F(p^2,n)$, unless n is of the form $\dfrac{p^{2t}-1}{p^2-1}$, $t \in \mathbf{N}$, in which case $F(p^2,n) \equiv 1 \pmod{p^2}$, or of the form $\dfrac{\sum_{j=1}^{s} c_j p^{2i_j+1}-1}{p^2-1}$, $i_1 < \cdots < i_s$, where $c_j > 0$ and $\sum_{j=1}^{s} c_j = p$, $s \geq 2$, in which case $F(p^2,n) \equiv \binom{p}{c_1, c_2, \ldots, c_s} \pmod{p^2}$ (the multinomial coefficient).*

Proof. By Lemma 5, if $F(p^2,n) \not\equiv 0 \pmod{p^2}$, then

$$v_p(F(p^2,n)) = \frac{\sigma_p((p^2-1)n+1)-1}{p-1} \leq 1, \tag{4}$$

which implies that $\sigma_p((p^2-1)n+1)$ is 1 or p.

If $\sigma_p((p^2-1)n+1)=1$, then $(p^2-1)n+1=p^k\equiv(-1)^k \pmod{(p+1)}$, therefore k must be even, say $k=2t$, and $n=\dfrac{p^{2t}-1}{p^2-1}$.

If $\sigma_p((p^2-1)n+1)=p$, then $(p^2-1)n+1=\sum_{k=1}^{p}p^{\alpha_k}$, $\alpha_1\le\alpha_2\le\cdots\le\alpha_p$, with $\alpha_1<\alpha_p$. It follows that

$$\begin{aligned}(p^2-1)n &\equiv -1+p\sum_{\alpha_i\ \text{odd}}1+\sum_{\alpha_i\ \text{even}}1\\ &\equiv -1+p^2-(p-1)\sum_{\alpha_i\ \text{even}}1\\ &\equiv -(p-1)\sum_{\alpha_i\ \text{even}}1 \pmod{(p^2-1)},\end{aligned}$$

so $\sum_{\alpha_i\ \text{even}}1$ must be divisible by $p+1$. Since $0\le\sum_{\alpha_i\ \text{even}}1\le p$, we see that $\sum_{\alpha_i\ \text{even}}1$ must be an empty sum. Therefore, all $\alpha_j=2i_j+1$. We obtain $n=\dfrac{\sum_{j=1}^{s}c_jp^{2i_j+1}-1}{p^2-1}$, $i_1<i_2<\cdots<i_s$, $\sum_{j=1}^{s}c_j=p$ and the first claim is proved.

For the second part of our theorem, we use the congruence $F(p^2,n)\equiv\binom{p^2n+1}{n} \pmod{p^2}$. Let $n_{-1}=0$. Consider $n=\dfrac{p^{2t}-1}{p^2-1}$. It follows that $n=(1010\cdots101)_p$, and since p^2n+1 attaches to this string the block 01 to the right, it is of the same form. Moreover, $M_i=N_{i-2}$ and $R_i!_p=1$, except for $R_{2t-1}!_p=(p-1)!$. By Granville's theorem, we get

$$F(p^2,n)\equiv p^{e_0}(-1)^{e_1}\frac{M_0!_pM_1!_p}{R_{2t-1}!_p} \pmod{p^2}. \tag{5}$$

Now, $M_0!_p=1, M_1!_p=(p-1)!$. Thus, (5) becomes $F(p^2,n)\equiv p^{e_0}(-1)^{e_1}\equiv1 \pmod{p^2}$, since $e_0=e_1=0$.

Consider $n=\dfrac{\sum_{j=1}^{s}c_jp^{2i_j+1}-1}{p^2-1}$, $i_1<\cdots<i_s$ and $\sum_{i=1}^{s}c_i=p$, with $s\ge2$. It follows that

$$\begin{aligned}n&=\frac{\sum_{j=1}^{s}c_j(p^{2i_j+1}-p)+p^2-1}{p^2-1}=\sum_{j=1}^{s}c_j\sum_{k=1}^{i_j}p^{2i_j-2k+1}+1\\ &=c_sp^{2i_s-1}+c_sp^{2i_s-3}+\cdots+(c_s+c_{s-1})p^{2i_{s-1}-1}+\cdots+1.\end{aligned} \tag{6}$$

By Kummer's theorem, there is a carry in this case, so $e_0 = e_1 = 1$. Also, $n_0 = 1$, $M_0!_p = 1$, $M_1!_p = (p-1)!$, $R_i!_p = 1$ except for $R_{2i_k}!_p = (c_k p)!_p = \dfrac{(c_k p)!}{c_k! p^{c_k}}$ and $R_{2i_k+1}!_p = (c_k)!_p = c_k!$, for $k = 1, 2, \ldots, s$. Applying Granville's theorem we get

$$\begin{aligned} F(p^2, n) &\equiv p^{e_0}(-1)^{e_1} \frac{M_0!_p M_1!_p}{R_0!_p \cdots R_{2i_s+1}!_p} \equiv (-1)\, p \, \frac{(p-1)!}{\prod_k (c_k p)!_p \, c_k!} \\ &\equiv \binom{p}{c_1, c_2, \ldots, c_s} \frac{(-1)}{\prod_k (c_k p)!_p} \equiv \binom{p}{c_1, c_2, \ldots, c_s} \pmod{p^2}, \end{aligned} \tag{7}$$

since $(c_k p)!_p \equiv (-1)^{c_k} \pmod p$ and $\sum_{k=1}^{s} c_k = p$. □

The following corollary gives a complete answer to the second question of Hough and Simion.

Corollary 7. *$F(4, n)$ is divisible by 4, unless n is of the form $\dfrac{2^{2t} - 1}{3}$, in which case $F(4, n) \equiv 1 \pmod 4$, or of the form $\dfrac{2^{2t+1} + 2^{2j+1} - 1}{3}$, for $t > j$, in which case $F(4, n) \equiv 2 \pmod 4$.*

In [10] we study products of factorials modulo p, and a consequence of one of our results is that the multinomial coefficients appearing in our previous theorem cover all residues of the form $p\,k$ modulo p^2, where $0 \le k \le p-1$ (except for $p = 5$). Therefore, the residues of $F(p^2, n)$ modulo p^2 will be $\{1\} \cup \{p\,k \mid 0 \le k \le p-1\}$ (except for $p = 5$, in which case the residues are $\{0, 1, 5, 10, 20\}$).

4 Divisibility of p^q-Catalan numbers

Let p prime and $q \ge 2$ fixed. For easy writing we denote by $\mathcal{S}$ the set of all positive integers of the form $\dfrac{\sum_{k=1}^{s} c_k p^{qt_k + j_k} - 1}{p^q - 1}$, $t_k \in \mathbf{N}$, $1 \le c_k < p$, $0 \le m < q$, $l = \sum_k c_k = m(p-1) + 1$, $d = \dfrac{\sum_k c_k p^{j_k} - 1}{p^q - 1} \in \mathbf{N}$, $qt_k + j_k \ne qt_i + j_i$ and $0 \le j_1 \le j_2 \le \cdots \le j_s < q$.

Concerning arbitrary powers of an odd prime, we prove

Theorem 8. *If p is a prime and $q \geq 3$, then p^q divides $F(p^q, n)$, unless n is of the form $\dfrac{p^{tq}-1}{p^q-1}$, for some $t \in \mathbf{N}$, in which case $F(p^q, n) \equiv 1 \pmod{p^q}$, or n is in $\mathcal{S}$, in which case, $F(p^q, n) \equiv \epsilon\, p^{e_0+m(p^{q-1}-1)} \dfrac{p^{q-1}!}{(lp^{q-1})!} \dbinom{l}{c_1, c_2, \ldots, c_s} \pmod{p^q}$, if $j_k \geq 1, t_k \geq 1$, where $\epsilon = 1$ for $p = 2$ and $\epsilon = (-1)^{e_{q-1}}$ for $p \geq 3$.*

Proof. By Lemma 5, if $p^q \not| F(p^q, n)$, then $v_p(F(p^q, n)) = \dfrac{\sigma_p((p^q-1)n+1)-1}{p-1} \leq q-1$, so $\sigma_p((p^q-1)n+1) = m(p-1)+1$, for some $0 \leq m \leq (q-1)$. If $m = 0$, then $(p^q-1)n+1 = p^{tq+i}$, for some $0 \leq i \leq q-1$. Working modulo $p^q - 1$ implies $i = 0$. Thus, $n = \dfrac{p^{tq}-1}{p^q-1}$. Next, assume $0 < m \leq q-1$. We obtain, for $l = m(p-1)+1$, $(p^q-1)n+1 = p^{\alpha_1} + \cdots + p^{\alpha_l}, \alpha_1 \leq \cdots \leq \alpha_l$, where no p of the α_i's can be equal. Thus, $n = \dfrac{\sum_{k=1}^{s} c_k p^{qt_k+j_k} - 1}{p^q-1}$, $1 \leq c_k < q, 0 \leq j_1 \leq j_2 \leq \cdots j_s < q$, $\sum_{k=1}^{s} c_k = l$ and $\displaystyle\sum_k c_k p^{j_k} \equiv 1 \pmod{p^q-1}$, therefore, n is in $\mathcal{S}$.

To find the residues of the generalized Catalan numbers modulo p^q, we use the congruence $F(p^q, n) \equiv \binom{p^n n+1}{n} \pmod{p^q}$, Granville's theorem and the proof of Theorem 6. If $n = \dfrac{p^{tq}-1}{p^q-1}$, $m = p^q n + 1$ and $r = p^{tq}$. We observe that $M_{i+q} = N_i$. Moreover, $M_{iq} = 1$, $M_{iq+1} = p^{q-1}$, $M_{iq+2} = p^{q-2}$, $\cdots$, $M_{iq+q-1} = p$, $M_{(i+1)q} = 1$, for any $0 \leq i \leq t-1$; $R_i!_p = 1$ except for $R_{qt-q+1} = p^{q-1}$, $R_{qt-q+2} = p^{q-2}$, $\cdots$, $R_{qt-1} = p$. Since $e_0 = e_1 = 0$, we get

$$\begin{aligned} F(p^q, n) &\equiv p^{e_0} \frac{M_0!_p \cdot M_1!_p \cdots M_{q-1}!_p}{R_{qt-q+1}!_p \cdots R_{qt-1}!_p} \pmod{p^q} \\ &\equiv \frac{(p^{q-1})!_p \cdots (p^2)!_p \cdot p!_p}{(p^{q-1})!_p \cdots (p^2)!_p \cdot p!_p} \pmod{p^q} \equiv 1 \pmod{p^q}. \end{aligned}$$

We find the residues of $F(p^q, n)$ modulo p^q when $j_k \geq 1, t_k \geq 1$, for any k. In base p, using $j_k \geq 1, t_k \geq 1$, we get

$$\begin{aligned} n &= \sum_{k=1}^{s} c_k p^{j_k} \left(p^{q(t_k-1)} + \cdots + 1 \right) + d \\ &= \cdots + d(p^q - 1) + 1 + d = 1 + dp^q + \cdots = 1 + d' p^q + \cdots \end{aligned}$$

($0 \leq d' < p$ and the power of p in the missing terms is at least $q+1$). Moreover, $p^q n + 1 = 1 + p^q + d' p^{2q} + \cdots$. Thus, $M_0 = 1, M_1 = p^{q-1}, M_2 = p^{q-2}, \ldots, M_{q-1} = p$. Since $R = \sum_{k=1}^{s} c_k p^{q t_k + j_k}$, then $R_k!_p = 1$, except for $R_{q(t_i-1)+j_i+1} = c_i p^{q-1}$, $R_{q(t_i-1)+j_i+2} = c_i p^{q-2}, \ldots, R_{qt_i+j_i+1} = c_i$, for any i. By Granville's theorem, using $\sum_{i=1}^{s} c_i = l = m(p-1)+1$ and $(c_i\, p^k)!_p = \dfrac{(c_i\, p^k)!}{(c_i\, p^{k-1})!\, p^{c_i\, p^{k-1}}}$, we obtain

$$\begin{aligned} F(p^q, n) &\equiv \epsilon\, p^{e_0} \frac{\prod_{k=1}^{q-1} (p^k)!_p}{\prod_{i=1}^{s} \prod_{k=0}^{q-1} (c_i p^k)!_p} \equiv \epsilon\, p^{e_0} \frac{\prod_{k=1}^{q-1} \dfrac{(p^k)!}{(p^{k-1})!\, p^{p^{k-1}}}}{\prod_{i=1}^{s} c_i! \prod_{k=1}^{q-1} \dfrac{(c_i p^k)!}{(c_i p^{k-1})!\, p^{c_i p^{k-1}}}} \\ &\equiv \epsilon\, p^{e_0} \frac{\dfrac{p^{q-1}!}{p^{(p^{q-1}-1)/(p-1)}}}{\prod_{i=1}^{s} \dfrac{(c_i p^{q-1})!}{p^{c_i\, (p^{q-1}-1)/(p-1)}}} \equiv \epsilon\, p^{e_0 + m(p^{q-1}-1)} \frac{p^{q-1}!}{\prod_{i=1}^{s} (c_i\, p^{q-1})!} \qquad (8) \\ &\equiv \epsilon\, p^{e_0 + m(p^{q-1}-1)} \frac{p^{q-1}!}{(lp^{q-1})!} \binom{lp^{q-1}}{c_1 p^{q-1}, \ldots, c_s p^{q-1}} \\ &\equiv \epsilon\, p^{e_0 + m(p^{q-1}-1)} \frac{p^{q-1}!}{(lp^{q-1})!} \binom{l}{c_1, c_2, \ldots, c_s} \pmod{p^q}, \end{aligned}$$

where $\epsilon = 1$ for $p = 2$ and $\epsilon = (-1)^{e_{q-1}}$ for $p \geq 3$. □

We use in the next section the following

Corollary 9. *p^q divides $\frac{1}{(p^q-1)n+1}\binom{p^q n}{n}$ if and only if p^q divides $\binom{p^q n+1}{n}$.*

5 Squarefree Binomial Coefficients

In this section we study squarefree binomial coefficients of the form $\binom{p^q n+1}{n}$, by employing the previous results on the generalized Catalan numbers.

In [5], the authors proved that if $\begin{pmatrix} n \\ k \end{pmatrix}$ is squarefree, then n or $n-k$ must be small. Finding explicit bounds is a much more difficult task. They showed that $\begin{pmatrix} 2n \\ n \end{pmatrix}$ is squarefree for $n > 2^{1617}$, and used some clever arguments to simplify the computer's work, in checking the possible exceptions $n = 2^r$ up to 2^{1617}. In this section of the paper we rely on [5] and use some estimates on the Chebyshev's function $\sum_{d \le x} \Lambda(d)$, where $\Lambda(d)$ is the Von Mangoldt's function, $\Lambda(d) = \log r$, if $d = r^s$, r prime and $\Lambda(d) = 0$, otherwise, to show our results. Define $e(x) = e^x$ and $\psi(x) = 0$, if x is an integer, and $\psi(x) = \{x\} - \frac{1}{2}$, otherwise, where $\{x\}$ is the fractional part of x.

The following lemma proves to be very useful

Lemma 10. *If $p^q \le 99999$, the inequality*

$$\begin{aligned} &0.9999975\sqrt{p^q n+1} - 1.0000025\sqrt{(p^q-1)n+1} > \\ &21.683\, p^{\frac{23q}{48}}\, n^{\frac{23}{48}}\, \left(\log\left(256((p^q-1)n+1)\right)\right)^{\frac{11}{4}} + \frac{11}{8}(3\log n + 2q\log p). \end{aligned} \tag{9}$$

is true for $n \ge \tau_0$ sufficiently large.

Proof. First, $\left(\sqrt{x+1} + \sqrt{x-1}\right)^2 = 2x + 2\sqrt{x^2-1} \le 4x$, since $2\sqrt{x^2-1} \le 2x$. Thus, $\sqrt{1+x} + \sqrt{1+x-n} \le \sqrt{x+1} + \sqrt{x-1} \le 2\sqrt{x}$, for $2 \le n \le x+1$. Now, let $x' = 1+x$. We evaluate

$$(1-\alpha)\sqrt{1+x} - (1+\alpha)\sqrt{1+x-n} = \frac{(1-\alpha)^2 x' - (1+\alpha)^2(x'-n)}{(1-\alpha)\sqrt{x'} + (1+\alpha)\sqrt{x'-n}}$$

$$= \frac{n(1+\alpha)^2 - 4\alpha x'}{(1-\alpha)\sqrt{x'} + (1+\alpha)\sqrt{x'-n}} \ge \frac{n(1+\alpha)^2 - 4\alpha x'}{(1+\alpha)(\sqrt{x'} + \sqrt{x'-n})} \ge \frac{n\left(\frac{(1-\alpha)^2}{1+\alpha} - \frac{4\alpha}{n(1+\alpha)}x\right)}{2\sqrt{x}}.$$

Therefore,

$$(1-\alpha)\sqrt{1+x}-(1+\alpha)\sqrt{1+x-n}\geq\left(\frac{(1-\alpha)^2}{1+\alpha}-\frac{4\alpha}{1+\alpha}\frac{x}{n}\right)\frac{n}{2\sqrt{x}}. \tag{10}$$

Taking $x=p^q n$, $\alpha=\dfrac{1}{4\cdot 10^5}$, in (10), we get

$$\begin{aligned}&0.9999975\sqrt{p^q n+1}-1.0000025\sqrt{(p^q-1)n+1}\\&\geq\left(\frac{0.9999975^2}{1.0000025}-\frac{1}{100000.25}p^q\right)\frac{1}{2\sqrt{p^q}}n^{\frac{1}{2}}.\end{aligned} \tag{11}$$

If $p^q\leq 99999$, then (11) implies our claim that the inequality (9) is true for $n\geq\tau_0$ sufficiently large, since, by (11), the left side of (9) is $O(n^{\frac{1}{2}})$ and the right side is $O(n^{\frac{23}{48}})$. $\square$

As before, p is a prime and $q\geq 2$ is an integer. Also, τ_0 is the bound obtained in Lemma 10 and $\tau_1(p^q)=\max\left(\frac{e^{60}-1}{p^q-1},5^{10}p^{5q},\tau_0\right)$. Our main result of this section is stated in the next

Theorem 11. *Assume $p^q\leq 99999$. Then, $\binom{p^q n+1}{n}$ is not squarefree for $n\geq\tau_1(p^q)$. Moreover, if $n<\tau_1(p^q)$, $\binom{p^q n+1}{n}$ may be squarefree only for n of the form occurring in Theorem 8 ($q\geq 3$) or Theorem 6 ($q=2$).*

We proceed to the proof of the theorem. Let $P=n(p^q n-n+1)(p^q n+1)$. Corollary 3.2 (p. 82) of [5] implies

Lemma 12. *Suppose that $\binom{p^q n+1}{n}$ is squarefree. Then,*

$$\begin{aligned}&\left|\sum_{d\in I}\psi\left(\frac{p^q n+1}{d}\right)\Lambda(d)\right|+\left|\sum_{d\in I}\psi\left(\frac{n}{d}\right)\Lambda(d)\right|\\&+\left|\sum_{d\in I}\psi\left(\frac{(p^q-1)n+1}{d}\right)\Lambda(d)\right|\geq\frac{1}{2}\sum_{d\in I,(d,P)=1}\Lambda(d),\end{aligned} \tag{12}$$

where I is the set of integers d in the range $\sqrt{(p^q-1)n+1}<d\leq\sqrt{p^q n+1}$.

An immediate consequence of Lemma 7.1 of [5] (see also [16]) is

$$\left|\sum_{d\in I}\psi\left(\frac{X}{d}\right)\Lambda(d)\right| \le \frac{1}{2R+2}\sum_{d\in I}\Lambda(d) + \left(\sum_{0<|r|\le R}|a_r^{\pm}|\right)\max_{X\le x\le XR}\left|\sum_{d\in I}e\left(\frac{x}{d}\right)\Lambda(d)\right|,$$

where $a_r^{\pm} = \frac{i}{2\pi(R+1)}\left(\pi\left(1-\frac{|r|}{R+1}\right)\cot\left(\frac{\pi r}{R+1}\right)+\frac{|r|}{r}\right) \pm \frac{1}{2R+2}\left(1-\frac{|r|}{R+1}\right)$.

Taking $R=10$ and using *Mathematica*[1] we obtained $\sum_{0<|r|\le 10}|a_r^{\pm}| \sim 0.868 \le \frac{86}{99}$, which implies

Lemma 13.

$$\left|\sum_{d\in I}\psi\left(\frac{X}{d}\right)\Lambda(d)\right| \le \frac{1}{22}\sum_{d\in I}\Lambda(d) + \frac{86}{99}\max_{X\le x\le 10X}\left|\sum_{d\in I}e\left(\frac{x}{d}\right)\Lambda(d)\right|.$$

Using (12) and the previous lemma we get

$$\begin{aligned}
\frac{1}{2}\sum_{d\in I,\,(d,P)=1}\Lambda(d) \;&\le\; \left|\sum_{d\in I}\psi\left(\frac{p^q n+1}{d}\right)\Lambda(d)\right| + \left|\sum_{d\in I}\psi\left(\frac{n}{d}\right)\Lambda(d)\right| \\
&+\; \left|\sum_{d\in I}\psi\left(\frac{(p^q-1)n+1}{d}\right)\Lambda(d)\right| \\
&\le\; \frac{3}{22}\sum_{d\in I}\Lambda(d) + \frac{86}{99}\max_{p^q n+1\le x\le 10(p^q n+1)}\left|\sum_{d\in I}e\left(\frac{x}{d}\right)\Lambda(d)\right| \\
&+\; \frac{86}{99}\max_{(p^q-1)n+1\le x\le 10((p^q-1)n+1)}\left|\sum_{d\in I}e\left(\frac{x}{d}\right)\Lambda(d)\right| \\
&+\; \frac{86}{99}\max_{n\le x\le 10n}\left|\sum_{d\in I}e\left(\frac{x}{d}\right)\Lambda(d)\right| \\
&\le\; \frac{3}{22}\sum_{d\in I}\Lambda(d) + \frac{86}{33}\max_{n\le x\le 10(p^q n+1)}\left|\sum_{d\in I}e\left(\frac{x}{d}\right)\Lambda(d)\right|.
\end{aligned}$$

Since, $\sum_{d\in I,\,(d,P)>1}\Lambda(d) \le \log n + \log\left((p^q-1)n+1\right) + \log\left(p^q n+1\right) \le 3\log n + 2q\log p$, for $n \ge 2$, we obtain

$$\sum_{d\in I}\Lambda(d) \le \frac{43}{6}\max_{n\le x\le 10(p^q n+1)}\left|\sum_{d\in I}e\left(\frac{x}{d}\right)\Lambda(d)\right| + \frac{11}{8}(3\log n + 2q\log p). \tag{13}$$

[1] A Trademark of Wolfram Research

Schoenfeld [14], obtained, for $x \geq e^{30}$, (see also [11])

$$\left| \sum_{d \leq x} \Lambda(d) - x \right| < \frac{1}{4 \cdot 10^5} x.$$

Since $\sum_{d \in I} \Lambda(d) = \sum_{d \leq \sqrt{p^q n+1}} \Lambda(d) - \sum_{d \leq \sqrt{(p^q-1)n+1}} \Lambda(d)$, we obtain

$$\begin{aligned} \sqrt{p^q n+1} - & \frac{1}{4 \cdot 10^5} \sqrt{p^q n+1} - \sqrt{(p^q-1)n+1} - \frac{1}{4 \cdot 10^5} \sqrt{(p^q-1)n+1} \\ & - \frac{11}{8}(3 \log n + 2q \log p) = 0.9999975 \sqrt{p^q n+1} - \\ & 1.0000025 \sqrt{(p^q-1)n+1} - \frac{11}{8}(3 \log n + 2q \log p) \\ & < \frac{43}{6} \max_{n \leq x \leq 10(p^q n+1)} \left| \sum_{d \in I} e\left(\frac{x}{d}\right) \Lambda(d) \right|, \end{aligned} \tag{14}$$

for $n \geq \dfrac{e^{60}-1}{p^q-1}$.

Now, we apply Theorem 9 of [5], a consequence of some very important bounds on exponential sums.

Theorem 14 (Granville-Ramaré). *If $k > 0$ integer and $y \leq \frac{1}{5} x^{3/5}$, then*

$$\left| \sum_{y \leq d \leq y'} e\left(\frac{x}{d}\right) \Lambda(d) \right| \leq \frac{50}{3} y \left(\frac{x}{y^{\frac{k+3}{2}}} \right)^{\frac{1}{4(2^k-1)}} (\log 16y)^{\frac{11}{4}},$$

for any $y \leq y' \leq 2y$.

Since $\sqrt{p^q n+1} \leq 2\sqrt{(p^q-1)n+1}$, the above theorem of Granville and Ramaré applies, and we get for $n > 5^{10} p^{5q}$ (to have the bound $y \leq \frac{1}{5} x^{3/5}$),

$$\begin{aligned} \max_{n \leq x \leq 10(p^q n+1)} \left| \sum_{d \in I} e\left(\frac{x}{d}\right) \Lambda(d) \right| \leq \max_{n \leq x \leq 10(p^q n+1)} \frac{50}{3} \sqrt{(p^q-1)n+1} \times \\ \left(\frac{x}{((p^q-1)n+1)^{\frac{k+3}{4}}} \right)^{\frac{1}{4(2^k-1)}} \left(\log\left(16\sqrt{(p^q-1)n+1}\right) \right)^{\frac{11}{4}} \end{aligned}$$

$$= \frac{50}{3}\sqrt{(p^q-1)n+1}\left(\frac{10(p^qn+1)}{((p^q-1)n+1)^{\frac{k+3}{4}}}\right)^{\frac{1}{4(2^k-1)}} \times$$
$$\left(\log\left(16\sqrt{(p^q-1)n+1}\right)\right)^{\frac{11}{4}} \le \frac{50}{3}\, 11^{\frac{1}{4(2^k-1)}}\, p^{\frac{q}{4(2^k-1)}} \times$$
$$n^{\frac{1}{4(2^k-1)}}\left((p^q-1)n+1\right)^{\frac{1}{2}-\frac{k+3}{4^2(2^k-1)}}\, 2^{-\frac{11}{4}}\left(\log\left(256(p^q-1)n+1\right)\right)^{\frac{11}{4}} \le$$
$$\frac{50}{3}\, 2^{-\frac{11}{4}}\, 11^{\frac{1}{4(2^k-1)}}\, p^{q\left(\frac{1}{2}-\frac{k-1}{4^2(2^k-1)}\right)}\, n^{\frac{1}{2}-\frac{k-1}{4^2(2^k-1)}}\left(\log\left(256((p^q-1)n+1)\right)\right)^{\frac{11}{4}}.$$

We obtain (by taking $k = 2$ - that will suffice for our purpose)

$$\max_{n\le x\le 10(p^qn+1)}\left|\sum_{d\in I} e\left(\frac{x}{d}\right)\Lambda(d)\right| \le \frac{50}{3}\, 2^{-\frac{11}{4}}\, 11^{\frac{1}{12}}\, p^{\frac{23q}{48}}\, n^{\frac{23}{48}}\left(\log\left(256((p^q-1)n+1)\right)\right)^{\frac{11}{4}}$$

By combining (14) and the previous inequality, we get (by Lemma 12) that if $\binom{p^qn+1}{n}$ is squarefree, and $n \ge \max\left(\frac{e^{60}-1}{p^q-1}, 5^{10}p^{5q}\right)$, then

$$\begin{aligned} &0.9999975\sqrt{p^qn+1} - 1.0000025\sqrt{(p^q-1)n+1} \le \\ &21.683\, p^{\frac{23q}{48}}\, n^{\frac{23}{48}}\left(\log\left(256((p^q-1)n+1)\right)\right)^{\frac{11}{4}} + \frac{11}{8}(3\log n + 2q\log p), \end{aligned} \tag{15}$$

which is false for $n \ge \tau_0$ by Lemma 10. By taking $\tau_1(p^q) = \max\left(\frac{e^{60}-1}{p^q-1}, 5^{10}p^{5q}, \tau_0\right)$, the first claim of Theorem 11 follows. If $n < \tau_1(p^q)$, then we use the fact that p^q divides $\binom{p^qn+1}{n}$ (therefore it it is not squarefree, since $q \ge 2$), unless n is of the form $\dfrac{p^{tq}-1}{p^q-1}$, for any $t \in \mathbf{N}$, or of the form $\dfrac{\sum_{k=1}^{m(p-1)+1} p^{qt_k+j_k} - 1}{p^q-1}$, for any $t_i \in \mathbf{N}$, $1 \le m \le q-1$, $0 \le j_i \le q-1$, and $\sum_i p^{j_i} \equiv 1 \pmod{p^q-1}$. The proof of Theorem 11 is done.

Remark 15. The inequality (9) provides explicit bounds for n, for any choice of p and q, with $p^q \le 99999$. We can increase the bound for p^q, by using a weaker result of Schoenfeld [14]. However, in doing that we increase the bound on n as well, so we preferred a better bound on n.

If $q = 2$, using Theorem 6, we get better results for the number of exceptions up to the bound $\tau_1(p^2)$. We give here two samples in our next theorem.

Theorem 16. *Except for* 1, 3 *and* 45, $\binom{4n+1}{n}$ *is not squarefree. Except for* 1, 4 *and* 10, $\binom{9n+1}{n}$ *is not squarefree.*

Proof. If $(p,q)=(2,2)$, the inequality (9) changes into

$$\begin{aligned}&0.9999975\sqrt{4n+1}-1.0000025\sqrt{3n+1}>\\&42.1311\,n^{\frac{23}{48}}\,(\log(768n+1))^{\frac{11}{4}}+\frac{33}{8}\log n+1.65566,\end{aligned}\tag{16}$$

which is true for $n\geq 2^{1518}=\tau_0$. Observe that $\tau_1=\tau_0=\max\left(\frac{e^{60}-1}{3},5^{10}2^{10},\tau_0\right)$. Theorem 6 and Corollary 9 imply that the exceptions for $n<2^{1518}$ (if they exist) are of the form $\dfrac{2^{2t+1}+2^{2j+1}-1}{3}$, $j\leq t$ (we include $j=t$ in the count). Observe that the number of pairs (j,t), giving different numbers of the above form, is less than $\binom{761}{2}\sim 2^{18.2}$.

If $(p,q)=(3,2)$, the inequality (9) changes into

$$\begin{aligned}&0.9999975\sqrt{9n+1}-1.0000025\sqrt{8n+1}>\\&26.04\,n^{\frac{23}{48}}\,(\log(2048n+1))^{\frac{11}{4}}+\frac{33}{8}\log n+2.62417,\end{aligned}\tag{17}$$

which is true for $n\geq 3^{956}=\tau_0$. Observe that $\tau_1=\tau_0=\max\left(\frac{e^{60}-1}{8},5^{10}3^{10},\tau_0\right)$. As in the previous case, we get that the exceptions for $n<3^{956}$ (if they exist) are of the form $\dfrac{3^{2t+1}+3^{2j+1}+3^{2i+1}-1}{8}$, $i\leq j\leq t$ (we include $i=j=t$ in the count). Observe that the number of triples (i,j,t), giving different numbers of the above form, is less than $\binom{478}{3}$.

To check divisibility by squares up to τ_1, we need only concern ourselves with integers of the described forms. First, take the binomial $\binom{4n+1}{n}$. We used Granville's theorem (or Kummer's theorem) to find the power to which $p=3$ divides $\binom{m}{n}$, $m=4n+1$. We expanded n, m in base 3 and then compared their digits. The computation took about 6 hours on our PC (850Mhz with 256Mb of RAM). After the first run of the algorithm, we obtained that all binomial coefficients $\binom{4n+1}{n}$ (for n among the above values) are divisible by 3^2, except for the following integers $n=1$, 3, 13, 45, 85, 171, 181, 2731, 2733, 10965, 13653, 43861, 44741973, 181753173. For $n = 1$, we get $\binom{5}{1} = 5$. For $n = 3$, we get $\binom{10}{3} = 2{\cdot}11{\cdot}13$. For $n = 45$, we get $\binom{181}{45} = 2{\cdot}3{\cdot}5{\cdot}7{\cdot}13{\cdot}23{\cdot}29{\cdot}47{\cdot}53{\cdot}59{\cdot}71{\cdot}73{\cdot}79{\cdot}83{\cdot}89{\cdot}137{\cdot}139{\cdot}149{\cdot}151{\cdot}157{\cdot}163{\cdot}167{\cdot}173{\cdot}179{\cdot}181$. For the remaining values of n, the binomial coefficient $\binom{4n+1}{n}$ is divisible by 5^2 for $n = 171, 181, 2731, 2733, 10965, 13653, 43861, 44741973, 181753173$; divisible by 7^2 for $n = 13$; divisible by 13^2 for $n = 85$. The first claim follows.

We ran a similar program for $\binom{9n+1}{n}$, which unfortunately stopped (because of integer overflow) after a few days. Meanwhile, we wrote another program in Python (which we ran on a PC with the same features), based on the known fact that the number of carries is $\sum_{t=1}^{[\log_2 m]+1} \left(\left[\frac{m}{2^t}\right] - \left[\frac{n}{2^t}\right] - \left[\frac{m-n}{2^t}\right] \right)$. For each t, we checked if the number of carries is greater than or equal to 2, and if it is, we stopped the summation (the expression inside the sum is either 0 or 1). The output of our program (which ran for 9 days) is that the binomial coefficient $\binom{9n+1}{n}$ (for n among the above values) is divisible by 2^2, except for the following integers: 1, 4, 10, 34, 64, 274, 277, 280, 304, 334, 550, 5194, 24604, 199297, 199324, 201754, 202024, 145285144. Among these, for $n = 34$, 64, 274, 280, 304, 334, 24604, 199324, 201754, 202024, 145285144, we have divisibility by 5^2; for $n = 277, 550, 199297$, we have divisibility by 7^2; for $n = 5194$, we have divisibility by 11^2. For $n = 1, 4, 10$, we get $\binom{10}{1} = 2 \cdot 5$, $\binom{17}{4} = 3 \cdot 5 \cdot 7 \cdot 17 \cdot 37$, respectively, $\binom{91}{10} = 7 \cdot 11 \cdot 13 \cdot 17 \cdot 29 \cdot 41 \cdot 43 \cdot 83 \cdot 89$. The second claim follows. $\square$

Remark 17. Since we were successful in getting the computation done in Python, we experimented with a large set of integers n for divisibility of $\binom{4n+1}{n}$ and $\binom{9n+1}{n}$ by squares of various other primes r. It might be worth mentioning that the output in each case was a very short list of integers for which we have no divisibility by r^2 (or 2^2, respectively 3^2). Further investigation is needed to find a bound for the number of binomial coefficients of the form $\binom{p^q n+1}{n}$ (or more general $\binom{m}{n}$), which are not divisible by either p^2 or r^2 (p, r primes).

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