GENERATING TREES AND THE CATALAN AND SCHRÖDER NUMBERS

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ABSTRACT. A permutation $\pi \in S_n$ avoids the subpattern τ iff π has no subsequence having all the same pairwise comparisons as τ, and we write $\pi \in S_n(\tau)$. We present a new bijective proof of the well-known result that $|S_n(123)| = |S_n(132)| = c_n$, the n-th Catalan number. A generalization to forbidden patterns of length 4 gives an asymptotic formula for the vexillary permutations. We settle a conjecture of Shapiro and Getu that $|S_n(3142, 2413)| = s_{n-1}$, the Schröder number, and characterize the deque-sortable permutations of Knuth, also counted by s_{n-1}.

1. INTRODUCTION TO FORBIDDEN SUBSEQUENCES

We regard a permutation $\pi \in S_n$ as a sequence of n elements, $\pi = \{\pi(i)\}_{i=1}^n$. We say that π contains the 3-letter pattern 231 iff there is a triple $1 \le i < j < k \le n$ such that $\pi(k) < \pi(i) < \pi(j)$. Otherwise π *avoids* the pattern. We define τ-avoiding permutations similarly for every $\tau \in S_k$:

Definition 1.1. *For $\tau \in S_k$, a permutation $\pi \in S_n$ is τ-avoiding iff there is no $1 \le i_{\tau(1)} < i_{\tau(2)} < \ldots < i_{\tau(k)} \le n$ such that $\pi(i_1) < \pi(i_2) < \ldots < \pi(i_k)$. The subsequence $\{\pi(i_{\tau(j)})\}_{j=1}^k$ is said to have* type τ.

Two sequences, π, ρ of length n are evidently of the same type iff they have the same pairwise comparisons throughout, namely if $\pi(i) < \pi(j) \leftrightarrow \rho(i) < \rho(j)$. We denote by $S_n(\tau)$ the set of all permutations in S_n which avoid τ. If $R = \{\sigma_1, \sigma_2, \ldots, \sigma_q\}$, we abbreviate $S_n(R) = S_n(\sigma_1, \ldots, \sigma_q) = \bigcap S_n(\sigma_j)$. Fundamental questions are to determine $|S_n(R)|$ viewed as a function of n, and if $|S_n(R)| = |S_n(R')|$ to discover an explicit bijection between $S_n(R)$ and $S_n(R')$.

The most studied case has been to forbid a single pattern of length 3. Because of obvious symmetry arguments described below, there are only two distinct cases to enumerate, $|S_n(123)|$ and $|S_n(132)|$. It happens that these two functions are equal, $|S_n(123)| = |S_n(132)| = c_n = \frac{1}{n+1}\binom{2n}{n}$.

Historically, these two enumerative results were obtained independently [13], [10]. The first satisfactory bijection between the two cases was presented by Rodica Simion and Frank Schmidt [20], and a second was given by Dana Richards [15].

In section two, we present a new bijective proof that $|S_n(123)| = |S_n(132)|$. This proof has the advantage that the enumerative result also follows naturally. In section three, we generalize the result of section two to show that $|S_n(1234)| = |S_n(1243)| = |S_n(2143)|$. Permutations which avoid the pattern 2143 have been studied elsewhere under the name *vexillary permutations*. In section four, we use our techniques to settle a conjecture of Shapiro and Getu, namely that $|S_n(3142, 2413)| = s_{n-1}$, the $n-1$-th Schröder number. The Schröder numbers have many connections with the Catalan numbers.

To conclude this introduction, we detail the symmetry arguments that reduce somewhat the number of problems which can sensibly be posed. A more natural way to think of definition 1.1 is in terms of the familiar *permutation matrices*. If $\pi = \{\pi(i)\}_{i=1}^n$, let $M(\pi)$ be the $n \times n$ matrix with entries $m_{i,j} = \delta_{i,\pi(j)}$ in terms of the Kronecker delta. Then a permutation π contains τ as a subsequence if the corresponding matrix $M(\pi)$ contains $M(\tau)$ *as a submatrix*.

In addition to making the definition clearer, this point of view makes trivial the following observation: $M(\pi)$ contains $M(\tau)$ iff the transpose matrix $M(\pi)^T$ contains $M(\tau)^T$. The same may be said for simultaneously reflecting both the matrices $M(\pi)$ and $M(\tau)$ in either a horizontal or a vertical mirror. These operations together generate the dihedral group acting on the permutation matrices in the obvious way. Since $M(\tau)^T = M(\tau^{-1})$, it follows immediately that $|S_n(\tau)| = |S_n(\tau^{-1})|$. The operations corresponding to reflecting the permutation matrix in a mirror carry $\tau = \{\tau(i)\}_{i=1}^k$ into $\tau^| = \{\tau(k+1-i)\}$ and $\tau^- = \{k+1-\tau(i)\}$.

For subsequences of length 3, these elementary considerations provide that $|S_n(123)| = |S_n(321)|$, and that $|S_n(231)| = |S_n(132)| = |S_n(213)| = |S_n(312)|$, reducing the enumerative problem from six to just two cases. For length 4, the number of cases is reduced from 24 to seven, these being represented by 1234, 1243, 1324, 1432, 1423, 2413 and 2143.

2. A Catalan tree

For a given forbidden permutation τ, we define recursively a rooted tree in which the vertices on the n-th level are identified with the permutations of $S_n(\tau)$. Let the root be the permutation $(1) \in S_1(\tau)$, and let each $\pi \in S_n(\tau)$ be a child of the permutation $\pi' \in S_{n-1}(\tau)$ obtained from π by deleting the largest element, n. (Clearly, a deletion cannot introduce a forbidden τ.) Call the resulting tree $T(\tau)$.

Given $\pi \in S_n$, and given $i \in [n+1]$, let

$$\pi^i = (p_1, p_2, \ldots, p_{i-1}, n+1, p_i, p_{i+1}, \ldots, p_n),$$

we will call this *inserting* $n+1$ *into the site* i.

Definition 2.1. *With respect to a particular τ we will call site i of $\pi \in S_n(\tau)$ an* active site *if the insertion of $n+1$ into site i creates a permutation $\pi^i \in S_{n+1}(\tau)$.*

Clearly the children of π in $T(\tau)$ are just the elements π^i as i ranges over the active sites of π relative to τ. In all proofs involving a structural description of a tree $T(\tau)$, we will rely heavily on the following observations, valid for all π, i.

(1) If π^i does not contain sequences of type τ, neither does π.

(2) If π^i contains sequences of type τ but π does not, then new element $n+1$ participates in all such sequences.

(3) $n+1$ is the largest element of π^i; therefore if it participates in a sequence of type τ, it does so as the largest element of τ.

(4) If the site in π between p_k and p_{k+1} is not active, then neither is the site between p_k and p_{k+1} in π^i.

In the following structural lemmas, we characterize the trees $T(123)$ and $T(132)$. It will here be convenient to label each vertex of $T(\tau)$ with the number of its children (equally, with the number of active sites in the associated permutation). We use the following notation, a *succession rule*, to connect the label of a parent with the label of its t children:

$$(p) \longrightarrow (c_1)(c_2)(c_3)\cdots(c_t).$$

The label (p) will usually include information about the value of t, but in general this will not be sufficient information. It is always our goal to introduce labels leading to a family of succession rules, each globally applicable throughout the tree, and together fully determining its structure. For the trees presently under consideration, one succession rule suffices:

Lemma 2.2. *In* $T(123)$,

$$(t) \longrightarrow (2)(3)(4)\cdots(t+1).$$

Proof. Let π be any node in $T(123)$ having label t. Note that all sites to the left of the first ascent in π are active, but none to the right are. So p_t is the leftmost element which is not a left-to-right minimum. (If $t = n+1$, then π is the descending permutation.)

If $n+1$ is inserted into the leftmost site, the new permutation $\pi^\star = (n+1, p_1, p_2, \ldots, p_n)$ has $t+1$ active sites, namely all those to the left of p_t. On the other hand, if $n+1$ is inserted elsewhere to the left of p_t, say to form π^s, then $n+1$ itself becomes the new leftmost ascent. Hence π^s receives the label s.

The children of π in $T(123)$ are $\pi^\star, \pi^2, \pi^3, \ldots, \pi^t$, and these receive the labels $t+1, 2, 3, \ldots, t$ respectively. □

Example 2.3. *Consider the following typical node of* $T(123)$*, in which the active sites are numbered from left to right:*

$$\pi = ({}_1 5_2 3_3 1_4 4_\times 2_\times)$$

If we form π^3*, we are left with* 3 *active sites, those to the right vanishing:*

$$\pi^3 = ({}_1 5_2 3_3 \mathbf{6}_\times 1_\times 4_\times 2_\times)$$

Lemma 2.4. *In* $T(132)$,

$$(t) \longrightarrow (2)(3)(4)\cdots(t+1).$$

Proof. The active sites are no longer necessarily the first t sites, so suppose they are numbered from the left $a_1, a_2, \ldots, a_t$.

If inserting $n+1$ creates a 132, then $n+1$ plays the part of 3. This cannot happen if $n+1$ becomes the leftmost element, so site 1 is always active ($a_1 = 1$). Furthermore $\pi^\star = \pi^1$ has label $t+1$, because the t active sites of π remain active (and one new one is introduced preceding the new element). For consider inserting $n+2$ in any site of π^1. A subsequence $(n+1, n+2, p_j)$ cannot be of type 132. Hence any 132 created must be of form $(p_i, n+2, p_j)$, but this would have caused the site to be inactive in π.

On the other hand, suppose $n+1$ is inserted into active site a_s for $s \geq 2$. This will render inactive all the sites to the *left* of the insertion, except for the first site. This is because $(p_1, n+2, n+1)$ would be a forbidden sequence. This leaves $t-(s-1)$ to the right of $n+1$, plus the leftmost site, a total of $t-s+2$.

The children of π in $T(132)$ thus receive the labels $t+1, t, \ldots, 3, 2$ respectively as the active sites are considered in order from left to right. $\square$

Example 2.5. *Consider the following typical node of* $T(132)$*:*

$$\pi = ({}_1 5_2 3_\times 4_3 1_\times 2_4)$$

We insert at the third active site ($a_3 = 4$) *to form* π^4*, we are left with* 3 *active sites, those to the left vanishing:*

$$\pi^4 = ({}_1 5_\times 3_\times 4_\times \mathbf{6}_2 1_\times 2_3)$$

From these two lemmas, we conclude that $T(123)$ and $T(132)$ are isomorphic trees, and it is easy to see that the trees have trivial symmetry groups and so the isomorphism is unique. Since siblings receive distinct labels, a vertex can be uniquely determined in each tree by listing the labels of its ancestors.

Example 2.6. *We list on the left a node of* $T(123)$*, then the labels of its ancestors from the root down, then the corresponding node of* $T(132)$.

132	$(2,2,2)$	123
312	$(2,2,3)$	312
231	$(2,3,2)$	213
213	$(2,3,3)$	231
321	$(2,3,4)$	321

Example 2.7. *The vertices from the above examples,* $(536142) \in T(123)$ *and* $(534612) \in T(132)$ *are carried to each other by the unique bijection induced by the tree isomorphism.*

If a sequence of vertex labels $(f_1, f_2, \ldots, f_n)$, having the property that $f_1 = 2$ and $2 \leq f_i \leq f_{i-1} + 1$ is converted into a sequence $(a_1, a_2, \ldots, a_n)$ according to $a_i = i + 2 - f_i$, then the new sequence will be non-decreasing with $1 \leq a_i \leq i$. Such sequences are a familiar instance of the Catalan numbers, being naturally associated with non-diagonal-crossing lattice paths. We conclude

Theorem 2.8. *For all $n \geq 1$, $|S_n(123)| = |S_n(132)| = c_n = \frac{1}{n+1}\binom{2n}{n}$.*

References to the Catalan number are almost everywhere dense in the combinatorial literature; historically minded readers might be interested in [3] (but references go back at least to Euler and Segner, 1758). The first enumeration of $S_n(123)$ is in [13], for $S_n(132)$ see [10]. The first purely bijective proof that $|S_n(123)| = |S_n(132)|$ was presented in [20]. This bijection has the advantage of fixing the intersection of the two groups. The new bijection presented here does not; see elements 213 and 231 in the above table. On the other hand, we were able to produce the enumerative result with little extra effort.

We strengthen the enumerative result somewhat by counting the number of permutations avoiding 123, with length n and t active sites. First let $N(m,s)$ be the number of nodes on level $m+1$ with $m+2-s$ children. Small values of this function are given in table 1, the first column corresponding to the fact that the tree has one node on level one, labelled 2.

Since there will be exactly one permutation on level $n+1$ having label r for each permutation on level n having a label $\geq r-1$ it follows that (for all $m \geq 1$ and $1 \leq s \leq m$),

$$N(m,s) = \sum_{i=1}^{s} N(m-1,i) \tag{2.9}$$

$$= \sum_{i=1}^{s-1} N(m-1,i) + N(m-1,s) \tag{2.10}$$

$$= N(m,s-1) + N(m-1,s) \tag{2.11}$$

It follows that $N(n,s)$ counts the number of non-diagonal-crossing integer lattice paths from $(0,0)$ to (m,s), the number of these obeying the same recurrence, and the initial conditions imposed by the first column. In closed form, the number of such paths is well-known to be $\binom{m+s}{s} - \binom{m+s}{s-1}$. Hence

Theorem 2.12. *The number of $\pi \in S_n(123)$ having t active sites relative to 123 is*

$$\binom{2n-t}{n-t+1} - \binom{2n-t}{n-t}$$

The rooted trees $T(\tau)$ introduced here seem to be entirely natural objects, but do not appear widely in the literature. The technique appears to be original to Chung, Graham, Hoggatt and Kleiman, who introduce it to examine the reduced Baxter permutations in [4]. This paper explicitly suggests application to other classes of permutations, but we have not heard of any such work appearing in the 10 years between that paper and the beginnings of the present work.

The technique is now beginning to be more widely used, and the objects have aquired the name *generating trees*. Recent applications involving permutations include [23], [22], [8], [5]. If the objects generated are restricted permutations, we may wish to speak of *restricted permutation trees*. But there is no reason to stop here. Other classes of combinatorial objects for which generating trees have been produced include directed animals [25], binary trees [8], planar maps [5], and semiorders [below].

The particular object $T(123)$ (without the permutations attached) is an especially natural object; we hereby name it the *Catalan tree*. We imagine that this particular generating tree must appear in other settings; we would be interested to learn of any in addition to the following.

Although she does not use the fact, the mimimal semiorders introduced by Karen Stellpflug in [24], [1] are also generated by a Catalan tree. A partially ordered set is a semiorder iff it can be represented by a set of equal length open intervals in the real line, with the order relation $(a, b) < (c, d)$ iff $b \leq c$. A semiorder has representation number k if it has a representation in which all intervals have integer endpoints and the same length k, but has no such representation with intervals of length $k - 1$.

Stellpflug shows how to obtain the minimal k-representable semiorders inductively by the process of duplicating one minimal element. If a minimal k-representable semiorder has r minimal elements, it produces by her construction r minimal $k+1$-representable semiorders, having variously $2, 3, 4, \ldots, r+1$ minimal elements. Noting that this process forms a Catalan tree amounts to an alternate proof of Stellpflug's result that the number of these k-representable semiorders is c_k.

3. Trees for forbidden sequences of length 4

We repeat the arguments of the above section for certain $\tau \in S_4$, retaining the definition of an active site, but augmenting the notion of a label on a node. We begin with the tree $T(1234)$. To each node $\pi \in S_n(1234)$ we associate an ordered pair (x, y) as follows. Let x be the position of the first ascent in π. In the terminology of Schensted [17], x is the index of the first element of the second basic subsequence (or $n+1$ if none exists). Let y be the number of active sites in π. In this instance, y is the index of the first element of the third basic subsequence (or $n + 1$).

Lemma 3.1. *In* $T(1234)$,

$$(x, y) \longrightarrow (2, y+1)(3, y+1) \ldots (x, y+1)(x, x+1)(x, x+2) \ldots (x, y)(x+1, y+1)$$

Proof. Let π be a node of $T(1234)$ with label (x, y). The y active sites of π are the first y sites. By considering the new locations of the first elements of the second and third basic subsequences we verify that π^i is associated in $T(1234)$ with

$$(x+1, y+1) \quad \text{if } i = 1, \tag{3.2}$$
$$(i, y+1) \quad \text{if } 2 \leq i \leq x, \tag{3.3}$$
$$(x, i) \quad \text{if } x+1 \leq i \leq y. \tag{3.4}$$

□

Next consider the tree $T(1243)$, in which the nodes are again to be labelled (x, y) according as x is the position of the first ascent, and y is the number of active sites. The y active sites are no longer necessarily the first y sites on the left.

Lemma 3.5. *In* $T(1243)$,

$$(x, y) \longrightarrow (2, y+1)(3, y+1)\ldots(x, y+1)(x, x+1)(x, x+2)\ldots(x, y)(x+1, y+1)$$

Proof. Let the y active sites be numbered left-to-right as $a_1, a_2, \ldots, a_y$. Note that the first x sites are active, as an $n+1$ here cannot find an increasing pair to its left to form a 1243.

The insertion of $n+1$ into site a_i of π splits it into two sites, both potentially active. We may verify that if a site was active in π, it remains active in π^{a_i} unless it falls to the right of x and to the left of position a_i. It is then easy to check that a_i has the associated pair

$$(x+1, y+1) \quad \text{if } i = 1, \tag{3.6}$$
$$(i, y+1) \quad \text{if } 2 \leq i \leq x, \tag{3.7}$$
$$(x, x+y+1-i) \quad \text{if } x+1 \leq i \leq y. \tag{3.8}$$

□

We define the tree $T(2143)$ analogously, with x being the position of the first *ascent* and y being as usual the number of active sites. We can prove in an almost identical fashion that

Lemma 3.9. *In* $S_n(2143)$,

$$(x, y) \longrightarrow (2, y+1)(3, y+1)\ldots(x, y+1)(x, x+1)(x, x+2)\ldots(x, y)(x+1, y+1)$$

Combining the results of lemmas 3.1, 3.5 and 3.9, we hav the following theorem and its immediate corollary.

Theorem 3.10. $T(1234) \cong T(1243) \cong T(2143)$, *and these isomorphisms are unique.*

Proof. In each tree the root is labelled $(2,2)$. Applied recursively, the structural lemmas above ensure the isomorphisms.

From the labels (x,y), we can count the number of children of each node, and the number of children of each child. The sets of these are different for different pairs (x,y), and it is easy to check that no siblings ever have the same label. Therefore the trees have trivial symmetry groups. $\square$

Corollary 3.11. $|S_n(1234)| = |S_n(1243)| = |S_n(2143)|$, *for all positive* n.

Regev has shown [14] that $|S_n(1234)|$ is asymptotic to $c \cdot \frac{9^n}{n^4}$ for a constant c. This result can now be applied to the sets $|S_n(1243)|$ and $|S_n(2143)|$ as well. The permutations of $|S_n(2143)|$ in particular have been extensively studied, as these are precisely the *vexillary permutations*. The vexillary permutations are relevant to the theory of Schubert polynomials, and therefore to the cohomological structure of flag manifolds. They are a superset of the *dominant* and of the *Grassmannian* permutations. For alternative characterizations of the vexillary permutations, see section 2 of Lascoux and Schützenberger [11] and chapter one of MacDonald [12], which also defines the dominant and Grassmannian permutations.

There is considerable work still to be done with restricted permutation trees, even for single forbidden subsequences of length 4. The following questions were posed in [?]. The first was answered in the affirmative by Stankova [23], the others are believed to be open.

Question 3.12. *Is* $T(4132) \cong T(3142)$*?*

Question 3.13. *Is* $T(1342)$ *a proper subtree of* $T(1432)$*?*

Question 3.14. *Is* $T(1423)$ *a proper subtree of* $T(1324)$*?*

This is also a convenient place to mention Ira Gessel's conjecture that $S_n(R)$ is P-recursive, for any set R of restrictions [6].

4. A Schröder tree

The Schröder numbers are closely related to the Catalan numbers, but less well known. Like the Catalan numbers, they have many combinatorial interpretations, including one in terms of lattice paths. For some references see [18], [2], [9], [16], [21]. The Catalan numbers, as seen above, count the number of non-diagonal-crossing lattice paths from the origin to (n,n) composed of the vectors $(0,1)$ and $(1,0)$. The Schröder numbers count the number of non-diagonal-crossing lattice paths from the origin to (n,n) composed of the vectors $(0,1)$, $(1,0)$ and $(1,1)$. They are thus the diagonal elements in table 2.

This characterization leads directly to the following formula for the n-th Schröder number, the i-th term being the number of paths using i diagonal steps $(1,1)$.

$$s_n = \sum_{i=0}^{n} \binom{2n-i}{i} c_{n-i}$$

Lou Shapiro and Seyoum Getu conjectured [7] that s_{n-1} is the number of permutations of length n having no subsequence of type 3142 or 2413. We settle this conjecture in the affirmative. The result is of interest, as it is the first non-trivial enumerative result to be obtained for any problem involving forbidden subsequences of length $k \geq 4$. There have since been others obtained by the author, by Stankova, and by Dulucq, Gire and Guibert.

As in the Catalan-tree case, we begin by defining recursively a rooted tree, $T(3142, 2413)$. Let the root be (1), and let each $\pi \in S_n(3142, 2413)$ be a child of the permutation $\pi' \in S_{n-1}(3142, 2413)$ obtained by deleting the largest element of π. Again, we label each vertex with the number of its children. We have the following structural lemma.

Lemma 4.1. *In* $T(3142, 2413)$,

$$(t) \longrightarrow (3)(4) \cdots (t)(t+1)(t+1)$$

Proof. Let π be an arbitrary element on level n of $T(3142, 2413)$, having label (t). We again consider active sites, but instead of clearing sites to left or right, an insertion will clear sites across the middle. By the middle of a permutation, we mean the position of the largest element, n. Note that the two sites immediately adjacent to n are always active: if placing $n+1$ here created a sequence of either type 3142 or 2413, then n would already play the same role in a like sequence, a contradiction.

Now divide the permutation π into *blocks* of contiguous elements, the blocks being separated by the active sites. We note that if a block B is right of n but left of C, then all the elements in B are larger than all those in C. Otherwise, a smaller element b in B and a larger element c in C would form a sequence n, b, c of type 312, rendering inactive the supposedly active site separating the blocks B and C. It follows that the values in the blocks to the right of n decline monotonically to the right of the middle. A symmetric claim can be made on the left of the middle.

In fact, we can say something stronger. Let v and w be two elements in the same block B right of the middle, and let x be an element left of the middle such that $v < x < w$. If v is to the left of w then x, n, w, v is of type $2,4,1,3$, a contradiction. It follows that within B all the elements larger than x are toward the middle, and all those smaller than x are away from the middle. Now consider the site separating these two bunches of elements. If this site is inactive, then it falls within a sequence of type $31|2$ or $2|13$. Suppose such a sequence exists; those elements playing the roles of 1 and 2 in this sequence must be within the block B, otherwise one of the active sites adjoining the block would also be deactivated. Since the near elements are all larger than the far elements, type $31|2$ is excluded. For type $2|13$, the element playing the role of 3 cannot be within the block B, since it would then be bigger than the element playing 2, nor can it be in a block to the right, by the remarks of the previous paragraph.

The conclusion is that each block is composed of consecutive elements from $[n]$. Therefore we can order the blocks by taking an arbitrary representative from each one; those $t-1$ elements just toward the middle from the t active sites will do. We call this the *inner subsequence*. This subsequence is unimodal (downwards), since it takes one representative from each block.

Number the active sites $0, 0, 1, 2, ..., t-2$ according as they are associated with the largest, next largest, etc. members of the inner subsequence. We claim that insertion of $n+1$ into the site thus numbered q produces a permutation with $t+1-q$ active sites. For insertion splits one active site into two sites (both automatically active because associated with the largest element), and then q sites are deactivated. The deactivated sites are precisely those which were numbered $< q$, except for the highest-numbered one in this set which is on the far side of the middle.

To see this, let the element associated with the insertion site be x, and assume w.l.o.g. that the insertion is left of the middle. Then $n+1, x, n$ forms a sequence of type 312, deactivating all sites between x and the middle. Likewise, the site right of the middle with the highest number $< q$ is associated with an element y which is greater than x and so the sequence $n+1, x, y$ provides the same service. But the site associated with y is itself not deactivated, nor are those further to the right of y (or left of x).

We check this last claim for sites right of y: if one of these is deactivated, it is because of a sequence involving $n+1$, therefore some $n+1, v, w$ with $v < w$ and v between x and y and w to the right of y. First note that w cannot be greater than y because it is located in a block to the right of y's block. If $w < x$, then x, v, y, w is of type 3142, a contradiction. If $w > x$, then the element of the inner subsequence in its block is likewise greater than x (and less than y). But this contradicts our choice of y as the smallest element of the inner subsequence larger than x and right of the middle. (The claim for sites to the left of x is easy to check from the descending block structure.)

□

As we did with the Catalan trees, we determine a recurrence for the number of permutations on the n-th level of the Schröder tree having t children. Again for simplicity, we let $m = n-2$ and $s = n+1-t$, then seek $f(m, s)$. From the lemma, we can see that

$$f(m, s) = (\sum_{i=0}^{s-1} f(m-1, i) + 2f(m-1, s).$$

Into this we substitute the formula for $f(m, s-1)$ to obtain $f(m, s) = f(m, s-1) - f(m-1, s-1) + 2f(m-1, s)$. We take for our boundary conditions $f(2, s) = 2\delta_{2s}$, since there are 2 permutations on level $n = 2$, each having 3 children. We illustrate this in table 3. In this figure, it is apparent that the diagonal elements are (with one exception) the same as those in table 2, which was governed by the recurrence $g(m, s) = g(m, s-1) + g(m-1, s-1) + g(m-1, s)$. The following very elegant proof that the diagonals are identical was provided by Ian Goulden [personal commmunication].

Lemma 4.2. *$f(i, i) = g(i, i)$ for all $i \geq 1$.*

Proof. In table 2, let $G = \sum_{i=0}^{\infty} g(i, i) z^i$ be the generating function for the diagonal elements. These are the number of non-diagonal-crossing paths from the origin to (i, i). Note that every such path returns to the diagonal for a first time. If this is at (j, j) for some $j < i$, we can decompose the path into a non-diagonal-crossing path of length $j-1$ from $(1, 0)$ to $(j, j-1)$ and a non-diagonal-crossing path of length $i-j$ from (j, j) to (i, i). From this observation we derive the equation $G = zG^2 + zG + 1$. (The zG^2 term comes from the paths which begin with a step to the east; the zG term from those which begin with a step to the northeast.)

In the second table, begin by halving all the elements. Let $F = \frac{1}{2} \sum_{i=0}^{\infty} f(i, i) x^i$ be the generating function for the diagonal. F is the sum, taken over all paths beginning at the origin and using k of each of $(0, 1)$ and $(1, 0)$ and l of $(1, 1)$, of $(2)^k (1)^k (-1)^l x^{k+l}$. By the same argument as above, therefore, $F = 2xF^2 + (-1)xF + 1$. Verify that substituting $F = \frac{G+1}{2}$ into this equation produces the familiar generating function equation for the Schröder numbers, as desired. $\square$

Theorem 4.3. *$|S_n(3142, 2413)| = g(n-1, n-1) = s_{n-1}$, the $n-1$-th Schröder number.*

Proof. Lemma 4.2 shows that the number of permutations of length $n+1$ avoiding 3142 and 2413 and having 3 active sites is s_{n-1}. But there is exactly one node labelled 3 on level $n+1$ for *every* node on level n (if $n > 2$). So $|S_n(3142, 2413)|$ is also equal to s_{n-1}. $\square$

It is interesting that we were able to find a simpler expression for the numbers $|S_n(3142, 2413)|$ than for $|S_n(1234)|$. Why? We offer two suggestions.

First, the fact that 3142 is 2413 written in reverse means that $T(3142, 2413)$ is invariant under mirror reflection (if embedded in the plane with siblings arranged in lexicographic order). Perhaps this symmetry somehow enables us to obtain a single-parameter labelling, where two parameters were necessary for $T(1234)$.

Second, we see that the permutation matrices corresponding to 3142 and 2413 form a complete symmetry class under the action of the dihedral group D_4. It seems combinatorially more natural to forbid this entire set of objects than to impose a single restriction. Indeed, Shapiro and Getu's attention was drawn to this case by considering a class of permutations characterized by avoiding these two submatrices (and thus invariant under the action of the dihedral group). We ask, therefore, whether more natural enumerative results may be obtained by forbidding entire symmetry classes of permutations.

We offer a possible method for a second proof that $|S_n(3142, 2413)| = s_{n-1}$. From the generating function equation $G = zG^2 + zG + 1$, it is easy to derive the following recurrence for the Schröder numbers:

$$s_n = s_{n-1} + \sum_{j-1}^{n} s_{j-1} \cdot \ s_{n-j}$$

The corresponding equation for the Catalan numbers is

$$c_{n+1} = \sum_{j=0}^{n} c_j \cdot c_{n-j}$$

and can be used to prove that $|S_n(132)| = c_n$. To count the permutations of length $n + 1$ which avoid 132, suppose $n + 1$ is fixed in position $j + 1$. Then there are j elements to the left, and $n - j$ to the right. Which these elements are is precisely determined by the observation that no element on the left can be smaller than any on the right. How they may be arranged is determined recursively: $|S_{j-1}(132)|$ ways on the left and $|S_{n-j}(132)|$ ways on the right. If either side is empty, we have the base case $S_0(123) = 1 = c_0$. Summing over j, the recurrence follows.

The data we have examined support the following conjecture.

Conjecture 4.4. *Among all the permutations of $S_n(3142, 2413)$, take those in which 1 appears in position j. For each of these, count 1 less than the number of active sites (with respect to 3142 and 2413). Then the total is $s_{j-1} \cdot \ s_{n-j}$.*

The Schröder recurrence 4 would follow immediately from this conjecture. By counting active sites, we are tallying the permutations of $S_{n+1}(3142, 2413)$. The terms inside the sum come from letting the position of n range along the permutation of length n. Subtracting 1 for each permutation produces the extra term of s_n outside the sum.

5. Knuth's deque permutations

It has been seen that the framework of forbidden subsequences unifies various problems from the literature which have to do with excluded configurations. For instance, the permutations which can be sorted by passage through a stack are those of $S_n(231)$ [10]. The matrices corresponding to $S_n(3142, 2413)$ are exactly those which do not fill up under 'bootstrap percolation' [19].

We offer a characterization of the permutations which can be sorted by an output restricted double-ended queue, the number of which is also a Schröder number. In [10], Knuth characterizes by $S_n(312)$ those permutations that can be *realized* using a stack. This is equivalent to saying that the permutations which avoid the inverse permutation, 231, are those which can be *sorted* using a stack. The fashion recently is to adopt the latter viewpoint.

In the same source, Knuth introduces the permutations which can be sorted (realized) using an output-restricted double-ended queue. That is, we are given a queue with three permitted operations, S to insert an element on the left, Q to insert on the right, and X to remove from the left. We ask for which n-permutations can a sequence of $2n$ operations be specified which produces the identity.

Knuth shows that the number of such deque-sortable permutations is the Shröder number: $|\,\mathrm{Deq}_n\,| = s_{n-1}$. We will show that they form precisely the set $S_n(2431, 4231)$. First, note that neither 2431 nor 4231 is deque-sortable. This can be done by trying all sorting sequences of S, Q, and X, and noting that the identity is never produced. Then observe that any permutation containing a subsequence of one of these types cannot be deque-sorted either, because introducing new elements does nothing to undo the essential knot produced by these 4. Thus $\mathrm{Deq}_n \subseteq S_n(2431, 4231)$. We now show that the $|S_n(2431, 4231)| = s_{n-1}$, establishing the equality of the two sets.

In section one we remarked that $|S_n(\tau)| = |S_n(\tau^{-1})|$; therefore $|S_n(2431, 4231)| = |S_n(4132, 4231)|$. But we have

Lemma 5.1. *In* $T(4132, 4231)$,

$$(t) \longrightarrow (3)(4)\cdots(t)(t+1)(t+1)$$

Proof. Let π be an arbitrary node on level n of $T(4132, 4231)$, with label (t). A site is inactive, if and only if there is either a 231 or a 132 to its right. Thus if a site w is inactive, any site v left of w is also inactive, under the influence of the same 231 or 132 which deactivated w. It follows that the t active sites are those furthest to the right in π.

Inserting $n+1$ in the rightmost site creates no 231 or 132, hence deactivates no sites. It therefore gives rise to a child permutation with label $(t+1)$. On the other hand, inserting $n+1$ in any other active site (except the very leftmost) does create some 231s and/or 132s, the rightmost of which begins in the lefthand neighbour of the insertion. Hence all sites left of this point are rendered inactive. If the insertion is into site number $2, 3, \ldots, t$, counting from the right, only the $3, 4, \ldots, t+1$ right of the lefthand neighbour of the insertion remain active. $\square$

Therefore $T(4132, 4231) \cong T(3142, 2413)$, whence $|S_n(2431, 4231)| = s_{n-1}$. We conclude

Theorem 5.2.

$$\mathrm{Deq}_n = S_n(2431, 4231)$$

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