Chain Lengths in the Dominance Lattice

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Abstract

We find the size of the largest union of two or three chains in the lattice of partitions of n under dominance order. We also present some partial results and conjectures on chains and antichains in this lattice.

1 Introduction

Let P_n denote the poset of partitions of the positive integer n, ordered by dominance (aka majorization), i.e. $\mu \leq \nu$ if $\mu_1 + \mu_2 \cdots + \mu_k \leq \nu_1 + \nu_2 + \cdots + \nu_k$ for all k. This poset is a lattice, and is self-dual under conjugation. P_n is not graded for $n \geq 7$, since there exist saturated chains from $\{n\}$ to $\{1^n\}$ of all lengths from $2n - 3$ to $cn^{3/2}$ [2, 6].

Given any poset P, there exists a partition $\lambda(P)$ such that the sum of the first k parts of λ is the maximal number of elements in a union of k chains in P. In fact, the conjugate of λ has the same property with chains replaced by antichains [1, 4, 5]. Let $\lambda_k(P)$ denote the kth part of this partition.

The length $h(P_n)$ of the longest chain in P_n has been known for some time [6]. If $n = \binom{m+1}{2} + r$, $0 \leq r \leq m$, then $h(P_n) = \frac{m^3 - m}{3} + rm$. In other words, $\lambda_1(P_n) = \frac{m^3 - m}{3} + rm + 1$. Our main results are the following theorems.

Theorem 1. *For $n > 16$, $\lambda_2(P_n) = \lambda_1(P_n) - 6$.*

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Theorem 2. *For $n > 135$, $\lambda_3(P_n) = \lambda_2(P_n) - 6$.*

Consider the subposet Q_n of P_n consisting of the partitions that appear in chains of length $h(P_n)$. Clearly Q_n is self-dual under conjugation, since conjugation takes a decreasing chain to an increasing chain of the same length. It seems likely that Q_n is a graded lattice, but for our purposes it will suffice to use a weaker statement, namely: for $\mu \in Q_n$, define $r(\mu)$ to be the length of the longest chain from $\{n\}$ to μ; then $\mu \neq \{1^n\}, \{n\}$ is covered by an element ν such that $r(\nu) = r(\mu) - 1$ and covers an element ν such that $r(\nu) = r(\mu) + 1$. In other words, every element of Q_n is on a fixed level. Figure 1 shows an example of a poset Q with this property that is not graded. Note that the top element is level 0, and the levels increase as we move down.

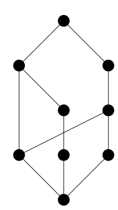

Figure 1: A non-graded poset with well-defined levels.

The covering relation in P_n comes in two flavors. Following the methods of [6], we represent Ferrers diagrams with vertical parts, as illustrated in Figure 2. We say μ covers ν by an H-step if there exists i such that $\nu_i = \mu_i - 1$, $\nu_{i+1} = \mu_{i+1} + 1$, and $\nu_k = \mu_k$ for $k \neq i, i+1$. In terms of Ferrers diagrams, this corresponds to moving a box horizontally one space to the right (and down some distance). The other flavor is a V-step, which is an H-step on the conjugate, and corresponds to moving a box vertically one space down (and right some distance). Chains from $\{n\}$ to $\{1^n\}$ consisting of H-steps followed by V-steps are maximal.

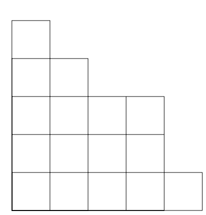

Figure 2: The partition $\{5, 4, 3, 3, 1\}$.

2 Down to work

First we focus on Theorem 1. The cases where $n \leq 16$ will be handled separately, so for now assume $n > 16$.

We will prove Theorem 1 by showing that there exist two disjoint chains in Q_n of lengths $h(P_n)$ and $h(P_n) - 6$. Since Q_n is a subposet of P_n, these are also chains in P_n. Since there are six elements of P_n in saturated antichains of size 1, this is clearly the maximum possible number of elements in two chains, thus giving $\lambda_2(P_n)$ exactly.

To that end, we seek two disjoint chains in Q_n from $\{n-2, 1, 1\}$ and $\{n-3, 3\}$ to $\{2, 2, 2, 1^{n-6}\}$ and $\{3, 1^{n-3}\}$. Let Q_n^* denote Q_n without the top three and bottom three elements.

Lemma 1. *If Q_n^* has at least two elements on every level, then it has two disjoint chains of maximal length.*

Proof: Clearly we can start two chains with the two elements in the top level, so proceed by induction. The only potential problem is if we reach two elements on level k that both cover only one and the same element on level $k+1$. In that case, take a second element on level $k+1$ and a maximal chain ending at it. This chain has a lowest point of intersection with one of the two old chains, so just replace that old chain with the new one from that point on. See Figure 3. □

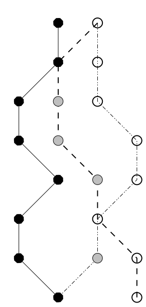

Figure 3: Salvaging a dead end.

Since Q_n^* is self-dual, it will suffice to show that the first half of its levels have at least two elements. We do this by explicitly constructing two disjoint chains to the halfway point. As a first approximation of these chains, take the following construction.

The left chain starts at $\{n-2, 1, 1\}$. At every step, we take the right-most possible H-step, e.g. the next partition is $\{n-3, 2, 1\}$. The right chain starts at $\{n-3, 3\}$. At every step, we take the left-most possible H-step, e.g. the next partition is $\{n-4, 4\}$. The names come from the relative positions of the chains when plotted, as in Figure 7. Both chains will eventually reach $\{m, m-1, \ldots, r+1, r, r, r-1, \ldots, 2, 1\}$, which is at least the halfway point [6], so the idea is to modify the left chain as little as possible to make it reach the halfway point without intersecting the right chain.

Once we've done that, we can apply Lemma 1 to get two disjoint chains of length $h(P_n) - 6$, then append the top and bottom three elements to one of them two get the desired chains. The following proposition will be used to prove several lemmas concerning the right chain.

Proposition 1. *If $\mu = \{\mu_1, \mu_2, \ldots, \mu_k\}$ is in the right chain, then $\mu_i - \mu_{i+1} \leq 2$ for $i = 1, 2, \ldots, k-2$. In other words, only the last difference can be greater than 2. Moreover, excluding the last difference, μ cannot have more than one difference equal to 2.*

Proof: By construction, we are always doing the left-most possible H-step. At first there is nothing to prove, since $k = 2$ through $\{\frac{n}{2}, \frac{n}{2}\}$ or $\{\frac{n+1}{2}, \frac{n-1}{2}\}$. Think in terms of partition diagrams as in the definition of H-steps. If there are no differences greater than 1 (excluding the last one), then push one box from μ_{k-1} to increase the last part (or from μ_k increase the number of parts). Now move to the left, pushing one box at a time until $\mu_i - \mu_{i+1} < 2$ for $i = 1, 2, \ldots, k-2$ again. Clearly we never get a difference greater than 2 or more than one difference of 2 unless we had one before, so the result follows by induction. □

3 Proof of Theorem 1

The proof comes in six cases, depending on r. We begin with general calculations that will be used in multiple cases. If $\mu = \{\mu_1, \mu_2, \mu_3, \ldots\}$ is reachable from $\{n\}$ by only H-steps, such as the elements of the left and right chains, then $r(\mu) = \mu_2 + 2\mu_3 + 3\mu_4 + \cdots$, since each box in μ_i had to be moved horizontally $i-1$ times.

Note that any μ in the left chain with $\mu_1 - \mu_2 > 2$ is not in the right chain by Proposition 1. This means that the left chain makes it safely to the partition $\{m+r, m-1, m-2, \ldots, 2, 1\}$ at level $\frac{m^3-m}{6}$ for $r \geq 2$. For $r > 2$, we can continue safely to $\{m+2, m-1, m-2, \ldots, r-2, r-2, \ldots, 2, 1\}$ (using both assertions in Proposition 1) for an additional $m(r-2) - \binom{r-2}{2}$ levels.

So we're done if $2(m(r-2) - \binom{r-2}{2}) \geq rm$. For $r > 4$, this comes down to $m \geq \frac{r^2-5r+6}{r-4} = r - 1 + \frac{2}{r-4}$. For $r = 5$, this means $m \geq 6$. In fact $m = 5$ also works, since we really just needed $m(r-2) - \binom{r-2}{2} \geq \left\lfloor \frac{rm}{2} \right\rfloor$. For $r > 5$, we just need $m \geq r$ (since m must be an integer), but that's as general as possible since $r \leq m$ by definition. Thus we've established Theorem 1 when $r \geq 5$.

If $r = 4$, then the above construction gets us to one level shy of where we need to be, since we only reach $\{m+2, m-1, m-2, \ldots, 3, 2, 2, 1\}$ safely. Since $h(P_n)$ is always even when r is even, the middle level consists of self-conjugate partitions. Note that not all self-conjugate partitions are in Q_n, but one will be if it is covered by an element of Q_n since by duality it covers the conjugate of that element. Now we simply observe that $\{m+2, m-1, m-2, \ldots, 3, 2, 2, 1\}$ covers the self-conjugate partition $\{m+2, m-1, m-2, \ldots, 3, 2, 1, 1, 1\}$. This partition cannot be in the right chain by Proposition 1 (it is also not H-reachable from $\{n\}$ [6]), so this establishes Theorem 1 when $r = 4$. Alternatively, the step to $\{m+1, m, m-2, \ldots, 3, 2, 2, 1\}$ is safe for $m > 4$, which will be more useful for proving Theorem 2.

If $r = 0$, then we safely reach $\{m+1, m-2, m-2, \ldots, 2, 1\}$, one level shy again. Once again, we simply observe that this covers the self-conjugate partition $\{m+1, m-2, m-2, \ldots, 3, 1, 1, 1\}$, which is not in the right chain by Proposition 1, so this establishes Theorem 1 when $r = 0$.

The remaining cases each require a lemma to get past the shortfall in the above argument.

If $r = 1$, then we safely reach $\{m+2, m-2, m-2, m-3, \ldots, 2, 1\}$, but in fact we can go further along the left chain.

Lemma 2. *The partitions* $\{m+1, m-1, m-2, \ldots, 2, 1\}$ *and* $\{m, m-1, \ldots, k+1, k, k, k-2, \ldots, 2, 1\}$, $5 \leq k \leq m$, *do not occur in the right chain.*

Proof: If $\{m+1, m-1, m-2, \ldots, 2, 1\}$ occurred in the right chain, then it would have to be preceded by $\{m+2, m-2, m-2, \ldots, 2, 1\}$ or $\{m+1, m, m-3, \ldots, 2, 1\}$ (otherwise we couldn't have done the left-most H-step), both of which violate Proposition 1.

If $\{m, m-1, \ldots, k+1, k, k, k-2, \ldots, 2, 1\}$ occurred in the right chain, then it would have to be preceded by $\{m, m-1, \ldots, k+1, k, k, k-1, k-4, k-4, \ldots, 2, 1\}$ (note this works even for $k = m$) which violates Proposition 1 unless $k - 4 = 0$, hence the need for $k \geq 5$, or by $\{m, m-1, \ldots, k+$

$1, k+1, k-1, k-2, \ldots, 2, 1\}$. In this case, we can recursively work our way back to $\{m+1, m-1, m-2, \ldots, 2, 1\}$, which is not in the right chain since it would have to be preceded by $\{m+2, m-2, m-2, \ldots, 2, 1\}$ or $\{m+1, m, m-3, \ldots, 2, 1\}$, both of which violate Proposition 1. □

Now apply Lemma 2 to extend the left chain safely to $\{m, m-1, \ldots, 5, 5, 3, 2, 1\}$, which occurs at level $\frac{m^3+5m-24}{6}$. Since $h(P_n) = \frac{m^3+2m}{3}$, it suffices if $m^3+5m-24 \geq m^3+2m$, or $m \geq 8$. $m = 7$ also works since $h(P_{29}) = 119$ and we reach level 59. The case $m = 6$, $n = 22$ can be dealt with individually. The left chain gets to $\{6, 5, 5, 3, 2, 1\}$ at level 37, but intersects the right chain at level 38 with $\{6, 5, 4, 4, 2, 1\}$. However, the right chain reaches $\{6, 5, 4, 4, 3\}$ at level 37, which also covers the self-conjugate partition $\{5, 5, 5, 4, 3\}$, so this establishes Theorem 1 when $r = 1$.

If $r = 2$, then we safely reach $\{m+2, m-1, m-2, m-3, \ldots, 2, 1\}$, but in fact we can go further along the left chain.

Lemma 3. *The partitions* $\{m+1, m, m-2, m-3, \ldots, 2, 1\}$ *and* $\{m+1, m-1, m-2, \ldots, k+1, k, k, k-2, \ldots, 2, 1\}$, $1 \leq k \leq m-1$, *do not occur in the right chain.*

Proof: If $\{m+1, m, m-2, m-3, \ldots, 2, 1\}$ occurred in the right chain, then it would have to be preceded by $\{m+2, m-1, m-2, m-3, \ldots, 2, 1\}$, $\{m+1, m+1, m-3, m-3, \ldots, 2, 1\}$, or $\{m+1, m, m-1, m-4, m-4, \ldots, 2, 1\}$, all of which violate Proposition 1. Note that we are tacitly assuming that $m > 4$, but that's fine since $n > 16$, so $m \geq 5$.

Since $\{m+1, m-1, m-2, \ldots, k+1, k, k, k-2, \ldots, 2, 1\}$ has two differences of size 2 for $k > 2$, Proposition 1 takes care of those cases (note $k = m-1$ means the partition is $\{m+1, m-1, m-1, m-3, \ldots, 2, 1\}$). If $\{m+1, m-1, m-2, \ldots, 3, 2, 2\}$ occurred in the right chain, then it would have to be preceded by $\{m+2, m-2, m-2, \ldots, 3, 2, 2\}$ or $\{m+1, m, m-3, \ldots, 3, 2, 2\}$, both of which violate Proposition 1. $k = 1$ is similar. □

Now apply Lemma 3 to extend the left chain safely to $\{m+1, m-1, m-2, \ldots, 2, 1, 1\}$, which occurs at level $\frac{m^3+5m}{6}$. Since $h(P_n) = \frac{m^3+5m}{3}$, this establishes Theorem 1 when $r = 2$.

Finally, if $r = 3$, we safely reach $\{m+2, m-1, m-2, \ldots, 2, 1, 1\}$. Now we just modify Lemma 3. Note we could also show that the right chain has no elements ending in 1,1 until it's too late, but this method is cleaner.

Lemma 4. *The partitions* $\{m+1, m, m-2, \ldots, 2, 1, 1\}$ *and* $\{m+1, m-1, m-2, \ldots, k+1, k, k, k-2, \ldots, 2, 1, 1\}$, $4 \leq k \leq m-1$ *do not occur in the right chain.*

Proof: Exactly the same as Lemma 3, since the second 1 at the end never comes into play. □

Now apply Lemma 4 to extend the left chain safely to $\{m+1, m-1, m-2, \ldots, 5, 4, 4, 2, 1, 1\}$, which occurs at level $\frac{m^3+11m-18}{6}$. Since $h(P_n) = \frac{m^3+8m}{3}$, it suffices if $m^3 + 11m - 18 \geq m^3 + 8m$, or $m \geq 6$. When $m = 5$, we get to level 27, and $h(P_5) = 55$, so this case is fine as well. This establishes Theorem 1 when $r = 3$, and thus completes the proof. □

4 Proof of Theorem 2

At all times in this proof, we assume that n is arbitrarily large. Looking back on it, we'll see we never needed more than $n > 135$.

Once again we wish to construct disjoint chains to the middle level. We use the left and right chains constructed in the proof of Theorem 1, plus a middle chain which will start at $\{n-5, 4, 1\}$. By construction, we can easily tell that some partitions do not occur in the left chain, in analogy with Proposition 1.

Proposition 2. *If $\mu = \{\mu_1, \mu_2, \ldots, \mu_k\}$ is in the left chain, then $\mu_i - \mu_{i+1} \leq 2$ for $i = 2, \ldots, k-1$. In other words, only the first difference can be greater than 2. Moreover, excluding the first difference, μ cannot have more than one difference equal to 2.*

Thus we will try to keep the middle chain safe by keeping the second difference greater than 2, or having two differences equal to 2 somewhere in the middle. Unfortunately, Lemma 1 does not generalize in the most obvious way for finding three chains, due to posets such as the ones shown in Figure 4. Consider the subposet R_n of P_n consisting of the partitions that appear in chains formed from top to bottom by a block of H-steps followed by a block of V-steps (note some steps may be both H-steps and V-steps). This is a subposet of Q_n, and is self-dual under conjugation (which switches H-steps and V-steps) [6]. Let R_n^* denote R_n without the top six and bottom six levels.

Lemma 5. *If R_n^* has at least three elements on every level, then it has three disjoint chains of maximal length.*

Proof: We can show this inductively, as in Lemma 1, if we can show that we do not have any two consecutive levels with connecting relations as shown in Figure 4. The pairs of bold lines indicate that we could have more lines like them without creating a third chain (e.g. one element covering three or four others, rather than just the two shown). Aside from the bold relations, there must be no other partitions in or relations between the two levels shown.

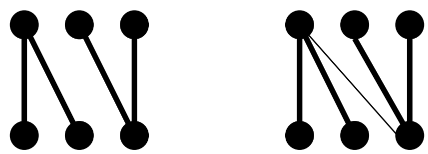

Figure 4: Three elements on every level but no three disjoint chains of maximal length.

Proving that these levels do not arise is extremely tedious, so we will not show all the details here. We'll just do a partial case to show how the general argument works; in particular we show that, with a few exceptions that can be ignored, we cannot get either of the subposets in Figure 5, where there are no other covering relations between these two levels involving α, β, or γ. This will prove that the first scenario shown in Figure 4 does not occur with either part as in Figure 5. By duality, it suffices to show that the one with H-steps does not occur.

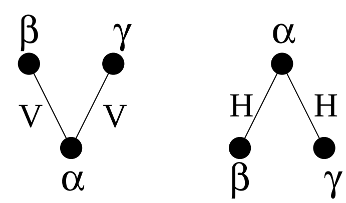

Figure 5: Two special cases.

Since an H-step from α is possible only where α has a difference greater than 1, or last part greater than 1, and since any H-step from α will stay in R_n, α must be of the form $\{a, a-1, \ldots, a-i, b, b-1, \ldots, b-j, c, c-1, \ldots, 1\}$ or $\{a, a-1, \ldots, a-i, b, b-1, \ldots, b-j\}$ where $a-i-b > 1$, $b-j-c > 1$ or $b-j > 1$, and each run ($a, \ldots, a-i$, $b, \ldots, b-j$, and $c, \ldots, 1$) has at most one repeat.

If α is $\{a, a-1, a-1, \ldots\}$, $\{a, a-1, \ldots\}$ (but not $\{a, a-1, a-1, \ldots\}$) or $\{a, a, \ldots\}$, then the partition γ we get by taking the right-most possible H-step is also covered by a partition of the form $\{a, a, a-2, \ldots\}$, $\{a+1, a-2, \ldots\}$, or $\{a+1, a-1, \ldots\}$, respectively, where the $\ldots$ ending matches the end of γ. Thus the first run of α must be just a. Similarly the second run must have just one or two numbers in it, and the third run is just $c = 1$ or empty. Thus α is $\{a, b\}$, $\{a, b, 1\}$, $\{a, b, b\}$, $\{a, b, b-1\}$, $\{a, b, b, 1\}$ or $\{a, b, b-1, 1\}$.

If $\alpha = \{a, 2\}$, then $\beta = \{a-1, 3\}$, $\gamma = \{a, 1, 1\}$, and we have an exception to the claim. This exception can be ignored, though, since it happens in the upper levels of R_n that are not in R_n^*. If $\alpha = \{a, b\}$ where $b > 2$, then $\beta = \{a-1, b+1\}$, $\gamma = \{a, b, b-1\}$, and γ is also covered by $\{a+1, b-1, b-1\}$. Working through the other cases, we similarly find only the isolated exceptions $\alpha = \{5, 3, 1\}$, $\{4, 2, 2\}$, $\{5, 3, 3, 1\}$. These are easily ignored since, for such small values of n, R_n^* does not have at least three elements on every level.

If α covered three or more partitions by H-steps, then the above argument would actually be simplified, thanks to the additional runs and large differences. Similarly, it really does suffice just to rule out the subposets in Figure 4.

A partition μ is H-reachable (i.e. there exists a chain of H-steps from $\{n\}$ to μ) if the parts of μ that come in runs with differences of at most 1 each have at most one repeated part, and that part appears no more than two times. μ is V-reachable (i.e. there exists a chain of V-steps from μ to $\{1^n\}$) if its conjugate is H-reachable, i.e. μ has no differences greater than 2, and any two differences of size 2 must have a repeated part between them [6].

To handle cases involving both H-steps and V-steps, we can use the fact that R_n only has H-steps between H-reachable partitions, and V-steps between V-reachable partitions. Of course, some partitions are both, and as long as $r > 0$ the number of such partitions will grow with m. With sufficient patience and brute force, the rest of the proof is straightforward. □

Note that this lemma will not apply when $r = 0$, but that's ok since the three chains we construct in that case end on self-conjugate partitions, and hence can be extended to their full length by conjugation. We only need Lemma 5 when r and m are both odd, so that $h(P_n) = \frac{m^3-m}{3} + rm$ is odd, i.e. the number of levels of P_n is even.

Just to take the first few steps along the middle chain without intersecting the right chain, we need $n > 14$ so that we can go through $\{n-5, 4, 1\}$, $\{n-6, 5, 1\}$, $\{n-7, 6, 1\}$, $\{n-7, 5, 2\}$, and $\{n-7, 5, 1, 1\}$, after which we just keep the second difference greater than 2 as long as possible. First we deal with the cases where $r > 4$. We safely reach $\{m+r-2, m+1, m-2, \ldots, 2, 1\}$, and continue on to $\{m+2, m+1, m-2, \ldots, r-4, r-4, \ldots, 2, 1\}$ at level $\frac{m^3-m}{6} + 2 + m(r-4) - \binom{r-4}{2}$. We want this to be at least half of $\frac{m^3-m}{3} + rm$, or $m \geq r - 1 + \frac{8}{r-8}$ for $r > 8$. So if $r \geq 16$, then this works for all m (since $m \geq r$ by definition), and for $8 < r < 16$ we just have to exclude finitely many values of m.

Now we can take another step, to $\{m+2, m, m-1, m-3, m-4, \ldots, r-4, r-4, \ldots, 2, 1\}$. This is not in the right chain by Proposition 1, and one must check that it is not in the left chain, but the usual argument of looking at possible predecessors and seeing they all violate Proposition 2 works. From now on, we will say that such a partition is "safe by the usual methods." From there we continue along to $\{m+1, m+1, m-1, m-3, m-4, \ldots, r-4, r-4, \ldots, 2, 1\}$, $\{m+1, m+1, m-2, m-2, m-4, \ldots, r-4, r-4, \ldots, 2, 1\}$, and on down to $\{m+1, m+1, m-2, m-3, m-4, \ldots, r-3, r-3, \ldots, 2, 1\}$. This is $m-r+2$ levels beyond where we last computed. One more step to $\{m+1, m, m-1, m-3, m-4, \ldots, r-3, r-3, \ldots, 2, 1\}$ is safe by the usual methods, and on down to $\{m+1, m, m-2, m-3, m-4, \ldots, r-1, r-1, r-3, r-3, r-4, \ldots, 2, 1\}$ (here we need $r-1 \leq m-2$, or $m > r$), and finally one more to $\{m+1, m-1, m-1, m-3, m-4, \ldots, r-1, r-1, r-3, r-3, r-4, \ldots, 2, 1\}$. That's another $m-r+2$ steps, for a grand total of $\frac{m^3-m}{6} + 2 + m(r-4) - \binom{r-4}{2} + 2m - 2r + 5$. We want this to be at least half of $\frac{m^3-m}{3} + rm$, or $m(r-4) \geq r^2 - 5r + 4$, or $m \geq r-1$ for $r > 4$. Since we already assumed $m > r$, this means the only possible bad cases are where $m = r = 5, 6, \ldots, 15$. Since we only care about large n, we ignore these 11 cases, and we've established Theorem 2 when $r \geq 5$.

For $r = 0$, we safely go through $\{m+1+(m-4), m, m-3, m-4, \ldots, 2, 1\}$, and eventually reach $\{m+1, m, m-3, m-4, m-4, \ldots, 2, 1\}$. One more step to $\{m+1, m-1, m-2, m-4, m-4, \ldots, 2, 1\}$ is safe by the usual methods, and then we take two more steps to reach $\{m, m, m-3, m-3, m-4, \ldots, 2, 1\}$ (note the order of these steps actually matters, we must make the m, m first). Now we're just one step away from the middle level of self-conjugate partitions, so we simply observe that this covers the self-conjugate partition $\{m, m, m-3, m-3, m-4, \ldots, 3, 2, 2, 2\}$, and we're done.

The cases $r = 2, 3, 4$ are each straightforward with lemmas such as those used in proving Theorem 1. The case $r = 1$, however, requires something more clever. The trick we use turns out to give quick proofs for the other three cases as well, so we just use one lemma to settle all four cases.

Lemma 6. *The partitions $\{m, m-1, m-2, m-3, \ldots, r+5, r+4, r+4, r+3, \ldots, 4, 1, 1\}$, $r = 1, 2, 3, 4$, can be reached safely on the middle chain.*

Proof: We certainly reach $\{m+1+(m+r-4), m, m-3, m-4, \ldots, 2, 1\}$ safely. Now the trick is to move the difference of 3 to the right. This will make finding a fourth disjoint chain much more difficult, but fortunately we're only trying to construct three chains. First we continue as before to $\{m+1+(m+r-5), m, m-3, m-4, \ldots, 2, 1, 1\}$. Then it's on to $\{m+1+(m+$

$r-6), m, m-3, m-4, \ldots, 3, 3, 1, 1\}$ (do not proceed to $\{\ldots, 3, 2, 2, 1\}$), and eventually $\{m+1+r, m, m-3, m-3, m-4, \ldots, 4, 3, 1, 1\}$. From there we go to $\{m+r, m+1, m-3, m-3, m-4, \ldots, 4, 3, 1, 1\}$, then $\{m+r, m, m-2, m-3, m-4, \ldots, 4, 3, 1, 1\}$, then $\{m+r, m-1, m-1, m-3, m-4, \ldots, 4, 3, 1, 1\}$ (safe thanks to two differences of size 2 in the middle), and continue along until reaching $\{m+r, m-1, m-2, m-3, \ldots, 4, 4, 1, 1\}$, and finally on to $\{m, m-1, m-2, m-3, \ldots, r+5, r+4, r+4, r+3, \ldots, 4, 1, 1\}$. $\square$

For $r = 1$, we reach $\{m, m-1, m-2, m-3, \ldots, 5, 5, 4, 1, 1\}$ by Lemma 6 at level $\frac{m^3+5m}{6} - 5$, which is at least the halfway point, namely $\frac{m^3+2m}{6}$, for $m \geq 10$.

For $r = 2$, we reach $\{m, m-1, m-2, m-3, \ldots, 6, 6, 5, 4, 1, 1\}$ by Lemma 6 at level $\frac{m^3+11m}{6} - 10$, which is at least the halfway point, namely $\frac{m^3+5m}{6}$, for $m \geq 10$.

For $r = 3$, we reach $\{m, m-1, m-2, m-3, \ldots, 7, 7, 6, 5, 4, 1, 1\}$ by Lemma 6 at level $\frac{m^3+17m}{6} - 16$, which is at least the halfway point, namely $\frac{m^3+8m}{6}$, for $m \geq 11$.

For $r = 4$, we reach $\{m, m-1, m-2, m-3, \ldots, 8, 8, 7, 6, 5, 4, 1, 1\}$ by Lemma 6 at level $\frac{m^3+23m}{6} - 23$, which is at least the halfway point, namely $\frac{m^3+11m}{6}$, for $m \geq 12$. Other arguments can handle this case for $m \geq 6$, but even $m = 12$, $r = 4$ gives us $n = 76 < 136$.

The largest case we did not settle was $m = r = 15$, or $n = 135$. This completes the proof of Theorem 2. $\square$

5 Smaller cases and related questions

The smaller n for which $\lambda_2(P_n) = \lambda_1(P_n) - 6$ are 10, 13, 14, and 15. Figure 6 shows P_{13} and Q_{13}, to give some idea of what's going on. In fact, $R_{13} = Q_{13}$, though this equality does not hold in general. For example, $\{5, 2, 1, 1, 1\}$ is in Q_{10} but not in R_{10}. Figure 7 shows Q_{16}. Since there are levels of size 1 in the middle, P_{16} cannot possibly have two chains of the desired lengths. Due to the size of the posets we are working with, we do not attempt to classify all n for which three chains of the desired lengths exist.

More generally, Table 1 shows the partitions of chain lengths for P_n, $1 \leq n \leq 14$. It is interesting to note that in all of these cases, the elements added between $\lambda_{k-1}(P_n)$ and $\lambda_k(P_n)$ form a chain that is added to the previous $k-1$ chains (and similarly for antichains). This is not the case for arbitrary

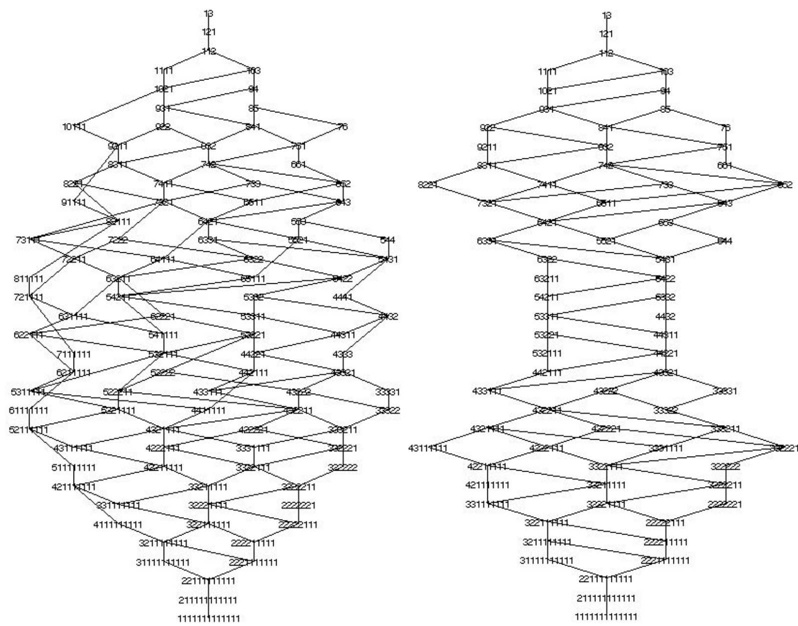

Figure 6: P_{13} (left) and Q_{13} (right).

posets, such as Figure 8. The proofs of Theorems 1 and 2 show that this is the case for every P_n when $k = 2$ or 3; it would be interesting to know if it holds for all k.

While the proof of Theorem 1 is constructive in the cases where $h(P_n)$ is even, so that the middle level consists of self-conjugate partitions, it is not constructive when $h(P_n)$ is odd, since in those cases the proof relies on Lemma 1. It would be interesting to give an explicit construction of two long chains in those cases, and similarly for three or more chains.

Note that each chain constructed so far was guaranteed to be disjoint from the others by where it had differences greater than 1. We can thus hope to construct arbitrarily many disjoint chains to the middle level for large n, though of course the argument grows more technically difficult with each chain. This idea motivates the following conjecture.

Conjecture 1. *For large n, $\lambda_i(P_n) - \lambda_{i+1}(P_n)$ depends only on i.*

Note that $\lambda_i(P_n) - \lambda_{i+1}(P_n)$ need not always be 6. It appears that the fourth chain starts just one level further down, so we conjecture that $\lambda_3(P_n) - \lambda_4(P_n) = 2$ for large n. Indeed, we can show this for large r by taking a

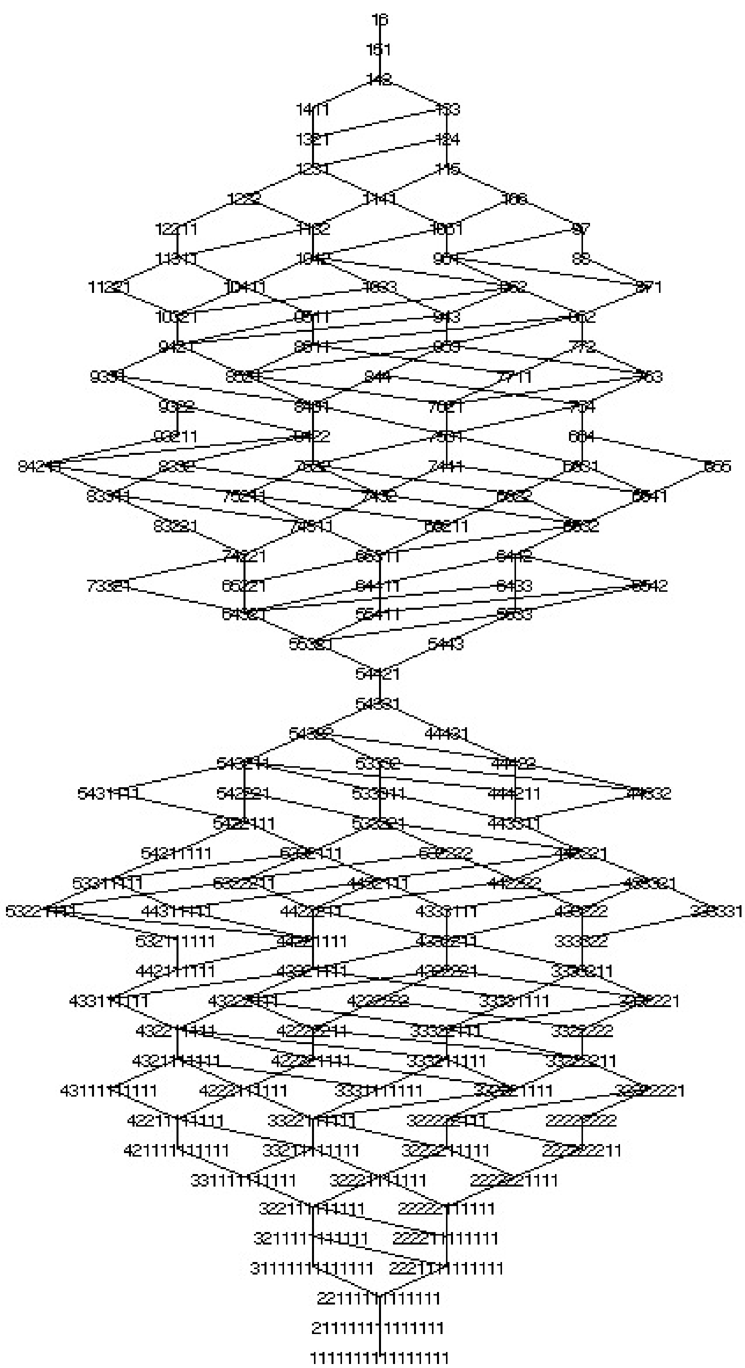

Figure 7: Elements of P_{16} on maximal chains.

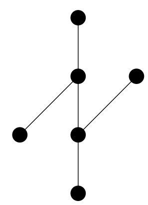

Figure 8: A poset P such that the largest chain is not one of the largest two chains. $\lambda(P) = \{4, 2\}$.

n	$\lambda(P_n)$
1	$\{1\}$
2	$\{2\}$
3	$\{3\}$
4	$\{5\}$
5	$\{7\}$
6	$\{9, 2\}$
7	$\{12, 3\}$
8	$\{15, 7\}$
9	$\{18, 9, 3\}$
10	$\{21, 15, 4, 2\}$
11	$\{25, 18, 10, 3\}$
12	$\{29, 21, 13, 10, 4\}$
13	$\{33, 27, 18, 14, 6, 3\}$
14	$\{37, 31, 24, 19, 15, 6, 3\}$

Table 1: Known values of $\lambda(P_n)$.

chain of partitions where the third difference is greater than 2, the technical difficulties arise when r is small.

Let M be the transition matrix from the bases $\{e_\mu\}$ to $\{m_{\mu'}\}$ of homogeneous symmetric functions of degree n. Since $M_{\mu\nu} > 0$ iff $\nu \le \mu'$, it is a theorem of Gansner and Saks (independently) that a generic matrix with the same 0 entries will have jordan blocks whose sizes are exactly the parts of $\lambda(P_n)$ (see [1, 3, 7]). Using Table 1 and Maple, one can verify that M is sufficiently generic at least for $n \le 13$.

Another open problem is to find the size $a(n)$ of the largest antichain in P_n. Let $p(n)$ be the number of partitions of n. There is the obvious upper bound $a(n) \le p(n)$. By Dilworth's theorem, $a(n) \ge p(n)/(h(P_n)+1)$, so we have $\Omega(n^{-5/2}e^{\pi\sqrt{2n/3}}) \le a(n) \le O(n^{-1}e^{\pi\sqrt{2n/3}})$. It would be interesting to find a constructive proof that $a(n)$ is at least as large as the lower bound. In addition to the values of $a(n)$ implied by Table 1, we can see that $a(15) = 9$. Moreover, $\lambda_9(P_{15}) = 2$, with the long antichains being $71^8, 6221^5, 541^6, 53221^3, 52^5, 4431^4, 442221, 433311, 3^5$ and their conjugates. One can also verify that $a(16) = 10$, with $\lambda_{10}(P_{16}) = 5$. The sequence of $a(n)$'s is number A076269 in [8].

One construction that shows $a(n)$ has a lower bound of the form $e^{c\sqrt{n}}$ is as follows. Begin with the antichain 7321^4, 722221, 651^5, 642211, 63322, 553111, 55222, 54421, 4444 in P_{16}. Let $\nu + 7n$ denote a partition ν from the list with $7n$ added to each part. Consider ν to have 7 parts, so some of them might be 0. Then $\{\nu + 7n, \nu + 7(n-1), \ldots, \nu + 7, \nu\}$ is a partition of $N = 16(n+1) + 49\frac{n^2+n}{2} = \frac{49}{2}n^2 + O(n)$. There are 9^{n+1} choices for the ν's, yielding an antichain of size 9^{n+1} in P_N. This yields a lower bound for $a(n)$ of $e^{c\sqrt{n}}$ where $c = \ln 9\sqrt{2/49} = 0.4439\ldots$. By starting with a 28-element antichain in P_{27} where each ν has at most 9 parts, and largest part at most 8, one can similarly get $c = \frac{\ln 28}{6} = 0.555\ldots$. This is still a long way from $\pi\sqrt{2/3} = 2.565\ldots$, but at least it's constructive.

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