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An Interesting Lemma for Regular C-fractions

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Abstract

In this short note we give an interesting lemma for regular C-fractions. Applying this lemma we obtain some congruence properties of some classical numbers such as the Springer numbers of even index, the median Euler numbers, the median Genocchi numbers, and the tangent numbers.

1 The interesting lemma

A regular C-fraction is a continued fraction of the form

$$\begin{aligned} a_0 + \mathbf{K}_{n=1}^{\infty}(a_n z/1) &= a_0 + \frac{a_1 z}{1} {}_{+} \frac{a_2 z}{1} {}_{+} \frac{a_3 z}{1} {}_{+} \cdots \\ &= a_0 + \cfrac{a_1 z}{1 + \cfrac{a_2 z}{1 + \cfrac{a_3 z}{\ddots}}}, \end{aligned}$$

where $a_n \in \mathbb{C}$.

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathbb{C}[[z]]$ be a formal power series. It is known that there exists a one-to-one correspondence between regular C-fractions $a_0 + \mathbf{K}_{n=1}^{\infty}(a_n z/1)$ and formal power series $\sum_{n=0}^{\infty} c_n z^n$ [6, pp. 252–265].

Now we assume that all coefficients are integral. The lemma we state here gives the division relation between the integral coefficients of the regular C-fraction and the integral coefficients of its corresponding formal power series.

Lemma 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{Z}[[z]]$ be an integral formal power series. Assume the corresponding uniquely determined regular C-fraction is*

$$\sum_{n=0}^{\infty} a_n z^n = \frac{b_0}{1+}\,\frac{bb_1 z}{1}\,\frac{}{+}\,\frac{bb_2 z}{1}\,\frac{}{+}\cdots, \tag{1}$$

where b and $(b_n)_{n\geq 0}$ are integral. Then a_n is divisible by $(b_0 b_1 b^n)$ for $n \geq 1$.

Proof. Setting $z = y/b$, Equation (1) becomes

$$\begin{aligned}\sum_{n=0}^{\infty} a_n \left(\frac{y}{b}\right)^n &= \frac{b_0}{1+}\,\frac{b_1 y}{1+}\,\frac{b_2 y}{1+}\,\frac{b_3 y}{1+}\cdots \\ &= b_0 - \frac{b_0 b_1 y}{1+b_1 y+}\,\frac{b_2 y}{1+}\,\frac{b_3 y}{1+}\cdots.\end{aligned}$$

Since $a_0 = b_0$, we have

$$\sum_{n=1}^{\infty} \frac{a_n}{b_0 b_1 b^n} y^n = \frac{-y}{1+b_1 y+}\,\frac{b_2 y}{1+}\,\frac{b_3 y}{1+}\cdots.$$

Since the right-hand side of the above identity can be uniquely expressed as a formal power series with integral coefficients, we conclude the proof. □

Let $f(t) = \sum_n a_n t^n$ and $g(t) = \sum_n b_n t^n$ $(n \geq 0)$ be two formal power series with integral coefficients. For a non-negative integer m we write

$$f(t) \equiv g(t) \pmod{m} \quad \text{iff} \quad a_n \equiv b_n \pmod{m} \quad \text{for all } n \geq 0. \tag{2}$$

Applying Lemma 1 we can obtain some congruence properties of some classical numbers such as the Springer numbers of even index, the median Euler numbers, the median Genocchi numbers, and the tangent numbers.

2 Applications

The Springer numbers ([1, p. 275]) are defined by

$$S(x) = e^x \operatorname{sech} 2x = \sum_{n=0}^{\infty} \frac{S_n x^n}{n!}. \tag{3}$$

The even (resp. odd) part of the Springer numbers is what Glaisher ([1, p. 276]) called the numbers P_n (resp. Q_n). That is to say,

$$\frac{\cosh x}{\cosh 2x} = \sum_{n=0}^{\infty} \frac{S_{2n} x^{2n}}{(2n)!}, \qquad \frac{\sinh x}{\cosh 2x} = \sum_{n=0}^{\infty} \frac{S_{2n+1} x^{2n+1}}{(2n+1)!}. \tag{4}$$

Springer introduced these numbers for a problem about root systems, and Arnold showed these numbers as counting various types of snakes ([4, p. 6–p. 7]).

Following the notation and the result in Corollary 3.3 of [1] we put

$$\begin{aligned} p(x) &= \sum_{n=0}^{\infty} S_{2n} x^{2n+1} = x - 3x^3 + 57x^5 - \dots \\ &= \frac{x}{1+}\,\frac{3x^2}{1}\,\frac{}{+}\,\frac{16x^2}{1}\,\frac{}{+}\,\frac{35x^2}{1}\,\frac{}{+}\cdots \\ &\quad + \frac{16n^2x^2}{1}\,\frac{}{+}\,\frac{(4n+1)(4n+3)x^2}{1}\,\frac{}{+}\cdots. \end{aligned} \tag{5}$$

Note that our definition of the Springer numbers S_{2n} differs from that in [1]. The unsigned sequence $(-1)^n S_{2n}$: 1, 3, 57, 2763, 250737, ..., is the sequence A000281 in [7]. Applying Lemma 1 we have S_{2n} is divisible by 3. Moreover, we have the following theorem.

Theorem 1. *For $n \geq 1$, the Springer number with even index S_{2n} is divisible by 3 and*

$$\frac{S_{2n}}{3} \equiv (-1)^n 3^{n-1} \pmod{16}. \tag{6}$$

Proof. Multiplying x into $p(x)$ and setting $t = x^2$, we have

$$\sum_{n=0}^{\infty} S_{2n} t^{n+1} = t - 3t^2 + 57t^3 - \dots = \frac{t}{1+}\,\frac{3t}{1+}\,\frac{16t}{1}\,\frac{}{+}\cdots.$$

Applying Lemma 1, S_{2n} is divisible by 3 for $n \geq 1$. And

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{S_{2n}}{3} t^{n+1} &= \frac{-t^2}{1+3t+}\,\frac{16t}{1}\,\frac{}{+}\,\frac{35t}{1}\,\frac{}{+}\cdots \\ &\equiv \frac{-t^2}{1+3t} \pmod{16} \\ &= \sum_{n=1}^{\infty} (-1)^n 3^{n-1} t^{n+1}. \end{aligned} \tag{7}$$

Comparing the coefficients of t^{n+1}, we have

$$\frac{S_{2n}}{3} \equiv (-1)^n 3^{n-1} \pmod{16}, \qquad n \geq 1.$$

□

Remark 1. Now we write Equation (7) as

$$\frac{-t^2}{1+3t+}\,\frac{16t}{1}\,\frac{}{+}\,\frac{35t}{1}\,\frac{}{+}\cdots = \frac{-t^2}{1+3t+}\,\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{c_n t}{1}\right),$$

where $c_{2n-1} = 16n^2$ and $c_n = (4n+1)(4n+3)$, for $n \geq 1$.

If we take the modulus $c_2 = 35$ instead of $c_1 = 16$ for Equation (7) in the above proof. Then we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{S_{2n}}{3} t^{n+1} &\equiv \frac{-t^2}{1+19t} \qquad \pmod{35} \\ &\equiv \frac{-t^2}{1-16t} \qquad \pmod{35} \\ &= \sum_{n=1}^{\infty} (-16^{n-1}) t^{n+1}. \end{aligned}$$

Comparing the coefficients of t^{n+1}, we have

$$\frac{S_{2n}}{3} \equiv -16^{n-1} \qquad \pmod{35}, \qquad n \geq 1. \tag{8}$$

Since $16^3 \equiv 1 \pmod{35}$, we can also write Equation (8) as follows: for $k \geq 1$,

$$\frac{S_{2n}}{3} \equiv \begin{cases} 34 & \pmod{35}, \quad \text{if } n = 3k-2; \\ 19 & \pmod{35}, \quad \text{if } n = 3k-1; \\ 24 & \pmod{35}, \quad \text{if } n = 3k. \end{cases} \tag{9}$$

Similarly, we take another c_n as the modulus for Equation (7), then we can get the congruences for $S_{2n}/3$ under the modulus c_n. $\square$

Let us define the Euler numbers E_n through the exponential generating function $E(x)$:

$$E(x) = \operatorname{sech} x + \tanh x = \sum_{n=0}^{\infty} \frac{E_n x^n}{n!}.$$

We construct the Seidel matrix $(a_{n,m})_{n,m\geq 0}$ associated with the sequence $(0, E_1, E_2, E_3, \ldots)$ as follows:

1. The first row $(a_{0,n})_{n\geq 0}$ of the matrix is the initial sequence $(0,\ E_1,\ E_2,\ E_3,\ \ldots)$.

2. Each entry $a_{n,m}$ of the n-th row is the sum of the entry immediately above and of the entry above and to the right of it:

$$a_{n,m} = a_{n-1,m} + a_{n-1,m+1}.$$

The resulting Seidel matrix is

$$\begin{array}{rrrrrrrr} 0 & 1 & -1 & -2 & 5 & 16 & -61 & \cdots \\ 1 & 0 & -3 & 3 & 21 & -45 & \cdots & \\ 1 & -3 & 0 & 24 & -24 & \cdots & & \\ -2 & -3 & 24 & 0 & \cdots & & & \\ -5 & 21 & 24 & \cdots & & & & \\ 16 & 45 & \cdots & & & & & \\ 61 & \cdots & & & & & & \\ \cdots & & & & & & & \end{array}$$

The absolute values of the upper diagonal sequence 1, 3, 24, 402, ... are called the median Euler numbers R_n (see [1, Section 4] or [7, Sequence A002832]). Using the same method as above, we have

Theorem 2. *For $n \geq 1$, the median Euler number R_n is divisible by 3 and*

$$\frac{R_n}{3} \equiv 3^{n-1} \pmod 5. \tag{10}$$

Proof. Since the ordinary generating function of the median Euler numbers R_n satisfies the continued fraction representation [1, Proposition 7]:

$$\begin{aligned} r(x) &= \sum_{n=0}^{\infty}(-1)^n R_n x^{n+1} = x - 3x^2 + 24x^3 - 402x^4 + 11616x^5 - \cdots \\ &= \frac{x}{1+}\,\frac{3x}{1+}\,\frac{5x}{1+}\,\frac{2\cdot 7x}{1+}\,\frac{2\cdot 9x}{1+}\cdots. \end{aligned} \tag{11}$$

Applying Lemma 1, R_n is divisible by 3 for $n \geq 1$. And

$$\begin{aligned} \sum_{n=1}^{\infty}(-1)^n \frac{R_n}{3} x^{n+1} &= \frac{-x^2}{1+3x+}\,\frac{5x}{1+}\,\frac{14x}{1+}\cdots \\ &\equiv \frac{-x^2}{1+3x} \pmod 5 \\ &= \sum_{n=1}^{\infty}(-1)^n 3^{n-1} x^{n+1}. \end{aligned}$$

Comparing the coefficients of x^{n+1}, we complete the proof. □

The Genocchi numbers G_n [7, Sequence A036968] are defined by

$$\frac{2x}{e^x+1} = \sum_{n=0}^{\infty} \frac{G_n x^n}{n!}.$$

The median Genocchi numbers H_{2n+1} (see [1, 2], or [7, Sequence A005439]) can be defined by $H_1 = 1$ and

$$H_{2n+1} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} G_{2n-2k}, \qquad n \geq 1,$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.

Theorem 3. *For $n \geq 1$, the median Genocchi number H_{2n+3} is divisible by 2^n and*

$$\frac{H_{2n+3}}{2^n} \equiv \begin{cases} 1 \pmod 6, & \text{if } n \text{ is odd;} \\ 4 \pmod 6, & \text{if } n \text{ is even.} \end{cases} \tag{12}$$

Proof. Since the ordinary generating function of the median Genocchi numbers H_{2n+1} satisfies the continued fraction representation [1, p. 295]

$$
\begin{aligned}
h(x) &= \sum_{n=0}^{\infty} H_{2n+1}x^{n+1} = x - x^2 + 2x^3 - 8x^4 + 56x^5 - \cdots \\
&= \frac{x}{1+}\,\frac{x}{1+}\,\frac{x}{1+}\,\frac{2^2x}{1}\,{+}\,\frac{2^2x}{1}\,{+}\,\frac{3^2x}{1}\,{+}\,\frac{3^2x}{1}\,{+}\,\cdots. \qquad (13)
\end{aligned}
$$

From [1, Lemma 1] we have

$$
\begin{aligned}
&\frac{x}{1+}\,\frac{c_1x}{1}\,{+}\,\frac{c_2x}{1}\,{+}\,\frac{c_3x}{1}\,{+}\,\cdots \\
&= x - \frac{c_1x^2}{1+(c_1+c_2)x-}\,\frac{c_2c_3x^2}{1+(c_3+c_4)x-}\,\frac{c_4c_5x^2}{1+(c_5+c_6)x-}\,\cdots \qquad (14)\\
&= \frac{x}{1+c_1x-}\,\frac{c_1c_2x^2}{1+(c_2+c_3)x-}\,\frac{c_3c_4x^2}{1+(c_4+c_5)x-}\,\cdots. \qquad (15)
\end{aligned}
$$

Then we can rewrite the continued fraction representation of $h(x)$ as

$$
h(x) = x - \frac{x^2}{1+2x-}\,\frac{2^2x^2}{1+2\cdot 2^2x-}\,\frac{2^2\cdot 3^2\cdot x^2}{1+2\cdot 3^2x-}\,\cdots\,{-}\,\frac{n^2(n+1)^2x^2}{1+2\cdot(n+1)^2x-}\,\cdots.
$$

Hence

$$
-\sum_{n=1}^{\infty} H_{2n+1}x^n = \frac{x}{1+2x-}\,\frac{2^2\cdot x^2}{1+2\cdot 2^2x-}\,\cdots.
$$

Now we apply Equation (15), and transform the above equation to

$$
-\sum_{n=0}^{\infty} H_{2n+3}x^{n+1} = \mathop{\mathbf{K}}_{n=0}^{\infty}\left(\frac{c_nx}{1}\right),
$$

where $c_0 = 1$, $c_{2n-1} = c_{2n} = n(n+1)$, for $n \geq 1$.

Applying Lemma 1, H_{2n+3} is divisible by 2^n for $n \geq 1$, and

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_{2n+3}}{2^n}x^n &= \frac{x}{1+x+}\,\frac{x}{1+}\,\frac{3x}{1}\,{+}\,\frac{3x}{1}\,{+}\,\frac{6x}{1}\,{+}\,\frac{6x}{1}\,{+}\,\cdots \qquad (16)\\
&\equiv \frac{x}{1+x+}\,\frac{x}{1+}\,\frac{3x}{1+3x} \pmod 6\\
&\equiv \frac{x}{3x^2+2x+1} \pmod 6\\
&\equiv \frac{x}{3x^2-4x+1} \pmod 6\\
&= \frac{x}{(3x-1)(x-1)} = \frac{1}{2}\cdot\frac{1}{1-3x} - \frac{1}{2}\cdot\frac{1}{1-x}\\
&= \sum_{n=0}^{\infty}\left(\frac{3^n-1}{2}\right)x^n.
\end{aligned}
$$

Comparing the coefficients of x^n, we have

$$\begin{aligned}\frac{H_{2n+3}}{2^n} &\equiv \frac{3^n-1}{2} \pmod 6\\ &= 3^{n-1}+3^{n-2}+\cdots+3+3^0.\end{aligned}$$

Since $3^n \equiv 3 \pmod 6$, for $n \geq 1$, we have

$$\frac{H_{2n+3}}{2^n} \equiv (n-1)\cdot 3+1 \equiv 3n-2 \pmod 6. \tag{17}$$

If $n = 2k-1$, for $k \geq 1$, then

$$\frac{H_{2n+3}}{2^n} \equiv 3(2k-1)-2 \equiv 1 \pmod 6.$$

If $n = 2k$, for $k \geq 1$, then

$$\frac{H_{2n+3}}{2^n} \equiv 3(2k)-2 \equiv 4 \pmod 6.$$

Hence we complete our proof. □

Using the similar method, we could get Barsky's result ([2, Theorem 1]): for $n \geq 1$,

$$\frac{H_{2n+3}}{2^n} \equiv \begin{cases} 3 \pmod 4, & \text{if } n \text{ is odd;} \\ 2 \pmod 4, & \text{if } n \text{ is even.} \end{cases} \tag{18}$$

The tangent numbers T_n are defined by

$$1+\tanh x = \sum_{n=0}^{\infty} \frac{T_n x^n}{n!}.$$

The unsign tangent numbers are the sequence [7, Sequence A009006]. The tangent numbers T_n are closely related to the Bernoulli numbers:

$$T_{2n-1} = 2^{2n}(2^{2n}-1)B_{2n}/2n. \tag{19}$$

Theorem 4. *For $n \geq 1$, the tangent number T_{2n+1} is divisible by 2^n and*

$$\frac{T_{2n+1}}{2^n} \equiv (-1)^n 4^{n-1} \pmod 6. \tag{20}$$

Proof. We use the classical continued fraction representation for the ordinary generating function of the tangent numbers T_n [1, Corollary 3.1]

$$\begin{aligned}\sum_{n=0}^{\infty} T_n x^{n+1} &= x + x^2 - 2x^4 + 16x^6 - 272x^8 + \dots \\ &= x + \frac{x^2}{1+}\,\frac{2x^2}{1+}\,\frac{6x^2}{1+}\cdots\frac{n(n+1)x^2}{1+}\cdots.\end{aligned} \tag{21}$$

Changing the variable x^2 as t we have

$$t + \sum_{n=1}^{\infty} T_{2n+1} t^{n+1} = \frac{t}{1+}\,\frac{2t}{1+}\,\frac{6t}{1+}\cdots\frac{n(n+1)t}{1}\,+\cdots.$$

Applying Lemma 1, T_{2n+1} is divisible by 2^n for $n \geq 1$. And

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{T_{2n+1}}{2^n} t^{n+1} &= \frac{-t^2}{1+t+}\,\frac{3t}{1+}\,\frac{6t}{1+}\cdots \\ &\equiv \frac{-t^2}{1+t+3t} \qquad \pmod 6 \\ &= \sum_{n=1}^{\infty} (-1)^n 4^{n-1} t^{n+1}.\end{aligned}$$

Comparing the coefficients of t^{n+1}, we complete the proof. □

The result that T_{2n+1} is divisible by 2^n is not new. Howard [5, Theorem 8] proved in an elementary way that for every $n \geq 1$ the number $(2^{n+1}(1-2^{2n})/2n)B_{2n}$ is an integer. That is to say, T_{2n-1} is divisible by 2^{n-1}. Ramanujan (see [3, p. 7]) proved some similar congruence properties, such as

$$\frac{2(2^{4n+2}-1)}{2n+1} B_{4n+2}, \quad \text{and} \quad \frac{-2(2^{8n+4}-1)}{2n+1} B_{8n+4}$$

are integers of the form $30k+1$, for $n \geq 0$. And it means that T_{4n+1}, T_{8n+3} are divisible by 2^{4n}, 2^{8n+1}, respectively, and

$$\frac{T_{4n+1}}{2^{4n}} \equiv \frac{-T_{8n+3}}{2^{8n+1}} \equiv 1 \pmod{30}.$$

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