Partition Coefficients of Acyclic Graphs *

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Abstract. We develop the concept of a "closure space" which appears with different names in many aspects of graph theory. We show that acyclic graphs can be almost characterized by the partition coefficients of their associated closure spaces. The resulting nearly total ordering of all acyclic graphs (or partial orders) provides an effective isomorphism filter and the basis for efficient retrieval in secondary storage.

1 Binary Partitions

In this paper we combine two mathematical threads and apply them in a graph-theoretic context. The first thread of binary partitions was studied by Euler as early as 1750. The second thread involving closure spaces is of more recent origin. A binary partition of a positive integer N is its expression as a sum of powers of 2. Mahler [16], and Churchhouse [3] [4] have studied binary partitions from a number theoretic point of view. Because our intention is to connect these partitions with closure spaces, we will confine our attention to the special case where N is also a power of 2.

By a *binary partition* of 2^n we mean a sequence of non-negative integers $< \cdots, a_k \cdots >$, $0 \leq k \leq n$ such that

$$a_n \cdot 2^n + a_{n-1} \cdot 2^{n-1} + a_{n-2} \cdot 2^{n-2} + \cdots + a_1 \cdot 2^1 + a_0 \cdot 2^0 = 2^n \qquad (1)$$

or $\sum_{k=0}^{n} a_k \cdot 2^k = 2^n$. The set of all such partitions we denote by $\mathcal{P}^n$. (From now on we frequently omit the adjective "binary".)

Several characteristics of (1) are readily apparent. First, $a_n \neq 0$ if and only if $a_k = 0$ for all $0 \leq k < n$. Second, since the right hand side is even and all terms $a_k \cdot 2^k$, $k > 0$ must be even, the coefficient a_0 must be even. Third, if $< \cdots, a_k, a_{k-1}, \cdots >$ is a partition, then $< \cdots, a_k - 1, a_{k-1} + 2, \cdots >$ must be as well. And fourth, if $< a_n, \cdots, a_k, \cdots, a_o >$ is a partition of 2^n then $< a_n, \cdots, a_k, \cdots, a_0, 0 >$ is a partition of 2^{n+1}.

With these observations, it is not difficult to write a process which generates all partitions in lexicographic order. Doing so, and displaying each partition, generates the following enumerations of $\mathcal{P}^3$ and $\mathcal{P}^4$. It is quite easy to verify by inspection that each sequence is a partition of 2^n. And because they are in lexicographic order, one can verify that all possible partitions have been generated.

If one were to run the same program with $n = 5$ there would be 202 generated partitions which are are impractical to display in a paper of this length.

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n = 3

1	0	0	0
0	2	0	0
0	1	2	0
0	1	1	2
0	1	0	4
0	0	4	0
0	0	3	2
0	0	2	4
0	0	1	6
0	0	0	8

n = 4

1	0	0	0	0
0	2	0	0	0
0	1	2	0	0
0	1	1	2	0
0	1	1	1	2
0	1	1	0	4
0	1	0	4	0
0	1	0	3	2
0	1	0	2	4
0	1	0	1	6
0	1	0	0	8
0	0	4	0	0
0	0	3	2	0
0	0	3	1	2
0	0	3	0	4
0	0	2	4	0
0	0	2	3	2
0	0	2	2	4

0	0	2	1	6
0	0	2	0	8
0	0	1	6	0
0	0	1	5	2
0	0	1	4	4
0	0	1	3	6
0	0	1	2	8
0	0	1	1	10
0	0	1	0	12
0	0	0	8	0
0	0	0	7	2
0	0	0	6	4
0	0	0	5	6
0	0	0	4	8
0	0	0	3	10
0	0	0	2	12
0	0	0	1	14
0	0	0	0	16

Fig. 1. $\mathcal{P}^3$ and $\mathcal{P}^4$

2 Closure Spaces

The preceding discussion of binary partitions will take on additional interest if we introduce the concept of a closure space. We let $\mathbf{U}$ denote some *universe* of elements of interest. Lower case letters $a, b, \cdots, x, y, z$ will denote individual elements of $\mathbf{U}$, and upper case letters will denote subsets. A set, $\mathbf{U}$, and a closure operator, φ, satisfying the following three closure axioms[2]

$$\begin{aligned} X &\subseteq X.\varphi \\ X &\subseteq Y \text{ implies } X.\varphi \subseteq Y.\varphi \\ X.\varphi.\varphi &= X.\varphi^2 = X.\varphi \end{aligned} \tag{2}$$

are said to be a *closure space* $(\mathbf{U}, \varphi)$, as in [12]. X is said to be *closed*[3] if $X.\varphi = X$. A closure operator, φ, is said to be *uniquely generated* if it also satisfies the following fourth axiom, which serves to distinguish it from a topological closure,

$$X.\varphi = Y.\varphi \text{ implies } (X \cap Y).\varphi = X.\varphi = Y.\varphi \tag{3}$$

Closure operators satisfying (3) above are uniquely generated in the sense that for any set Z, there exists a unique minimal set $X \subseteq Z$, called its *generator* and denoted $Z.gen$, such that $X.\varphi = Z.\varphi$.[4] The importance of uniquely generated

[2] We will write these expressions using the mixed infix/suffix form more common in algebra. That is, binary set operators will be written using infix and unary transformations will be written using suffix notation, as in $(X \cap Y).f$ to denote the image of $X \cap Y$ under f. This notation greatly simplifies expressions involving transformations of closure spaces; and the redundant dot delimiter is of great value when using computer parsing techniques.

[3] The family $\mathcal{C}$ of closed sets is closed under intersection, and this characterization is equivalent to (2), *c.f.* [9].

[4] Readily, if X_1 and X_2 were distinct minimal generators of $Z.\varphi$, then because $X_1.\varphi = X_2.\varphi = Z.\varphi$, we must have, by (3), $(X_1 \cap X_2).\varphi = Z.\varphi$ contradicting minimality.

closure spaces lies in the fact that in discrete systems they play a role that is in many respects analogous to the vector spaces of classical mathematics. We establish this parallel in the next paragraph.

A closure operator σ, satisfying the three closure axioms of (2), together with the Steinitz-MacLane *exchange* property

$$\text{if } \; y \notin X.\sigma \; \text{ then } \; y \in (X \cup \{x\}).\sigma \; \text{ implies } \; x \in (X \cup \{y\}).\sigma \tag{4}$$

can be shown to be the closure operator of a matroid [25] [26]. Recall that a *matroid* is a set system which generalizes the independent sets of a linear algebra, and a *vector space* is the closure, usually called the *spanning operator*, of one or more of these independent sets. Now (4) has the familiar interpretation: if y is not in the vector subspace spanned by X, but is in the vector space formed by adjoining x as a basis vector, then x must be in the vector space spanned when y is adjoined to X.

Similarly, a closure φ satisfying the three closure axioms and the *anti-exchange* property

$$\text{if } \; x, y \notin X.\varphi \; \text{ then } \; y \in (X \cup \{x\}).\varphi \; \text{ implies } \; x \notin (X \cup \{y\}).\varphi \tag{5}$$

is the closure operator of an anti-matroid [7] [15]. In [21] it is shown that

Theorem 1. *A closure operator is uniquely generated if and only if it satisfies the anti-exchange property (5).*

Therefore, uniquely generated closure spaces are precisely the analogs of vector spaces, but with respect to anti-matroids. From now on, we will simply call them *closure spaces*. Because they are uniquely generated, any closure space is completely characterized by enumerating its closed sets and their generators, that is by enumerating $[X.\varphi, X.gen], \forall X \subseteq \mathbf{U}$.

Closure spaces are fairly common in computer science and its applications, although they frequently have other names. Transitive closure, for example of the set of edges in an acyclic graph or of functional dependencies in an acyclic database schema, gives rise to a closure space. The term "convexity" is often applied to closure concepts, and many examples of convexity concepts occurring in graphs can be found in [8] [11] and [14]. Convexity in discrete geometries also yields a number of intuitively satisfying closure spaces. The convex hull operator is the closure operator. See [10] for an excellent treatment of *convex geometries*. Finally, numerous examples of anti-matroids, whose closure will yield a closure space, can be found in the survey of anti-matroids [7] or the text on *greedoids* [15] which generalize an important class of computer algorithms.

We have found that *ideal* and *interval* operators in partially ordered sets, or acyclic graphs provide an abundance of easily accessible examples. It is not hard to show that the path structure of an acyclic graph is uniquely generated [21]. That is, there is a unique, minimal representation[5] of any acyclic graph which

[5] Minimal in the sense that removal of any edge yields a graph with a different path structure, transitive closure, or partial order. We usually illustrate acyclic relationships with basic representations; they are far less cluttered.

we call a *basic graph* [18]. These are commonly used in the implementation of acyclic data structures and processes.

One can organize the *closed sets* of a closure space in many ways. The most natural is to partially order them by inclusion, in which case it can be shown that the partial order will be a lower semi-modular (or meet-distributive) lattice [17] [9]. A more interesting partial order, $\leq_\varphi$, of *all subsets* is given by

$$X \leq_\varphi Y \qquad \text{if and only if} \qquad Y \cap X.\varphi \subseteq X \subseteq Y.\varphi \qquad \forall X, Y \subseteq \mathbf{U}. \qquad (6)$$

which is described in [21]. The closure space with this partial order can be shown to be a lattice, $\mathcal{L}_{(U,\varphi)}$, called the *closure lattice* of $(\mathbf{U}, \varphi)$. Figure 2 illustrates the

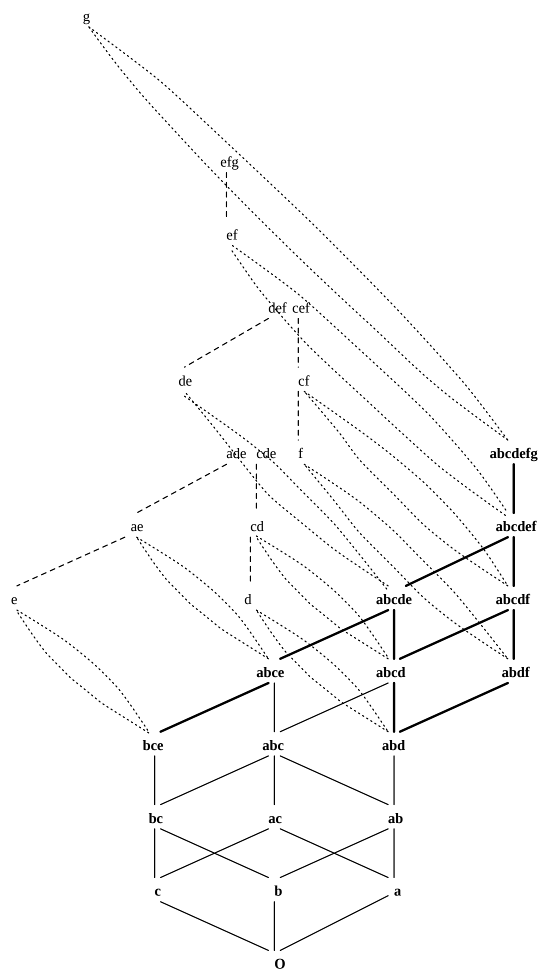

Fig. 2. The closure lattice $\mathcal{L}_{(U,\varphi)}$ of a small 7 point closure space.

structure of a small 7 point closure space. The closed sets of $(\mathbf{U}, \varphi)$ are set in bold face, and connected by solid lines. These closed sets form a sublattice whose partial order is by inclusion. It can be instructive to diagram the points and their set membership of this space. Since $\{g\}$ is the generator of $\mathbf{U} = \{abcdefg\}$, the closure of $\{g\}$, or any set containing the point g, is the whole space. The generator of $\{abce\}$ are the points $\{ae\}$, and so forth. There are 64 subsets whose closure is $\{abcdefg\}$; they constitute the lattice interval $[abcdefg, g]$. To avoid clutter, we simply denote all of them by a single dotted ellipse. Only one of its elements $\{efg\}$ is indicated. (From now on, we also ignore $\{\cdots\}$ delimiting sets of enumerated points.)

Closure lattices such as this have a number of unique properties which are explored in [20]. Central to this development is

Theorem 2. *The poset $\{Y_i | Y.\varphi \leq_\varphi Y_i \leq_\varphi Y.gen\}$, is a boolean algebra on n elements, where $n = |Y.\varphi| - |Y.gen|$.*

These boolean algebras, $[Y.\varphi, Y.gen]$ are denoted by dotted ellipses in Figure 2. It is not hard to see that each lattice interval, $[Y.\varphi, Y.gen] = \{Y_i | Y.gen \subseteq Y_i \subseteq Y.\varphi\}$, and that $|[Y.\varphi, Y.gen]| = 2^n$. Since every subset $Y \subseteq \mathbf{U}$ is an element of some closure/generator interval, the decomposition of $2^{\mathbf{U}}$ into these intervals is a binary partition of $2^{|U|}$, which we call the *partition coefficients* of $(\mathbf{U}, \varphi)$. For example, the binary partition corresponding to the closure space $(\mathbf{U}, \varphi)$ of Figure 2 is $< 0\ 1\ 0\ 1\ 3\ 4\ 0\ 8 >$ (where $a^n = a^7 = 0$ is the leading coefficient). There is a single interval, $[abcdefg, g]$, of size 2^6; so $a_6 = 1$. The three intervals of size 2^3, $[abcde, de]$, $[abcdf, cf]$, and $[abdf, f]$, imply that $a_3 = 3$. There are eight singleton elements, abc, ab, ac, bc, a, b, c and $\emptyset$, where $X.\varphi = X.gen$. Consequently, $a_0 = 8$. Those partitions for which $a_0 \neq 0$ we call *normal*. Customarily the closure of $\emptyset$ is empty[6], even though it is not required in a the general theory of closure spaces. The closure space of Figure 2 is normal.

These partition coefficients constitute an invariant of the closure space that is independent of representation or isomorphic mappings. It is evident from Theorem 2 and the preceding discussion that for every closure space there is a corresponding binary partition of $2^{|U|}$. It can also be shown that for every binary partition of 2^n there exists a closure space on n elements with that [*closed_set, generator*] structure.

3 Partition Coefficients of Acyclic Graphs

In this section, we apply the concept of closure spaces and their binary partition coefficients to the study of acyclic graphs and partially ordered sets.

With any graph one can postulate a number of invariants. They may be any of a variety of scalar quantities, such as covering or independence numbers [13] or various polynomial expressions, *e.g.* chromatic polynomials [1]. It is desirable

[6] That the convex closure of the empty set should be empty is so reasonable, it is taken to be an axiom in [10] and [11].

if the invariant conveys information about the graph. A fairly popular invariant of G is its *characteristic polynomial* [22]. In fact this terminology is slightly misleading. One is really associating the graph G with a linear transformation τ, for which the adjacency matrix of G is a representation. Now, the characteristic polynomial, eigenvalues, and eigenspaces of τ can be regarded as invariants of G [6].

We now do much the same. Given a poset, or acyclic graph $G = (P, E)$, one can use the path relation ρ to induce a partial order on the point set, P. Now we set $\mathbf{U} = P$, and let

$$\begin{aligned} Y.\varphi_L &= \{x | (x, y) \in \rho, \quad y \in Y\}, \\ Y.\varphi_R &= \{z | (y, z) \in \rho, \quad y \in Y\}, \quad \text{or} \\ Y.\varphi_C &= \{x | (y_1, x) \in \rho, (x, y_2) \in \rho, \quad y_1, y_2 \in Y\}. \end{aligned} \tag{7}$$

The first two closures are *ideal* operators on $\mathbf{U}$, and the last is an *interval* operator.[7]

For any acyclic graph G and uniquely generated closure φ, such φ_L, φ_R or φ_C above, we have an induced closure space. In Figure 3, we illustrate the three different closure spaces obtained by applying φ_L, φ_R, and φ_C to a single 5 point graph. Again, the sub-lattice of closed sets is denoted by solid lines. And, as usual, we will denote the $[closed\_set, generator]$ intervals by dashed ellipses. The partition coefficients of these three closure spaces are $< 0\ 0\ 1\ 3\ 3\ 2 >$, $< 0\ 1\ 0\ 1\ 4\ 4 >$, and $< 0\ 0\ 0\ 1\ 4\ 20 >$ respectively. Readily, different closure operators give rise to different partition coefficients.

We now treat the partition coefficients of this closure space as invariants of G. As observed above, this invariant depends on the closure operator. For the rest of this paper, we use only the ideal closure φ_L of (7). In Figure 4 we show $\mathcal{G}^4$, that is the collection of all basic, acyclic graphs on 4 points, together with the partition coefficients of their closure spaces. Because φ_L is path derived, any graph with additional edges, but the same transitive closure, must have the same associated closure space.

The graphs of $\mathcal{G}^4$ are not uniquely characterized by their coefficients; consider graphs (9) and (10) which both have $< 0\ 0\ 2\ 2\ 4 >$ as partition coefficients. But (9) is connected whereas (10) is disconnected. Unfortunately, this distinction is of little value. The connected, non-isomorphic graphs of Figure 5 both have partition coefficients $< 0\ 1\ 0\ 2\ 2\ 4 >$ with respect to φ_L. We would note that while the partition coefficients of Figures 5(a) and (b) are the same, their corresponding closure spaces, as illustrated by the lattices are distinct. This follows from

[7] In [11], φ_L is called *downset alignment* and φ_C is called *order convexity*, but just plain *convexity* in [17]. There are many conventions for drawing partially ordered sets. In an effort to distinguish between the underlying acyclic graph and its closure space, the author prefers to orient the former horizontally and the latter vertically. Because we illustrate with a left to right horizontal orientation, we use the subscripts, L(eft) and R(ight), to distinguish the ideal operators. The terms *upper/lower ideal* and $\downarrow$ operators are also encountered.

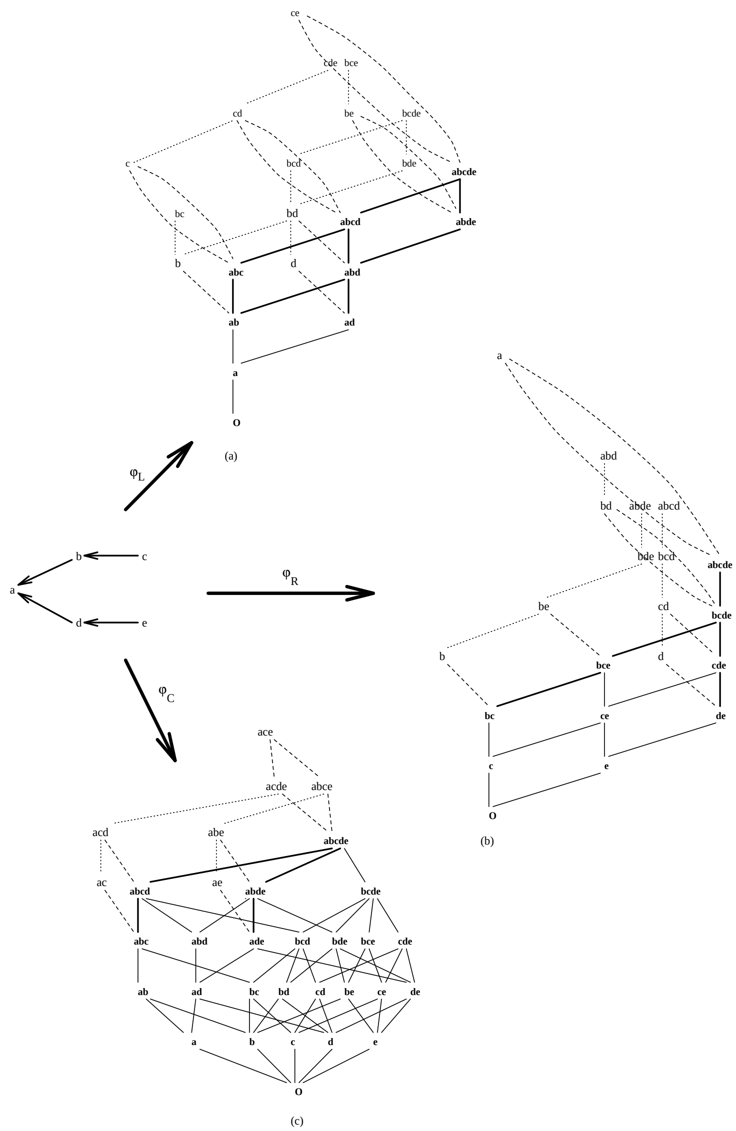

Fig. 3. Different closure spaces arising from the closure operators, φ_L (a), φ_R (b), and φ_C (c).

Theorem 3. Fundamental Theorem of Distributive Lattices *If* $(\mathbf{U}, \varphi)$ *is a finite closure space in which* $\mathbf{U}$ *is partially ordered and* φ *is an ideal operator, then the set of closed sets, partially ordered by inclusion, is a distributed lattice. Moreover, there is a one-to-one correspondence between the set of all distributive lattices and such closure spaces.*

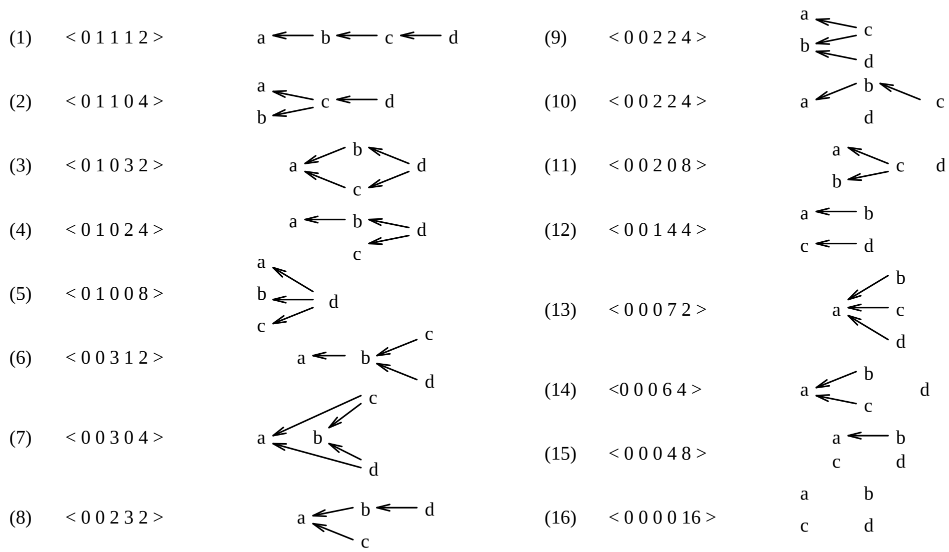

Fig. 4. All basic, acyclic 4 point graphs, $\mathcal{G}^4$ and their partition coefficients (w.r.t. φ_L)

Proof. See theorem 3.4.1 of [24] □

Distinct, non-isomorphic, graphs must have distinct closure spaces, but distinct closure spaces may have the same partition coefficients, just as two distinct linear transformations may have the same characteristic polynomial. Consequently, acyclic graphs cannot be completely characterized by their partition coefficients. Nevertheless, these coefficients convey significant information about the graphs and can be quite useful when manipulating them in computer systems.

The author has created one such computer system, capable of representing arbitrary graphs, whose primary purpose is the study of properties of graph transformations. For many of the studies of interest to us, we must generate all, or a large sample of, non-isomorphic graph on n points. Comparing binary partition coefficients is a useful filter for eliminating obviously non-isomorphic pairs. In Table 1 we display the expected number of acyclic graphs on n points that have the same identical binary partition coefficients, $exp(|G|\ per\ bp)$. For $n = 8$, there exist 16,999 distinct, non-isomorphic, acyclic graphs,[8] having 5,187 distinct partition coefficient sequences; so that an expected 3.277 have the same binary partition coefficients. But two graphs with the same partition coefficients need not have the same number of edges. They frequently do not. As shown on the next line of Table 1, the expected number of graph with identical partition coefficients and the same number of edges, $exp(|G|\ per\ bp\ and\ |E|)$, drops to

[8] The number of distinct n point acyclic graphs, or posets, grows exponentially. It is known that $|\mathcal{G}^n|$ is: 183,231 ($n = 9$), 2,567,284 ($n = 10$) and 46,794,427 ($n = 11$) [5]. No general enumeration formula is known.

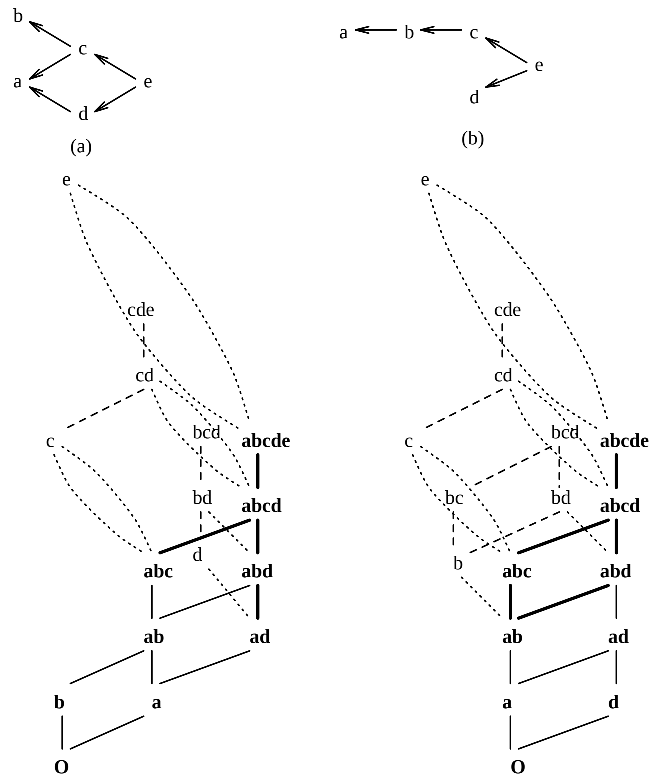

Fig. 5. Two graphs (a) and (b) having the partition coefficients $< 0\ 1\ 0\ 2\ 2\ 4 >$ together with their corresponding closure spaces

$n = \|P\|$	3	4	5	6	7	8
$\|\mathcal{G}^n\|$	5	16	63	318	2,405	16,999
$exp(\|G\|\ per\ bp)$	1.00	1.07	1.21	1.53	2.13	3.28
$exp(\|G\|\ per\ bp$ and $\|E\|)$	1.00	1.00	1.03	1.12	1.30	1.66

Table 1. Densities of acyclic graphs on n points when partitioned w.r.t. binary partition (bp) coefficients and w.r.t number of *edges*.

1.656. In practice, these expectations transalate into a effective filter. In a recent application that involved testing 1,034 M random pairs of 8-point graphs for isomorphism (equality), we first applied the edge cardinality filter; 193 M pairs passed this filter. Of these, only 148,762 had identical partition coefficients, and of these 87,710 were actually isomorphic. The probability of being isomorphic, given equal partition coefficients and numbers of edges was 1.69, compared to 1.66 as predicted by the table.

A quick measure of the effectiveness of invariant partition coefficients as an isomorphism filter can be attained by comparing it with other common filters. In Table 2, we count the number of equivalence classes generated in the family $\mathcal{G}^n$ of all n point acyclic graphs, assuming (a) partition coefficients alone, (b) partition coefficients plus equal edge cardinalities, (c) equal in (left) and out (right) degrees, (d) equal in (left) and out (right) ideals, and (e) equal ideals plus equal edge cardinalities. Readily, the expected number of graphs passing

		nbr of equivalence classes				
		(a)	(b)	(c)	(d)	(e)
n	$\lvert\mathcal{G}^n\rvert$	coeff	$+\lvert E\rvert$	degree	ideal	$+\lvert E\rvert$
4	16	15	16	16	15	16
5	63	52	61	63	52	61
6	318	208	285	125	208	284
7	2,045	962	1,570	432	951	1,551
8	16,999	5,187	10,263	1,588	4,932	9,863

Table 2. Comparison of isomorphism filters on graphs with n points

any filter, as in Table 1, is the expected number of graphs per equivalence class. The similarity of (a) and (b) with (d) and (e) is striking. This should not be too surprising, since φ_L is an ideal operator. But, it is a one-sided ideal operator, whereas (d) and (e) in Table 2 are based on two-sided ideals. Moreover, storage of filter (e) requires $2 \cdot n + 1$ integers whereas filter (b) consists of just $n + 1$ integers. In terms of information content, the partition coefficients are nearly twice as efficient. There may be more effective isomorphism filters, but we know of none with as dense information content.

A lexicographic ordering of the partition coefficients is an invariant, *nearly total* ordering of all acyclic graphs on n points. This can be of considerable value. In particular, we can use binary search to quickly obtain the neighborhood of any desired graph. The 4 point graphs of Figure 4 have been displayed in this order.

Another use of our graph manipulation system has been to gather various counts regarding basic, acyclic graphs on $n = |P|$ points with $e = |E|$ edges. Some of these results are summarized in Table 3. The numbers of trees on n points, connected graphs with $n - 1$ edges, is evident. We would observe that the counts are quite different from the similar table of [5] which has graphs with many more edges. They enumerate the *transitively closed* graphs (or partial orders) with e edges, whereas we enumerate the basic (or minimal) graphs with that order. Using the terminology of this paper, they count the edges in the closure of a partial order, while we count the edges in its generator.

The partition coefficients appear to encode a considerable amount of additional graph specific information. For example, it is not difficult to prove that:

$\|P\| =$	3		4		5		6		7		8	
$\|E\|$	nc	c	nc	c	nc	c	nc	c	nc	c	nc	c
0	1		1		1		1		1		1	
1	1		1		1		1		1		1	
2		3	4		4		4		4		4	
3				8	11		12		12		12	
4				2	2	27	43		46		47	
5						12	14	91	156		170	
6						5	5	87	110	350	670	
7								45	50	532	721	1,376
8								12	12	475	550	3,272
9								3	3	201	216	4,298
10										71	74	3,197
11										14	14	1,565
12										7	7	554
13												186
14												44
15												16
16												4
Totals	2	3	6	10	19	44	80	238	395	1,650	2,487	14,512
$\|\mathcal{G}^n\|$	5		16		63		318		2,045		16,999	

Table 3. Numbers of disconnected (nc) and connected (c) acyclic graphs on $|P|$ points with $|E|$ edges

Theorem 4. *If the closure operator is φ_L, then a_0 must be a power of two, whose exponent denotes the number of minimal (leftmost) elements.*

It also appears that partition coefficients encode a measure of connectivity information. After a tedious sequence of minor lemmas such as

Lemma 5. *Let φ be a path based closure and let $G^{(n)} = (P, E)$ on n points have the closure coefficients $< a_n,\ a_{n-1}, \cdots,\ a_0 >$. Then, $G^{(n+1)} = (P \cup \{x\}, E)$ has the closure coefficients $< 2 \cdot a_n,\ 2 \cdot a_{n-1}, \cdots,\ 2 \cdot a_0 >$.*

Lemma 6. *Let φ be the left (right) ideal closure. $G^{(n)} = (P, E)$ has a greatest (least) point if and only if the partition coefficient $a_{n-1} = 1$.*

one finally derives a curious result,

Theorem 7. *Let φ be an ideal closure. If all the binary partition coefficients of a graph, $< a_n,\ a_{n-1},\ \cdots\ ,\ a_1,\ a_0 >$ (w.r.t an ideal closure) are even, then the graph is disconnected or else there exists a disconnected graph with these binary partition coefficients.*

Suggested by this result, but not stated is the fact that if all the binary partition coefficients associated with a graph are even, then, with very high probability, the graph is disconnected. On the other hand, if even one coefficient is odd, the graph is probably connected.

Of major concern with the use of closure spaces and their binary partition coefficients as tools for the analysis and filtering of acyclic graphs, is the expected cost of generating them. The straightforward approach of generating all 2^n subsets and calculating their closures is clearly impractical for even moderate sized graphs. Fortunately, this is unnecessary. Given any closed set in a closure space, and its generator, one can easily determine all closed sets that it covers because,

Theorem 8. *If φ is uniquely generated, and if $X \neq \emptyset$ is closed, then $p \in X.gen$ if and only if $Z - \{p\}$ is closed.*

Proof. See Lemma 3.1, [20].

This theorem is treated as the defining property of *extreme points*, which are the generators of convex sets in [10]. It appears in one form or another in many efficient graph algorithms. For example, this property is used in [5] [2] to generate partial orders, where the universe **U** is the edge set; their closure is transitive closure; and a "cover" is a minimal edge set that generates the transitive closure. In [11] and [8] it is exploited to characterize properties of undirected graphs, and efficient algorithms to recognize them.

In our case, we use Theorem 8, to determine the closure space of a graph and its partition coefficients by first putting the entire point set P, which must be closed, in a queue. We then successively remove closed sets Y from the queue, verify that we have not already processed it,[9] then apply $generator(Y)$. We increment a_k where $k = |Y| - |Y.gen|$. For each $y \in Y.gen$, we add $Y - \{y\}$ to the queue. The cost of generating the closure space is approximately $|closedsets| \cdot cost_{generator(Y)}$. Assuming $cost_{generator(Y)}$ is nearly constant[10] given G, the cost of generating partition coefficients will be clearly dominated by the number of closed sets to be processed.

So, the key question becomes "what is the expected number of closed sets in an acyclic graph on n points?" The number of closed sets in any particular G is given by the sum of its partition coefficients, $\sum_{i=0}^{n} a_i$. Table 4 enumerates these expected values for $3 \leq n \leq 8$, first for those graphs with precisely $|E|$ edges, and then for all graphs in $\mathcal{G}^n$. Readily, the worst case behavior is $O(2^n)$; but this occurs only if $|E| = 0$. As $|E|$ increases the number of closed sets decreases towards an asymptote. If G must be connected, so that $|E| \geq n-1$, the number of number of closed sets is close to the asymptote itself.

4 Counting Binary Partitions

We close by once again considering binary partitions. The space of acyclic graphs grows exponentially. Is it reasonable to expect to characterize them by partition

[9] Because we use a queue, this is a level by level processing of the closed sets. It is possible to reach the same closed set twice. *C.f.* figure 2.

[10] With these small, finite graphs there is a hard upper bound for any n.

$\|E\|$	3	4	5	6	7	8
0	8.0	16.0	32.0	64.0	128.0	256.0
1	6.0	12.0	24.0	48.0	96.0	192.0
2	4.7	9.2	18.5	37.0	74.0	148.0
3		7.1	14.2	28.2	56.5	113.0
4		6.5	10.9	21.8	43.5	86.9
5			9.5	16.6	33.1	66.0
6			9.6	14.5	25.6	50.7
7				13.7	21.7	39.0
8				13.9	20.2	33.0
9				13.0	19.5	29.7
10					19.2	28.1
11					18.3	27.1
12					18.8	26.5
13						26.1
14						26.0
15						26.8
16						25.2
all graphs	5.6	8.4	12.2	17.1	23.6	32.3

Table 4. Expected numbers of closed sets in graphs with $|P|$ points and $|E|$ edges

coefficients as n becomes large? How many binary partitions are there on n points? It is customary to let $b(n)$, called the binary partition function, denote the *number* of binary partitions of n. As before, our interest is the number of binary partitions of 2^n, that is $b(2^n)$. In [3], it is shown that

$$b(2^n) = c_{n,0} + c_{n,1} \tag{8}$$

where $c_{1,0} = c_{1,1} = 1$, $c_{n+1,0} = c_{n,0} = 1$, and $c_{n+1,i} = \sum_{k=0}^{2i} c_{n,k}$. This particularly simple formulation was executed by Churchhouse on an Atlas computer in 1968 to obtain initial values of the binary partition function. A more complex, but somewhat faster code is given in [19]. With this code one can generate the following Table 5 of partitions of 2^n. The second column counts the number of *normal* partitions in which $a_0 \neq 0$, which in accordance with observation four in Section 1, is always $|\mathcal{P}^n| - |\mathcal{P}^{n-1}|$. Closure spaces associated with acyclic graphs must be normal. The third column counts the number of such non-isomorphic graphs on n points. It is easy to verify all sequences in Sloane's Handbook of Integer Sequences [23]. The point of this table is to illustrate that while the diversity of acyclic graphs on n points has exponential growth, the variety of closure spaces has super exponential growth; specifically $b(2^n) \sim (2^n)^{n/2}$. The concept of uniquely generated closure spaces is clearly rich enough to be embraced as a tool in the study of acyclic graphs and partially ordered spaces.

n	$b(2^n) = \|\mathcal{P}^n\|$	$\|normal\|$	$\|\mathcal{G}^n\|$
1	2	1	1
2	4	2	2
3	10	6	5
4	36	26	16
5	202	166	63
6	1,828	1,626	318
7	27,338	25,510	2,045
8	692,004	664,666	16,999
9	30,251,722	29,559,718	183,231
10	2,320,518,948	2,290,267,226	2,567,284

Table 5. Number of partitions of 2^n, of normal partitions, of acyclic graphs

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