# ENUMERATION OF SOLID 2-TREES

MICHEL BOUSQUET AND CEDRIC LAMATHE

ABSTRACT. The main goal of this paper is to enumerate solid 2-trees according to the number of edges (or triangles) and also according to the edge degree distribution. We first enumerate oriented solid 2-trees using the general methods of the theory of species. In order to obtain non oriented enumeration formulas we use quotient species which consists in a specialization of Pólya theory.

RÉSUMÉ. Le but de cet article est d'obtenir l'énumération des 2-arbres solides selon le nombre d'arêtes (ou de triangles) ainsi que selon la distribution des degrés des arêtes. Nous obtenons d'abord le dénombrement des 2-arbres solides orientés en utilisant les méthodes de la théorie des espèces. Pour obtenir le dénombrement des 2-arbres solides non orientés, nous utilisons la notion d'espèce quotient qui provient d'une spécialisation de la théorie de Pólya.

## 1. INTRODUCTION

**Definition 1.** Let $\mathcal{E}$ be a non-empty finite set of $n$ elements called *edges*. A *2-tree* is either a single edge (if $n = 1$) or a non-empty subset $\mathcal{T} \subseteq \wp_3(\mathcal{E})$ whose elements are called *triangles*, satisfying the following conditions:

1. For every pair $\{a, b\} = \{\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}\}$ of distinct elements of $\mathcal{T}$, we have $|a \cap b| \leq 1$, which means that two distinct triangles share at most one edge.
2. For every ordered pair $(a, b) = (\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\})$ of distinct elements of $\mathcal{T}$, there is a unique sequence $(t_0 = a, t_1, t_2, \ldots, t_k = b)$ such that for $i = 0, 1, \ldots, k-1$, we have $|t_i \cap t_{i+1}| = 1$, which means that each pair of consecutive triangles in this sequence share exactly one edge.

An edge $e$ and a triangle $t$ are *incident* to each other if $e \in t$. The *degree* of an edge is the number of triangles which are incident to that edge. The *edge degree distribution* of a 2-tree is described by a vector $\vec{n} = (n_1, n_2, \ldots)$, where $n_i$ is the number of edges of degree $i$. We denote by $\mathrm{Supp}(\vec{n})$, the *support* of $\vec{n}$ which is the set of indices $i$ such that $n_i \neq 0$. Figure 1 shows a 2-tree having 11 edges, 5 triangles and edge degree distribution given by $\vec{n} = (8, 2, 1)$.

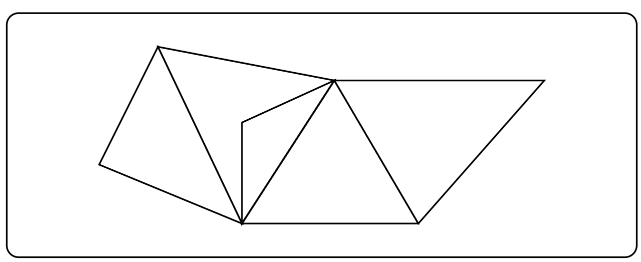

FIGURE 1. A 2-tree.

Several classes of 2-trees have been studied before. Beineke and Pippert enumerate some $k$-dimensional trees in [1] labelled at vertices. In [7], Harary and Palmer count unlabelled 2-trees. For the enumeration of plane 2-trees see [10], and for a classification of plane and planar 2-trees see [8]. More recently, in [5, 6], Fowler and al. work on general 2-trees and give asymptotical results. Here, we consider a new class of 2-trees, that is, *solid* 2-trees, *i.e.* 2-trees in which there is a cycle structure on the triangles around each edge.

**Lemma 1.** Let $m, n$ be two nonnegative integers, and $\vec{n} = (n_1, n_2, \ldots)$, an infinite vector of nonnegative integers. Then

1. There exists a 2-tree having $m$ triangles and $n$ edges if and only if $n = 2m + 1$.
2. There exists a 2-tree having $\vec{n}$ as edge degree distribution if and only if

$$\sum_i n_i = n \quad \text{and} \quad \sum_i i n_i = 3m. \tag{1}$$

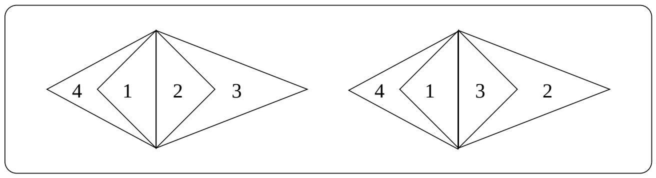


FIGURE 2. Two distinct solid 2-trees but the same 2-tree.

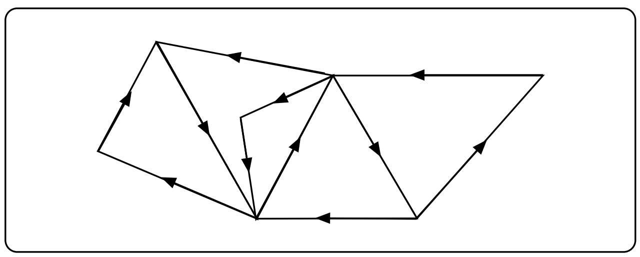

FIGURE 3. A well oriented 2-tree.

A *solid* 2-tree is a 2-tree in which there is a cyclic configuration of triangles around each edge. Figure 2 shows an example of two different solid 2-trees which are in fact the same 2-tree. As we can see, in the case of a solid 2-tree, one has to take into account the cyclic order of the triangles around each edge. A *well oriented* solid 2-tree is obtained from a solid 2-tree in the following way: first, pick any triangle and give a cyclic orientation on its edges. Then each triangle adjacent to the first triangle inherits a circular orientation (see Figure 3). This process is repeated until all edges receive an orientation. By the arborescent nature of the structure, there will be no conflict (the orientation of each edge will always be well defined). Figure 3 shows an example of a well oriented 2-tree. The species of non-oriented and well oriented solid 2-trees will be denoted respectively by $\mathcal{A}$ and $\mathcal{A}_o$. In order to analyze these two species, the following auxiliary species will be used:

- The species of *triangles* $X$: a single triangle will be denoted by $X$.
- The species of *edges* $Y$: a single edge will be denoted by $Y$.
- The species $L$ of *lists* or *linear orders*.
- The species $C$ and $C_3$ respectively denoting the species of oriented cycles and of oriented cycles of length 3.
- The species $\mathcal{A}^-$ and $\mathcal{A}_o^{\rightarrow}$ respectively denoting the species of non oriented and well oriented solid 2-trees *rooted at an edge.*
- The species $\mathcal{A}^{\Delta}$ and $\mathcal{A}_o^{\Delta}$ respectively denoting the species of non oriented and well oriented solid 2-trees *rooted at a triangle.*
- The species $\mathcal{A}^{\underline{\Delta}}$ and $\mathcal{A}_o^{\underline{\Delta}}$ respectively denoting the species of non oriented and well oriented solid 2-trees *rooted at a triangle having itself one of its edge distinguished.*
- Finally, the species $\mathcal{B}$ which consists of an oriented root edge $Y$ incident to a linear order ($L$-structure) of triangles $X$ each of which having its two remaining sides being themselves $\mathcal{B}$-structures. Therefore, the species $\mathcal{B}$ satisfies the following combinatorial equation

$$\mathcal{B}(X, Y) = YL(X\mathcal{B}^2(X, Y)), \tag{2}$$

as illustrated by Figure 4,

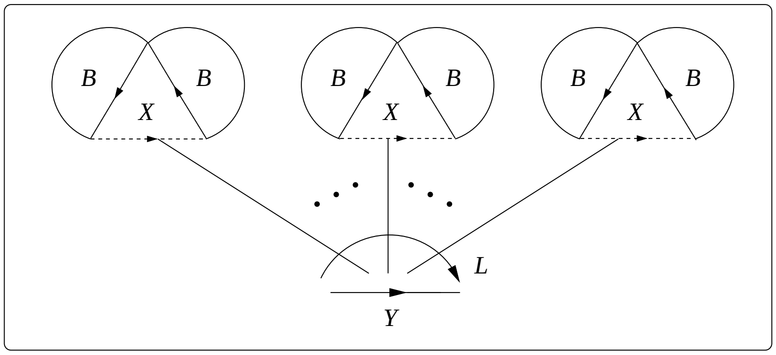


FIGURE 4. A $\mathcal{B}$-structure.

Note that $\mathcal{B}$ has been defined as a *two-sort* species where the sorts are $X$ and $Y$. Since the numbers of edges $n$ and of triangles $m$ are linked by the relation $n = 2m + 1$, equation (2) above can either be expressed as a one sort species in $X$ alone by setting $Y := 1$, or in $Y$ alone, by setting $X := 1$ respectively, giving the two following equations:

(3) $$\mathcal{B}(X) = L(X\mathcal{B}^2(X)),$$

(4) $$\mathcal{B}(Y) = YL(B^2(Y)).$$

Recall that setting $X := 1$ in a two sort species $F(X, Y)$ essentially means unlabelling the elements of sort $X$. The second form in equation (4) is more suitable for the use of Lagrange inversion formula. Therefore the species $Y$ of edges will be used as the base singleton species to make our computations. However, the results will be shorter and more elegant when expressed as a function of the number $m$ of triangles.

**• Lagrange Inversion Formula**

In this paper we make an extensive use of Lagrange inversion formula (see [2]): Let $A$ and $R$ be species satisfying $A(Y) = YR(A)$. If $F$ is another species, then

(5) $$[y^n]F(A(y)) = \frac{1}{n}[y^{n-1}]F'(t)R^n(t),$$

where $[y^n]F(A(y))$ denotes the coefficient of $y^n$ in $F(A(y))$. Another main tool used in this paper is the following dissymmetry theorem which has been proved in [5]. Note that in their paper, the authors made a proof for non solid 2-trees but obviously, the proof is also valid for both well oriented and non oriented solid 2-trees.

**Theorem 1.** The species $\mathcal{A}_o$ and $\mathcal{A}$ respectively of well oriented and (non oriented) solid 2-trees satisfy the following relations:

(6) $$\mathcal{A}_o^{\rightarrow} + \mathcal{A}_o^{\Delta} = \mathcal{A}_o + \mathcal{A}_o^{\underline{\Delta}},$$

and

(7) $$\mathcal{A}^{-} + \mathcal{A}^{\Delta} = \mathcal{A} + \mathcal{A}^{\underline{\Delta}}.$$

## 2. WELL ORIENTED SOLID 2-TREES

We begin this section by expressing the species appearing in the dissymmetry theorem (oriented case) in terms of the species $\mathcal{B}$.

**Theorem 2.** The species $\mathcal{A}_o^{\rightarrow}$, $\mathcal{A}_o^{\Delta}$ and $\mathcal{A}_o^{\underline{\Delta}}$ satisfy the following isomorphisms of species :

$$\mathcal{A}_o^{\rightarrow}(Y) = YC(\mathcal{B}^2(Y)), \tag{8}$$

$$\mathcal{A}_o^{\Delta}(Y) = C_3(\mathcal{B}(Y)), \tag{9}$$

$$\mathcal{A}_o^{\underline{\Delta}}(Y) = \mathcal{B}(Y)^3, \tag{10}$$

where $C$ and $C_3$ are the species of oriented cycles and of oriented cycles of length 3.

### 2.1. Enumeration according to the number of edges.

**• Labelled case**

Let $\mathcal{A}_o[n]$ be the number of edge labelled solid 2-trees over $n$ edges. We similarly define $\mathcal{A}_o^{\rightarrow}[n]$, $\mathcal{A}_o^{\Delta}[n]$ and $\mathcal{A}_o^{\underline{\Delta}}[n]$. Our first task is to determine $\mathcal{A}_o^{\rightarrow}[n]$. By applying Lagrange inversion with $F(t) = C(t^2) = -\log(1-t^2)$ and $R(t) = L(t^2) = (1-t^2)^{-1}$, we find

$$\begin{aligned} [y^n]\mathcal{A}_o^{\rightarrow}(y) &= [y^{n-1}]C(\mathcal{B}^2(y)), \\ &= \frac{2}{3(n-1)}\binom{3(n-1)/2}{n-1}. \end{aligned}$$

Hence, the number $\mathcal{A}_o^{\rightarrow}[n]$ of edge labelled solid 2-trees pointed at an edge over $n$ edges is given by

$$\mathcal{A}_o^{\rightarrow}[n] = n![y^n]\mathcal{A}_o^{\rightarrow}(y) = \frac{2}{3}n(n-2)!\binom{\frac{3(n-1)}{2}}{n-1}. \tag{11}$$

Now, using equation (9) and Lagrange inversion with $F(t) = C_3(t) = t^3/3$ and $R(t) = (1-t^2)^{-1}$, we obtain

$$\mathcal{A}_o^{\Delta}[n] = \frac{1}{3}(n-1)!\binom{\frac{3(n-1)}{2}}{n-1}. \tag{12}$$

To compute $\mathcal{A}_o^{\underline{\Delta}}[n]$, we use equation (10) and Lagrange inversion with $F(t) = t^3$ and $R(t) = (1-t^2)^{-1}$ and we get

$$\mathcal{A}_o^{\underline{\Delta}}[n] = (n-1)!\binom{\frac{3(n-1)}{2}}{n-1}. \tag{13}$$

Using equations (11), (12) and (13) and the dissymmetry theorem, we have:

**Proposition 1.** The number $\mathcal{A}_o[n]$ of well oriented edge-labelled solid 2-trees over $n$ edges is given by

$$\mathcal{A}_o[n] = \frac{2}{3}(n-2)!\binom{\frac{3(n-1)}{2}}{n-1}, \quad n > 1. \tag{14}$$

Note that if we express equation (14) as a function of $m$, the number of triangles, we obtain

$$\mathcal{A}_o[m] = \frac{m!}{3}\frac{1}{2m+3}\binom{3m+3}{m+1}, \quad m \geq 1. \tag{15}$$

**• Unlabelled case**

We first need to compute the generating series $\widetilde{\mathcal{A}_o}^{\rightarrow}(y)$. In order to accomplish this, we use the following property: let $F$ and $G$ be two species, then we have

$$\widetilde{F(G)}(x) = Z_F(\tilde{G}(x), \tilde{G}(x^2), \tilde{G}(x^3), \ldots), \tag{16}$$

where the *cycle index series* $Z_F$ of a species is defined by

$$Z_F(x_1, x_2, \ldots) = \sum_{k\geq 0}\frac{1}{k!}\sum_{\sigma\in\mathcal{S}_k}\mathrm{fix}F[\sigma]x_1^{\sigma_1}x_2^{\sigma_2}x_3^{\sigma_3}\cdots, \tag{17}$$

where $\mathcal{S}_k$ is the symmetric group of order $k$ and $\sigma_i$, the number of cycles of length $i$ in $\sigma$ and $\mathrm{fix}F[\sigma]$ is the number of $F$-structures left fixed under the relabelling induced by $\sigma$. For example, if $F = C$, the species of oriented cycles, we have

$$Z_C(x_1, x_2, \ldots) = \sum_{k\geq 1} \frac{\phi(k)}{k} \log\left(\frac{1}{1-x_k}\right). \tag{18}$$

Now, applying this to the species $\mathcal{A}_o^{\rightarrow} = YC(\mathcal{B}^2)$, we get

$$\begin{aligned}\widetilde{\mathcal{A}_o}^{\rightarrow}(y) &= yZ_C(\tilde{\mathcal{B}}^2(y), \tilde{\mathcal{B}}^2(y^2), \tilde{\mathcal{B}}^2(y^3), \ldots),\\ &= y\sum_{k\geq 1} \frac{\phi(k)}{k} \log\left(\frac{1}{1-\tilde{\mathcal{B}}^2(y^k)}\right).\end{aligned}$$

We note that since $\mathcal{B}$ is asymmetric (there are exactly $n!$ labelled structures for each unlabelled structures), we have $\tilde{\mathcal{B}}(y) = \mathcal{B}(y)$, hence

$$\begin{aligned}\widetilde{\mathcal{A}_o}^{\rightarrow}[n] &= [y^n]\widetilde{\mathcal{A}_o}^{\rightarrow}(y),\\ &= [y^{n-1}]\sum_{k\geq 1} \frac{\phi(k)}{k} \log\left(\frac{1}{1-\mathcal{B}^2(y^k)}\right).\end{aligned}$$

But

$$\begin{aligned}[y^{n-1}]\log\left(\frac{1}{1-\mathcal{B}^2(y^k)}\right) &= \frac{2k}{n-1}[t^{\frac{n-1}{k}-2}](1-t^2)^{-\frac{n-1}{k}-1},\\ &= \frac{2k}{3(n-1)}\binom{3(n-1)/2k}{(n-1)/k}.\end{aligned}$$

Obviously, $k$ must divide $n-1$ and $(n-1)/k$ must be even. Letting $d = (n-1)/k$, we finally get

$$\widetilde{\mathcal{A}_o}^{\rightarrow}[n] = \frac{2}{3(n-1)} \sum_d \phi((n-1)/d)\binom{3d/2}{d}, \tag{19}$$

the sum being taken over all even divisors $d$ of $n-1$. To compute $\widetilde{\mathcal{A}_o}^{\triangle}[n]$, we use equation (9) and the fact that

$$Z_{C_3}(y_1, y_2, \ldots) = \frac{1}{3}(y_1^3 + 2y_3).$$

We have

$$[y^n]\mathcal{B}^3(y) = \frac{1}{n}\binom{3(n-1)/2}{n-1},$$

and

$$[y^n]\mathcal{B}(y^3) = [y^{n/3}]\mathcal{B}(y) = \frac{3}{n}\binom{(n-3)/2}{n/3-1},$$

so that

$$\widetilde{\mathcal{A}_o}^{\triangle}[n] = \frac{1}{3n}\binom{\frac{3(n-1)}{2}}{n-1} + \frac{2}{n}\chi(3|n)\binom{\frac{(n-3)}{2}}{\frac{n}{3}-1}, \tag{20}$$

where $\chi(3|n) = 1$ if 3 divides $n$ and 0 otherwise. It can be easily shown, by a very similar way that

$$\widetilde{\mathcal{A}}_o^{\triangle}[n] = \frac{1}{n}\binom{\frac{3(n-1)}{2}}{n-1}. \tag{21}$$

And we get the following result:

**Proposition 2.** The number of unlabelled well oriented solid 2-trees over $n$ edges is given by

$$\widetilde{\mathcal{A}_o}[n] = \frac{2}{3(n-1)}\sum_d \phi\left(\frac{n-1}{d}\right)\binom{3d/2}{d} + \chi(3|n)\frac{2}{n}\binom{\frac{n-3}{2}}{\frac{n}{3}-1} - \frac{2}{3n}\binom{\frac{3(n-1)}{2}}{n-1}, \tag{22}$$

the first sum being taken over all even divisors $d$ of $n-1$.

We can also write $\widetilde{\mathcal{A}}_o[m]$, in function of the number $m$ of triangles, as follows

$$\widetilde{\mathcal{A}}_o[m] = \frac{1}{3m}\sum_{d|m}\phi\left(\frac{m}{d}\right)\binom{3d}{d} + \chi(3|2m+1)\frac{2}{2m+1}\binom{m-1}{\frac{2m-2}{3}} - \frac{2}{3(2m+1)}\binom{3m}{m}.$$

Note that this expression is also the number of unlabelled 3-gonal cacti on $m$ 3-gones (see [3]). The sequence of these numbers is known as sequence A054423 in the on-line encyclopedia of integers sequences ([11]).

## 2.2. **Enumeration according to edge degree distribution.**

Let $r = (r_0, r_1, r_2, \dots)$ be an infinite set of formal variables. In order to keep track of the edge degree distribution, we introduce, for a given number $n$ and $F$, any species, the following weight function:

$$w: \begin{array}{ccc} F[n] & \longrightarrow & Q[r_1, r_2, \dots] \\ s & \mapsto & w(s) \end{array} \tag{23}$$

where $Q[r_1, r_2, \dots]$ is the ring of polynomials over $Q$ in the variables $r_1, r_2, \dots$ and where the weight of a given structure $s$ is defined by $w(s) = r_1^{n_1} r_2^{n_2}\cdots$, where $n_i$ is the number of edges of degree $i$ in $s$. Equations (2), (8), (9) and (10) have the following weighted versions:

$$\mathcal{B}_r = Y L_{r'}(B_r^2), \tag{24}$$

and

$$\begin{aligned} \mathcal{A}_{o,w}^{\rightarrow}(Y) &= Y C_r(\mathcal{B}_r^2), & (25)\\ \mathcal{A}_{o,w}^{\Delta}(Y) &= C_3(B_r), & (26)\\ \mathcal{A}_{o,w}^{\underline{\Delta}}(Y) &= \mathcal{B}_r^3, & (27) \end{aligned}$$

where $C_r$ is the weighted species of cycles such that a cycle of length $i$ has the weight $r_i$, and its derivative $L_{r'}$ which is the species of lists where a list of length $i$ has the weight $r_{i+1}$. These species have the following generating series:

$$C_r(y) = r_1 y + \frac{r_2}{2}y^2 + \frac{r_3}{3}y^3 + \cdots,$$

and

$$L_{r'}(y) = r_1 + r_2 y + r_3 y^2 + \cdots.$$

Let $\vec{n} = (n_1, n_2, n_3, \dots)$ be a vector of nonnegative integers. Recall that there exists a 2-tree having a total of $n$ edges and $n_i$ edges of degree $i$ if and only if the following relation is satisfied:

$$\sum_i n_i = n \quad \text{and} \quad \sum_i i n_i = 3\left(\frac{n-1}{2}\right). \tag{28}$$

- **Labelled case**

Let $\vec{n}$ be a vector satisfying (28). Then the number $\mathcal{A}_o^{\rightarrow}[\vec{n}]$ of well oriented edge labelled solid 2-trees pointed at an edge, and having $\vec{n}$ as edge degree distribution, is given by

$$\mathcal{A}_o^{\rightarrow}[\vec{n}] = n![y^n][r_1^{n_1} r_2^{n_2}\cdots]\mathcal{A}_{o,w}^{\rightarrow}(y). \tag{29}$$

We have

$$\begin{aligned} [y^n]\mathcal{A}_{o,w}^{\rightarrow}(y) &= \frac{1}{n-1}[t^{n-2}]\frac{d}{dt}(C_r(t^2))\cdot L_{r'}^{n-1}(t^2), \\ &= \frac{2}{n-1}[t^{n-3}](r_1 + r_2 t^2 + r_3 t^4 + \cdots)^n, \\ &= \frac{2}{n-1}[t^{n-3}]\sum_{\ell_1+\ell_2+\cdots=n}\binom{n}{\ell_1, \ell_2, \dots} r_1^{\ell_1} r_2^{\ell_2}\cdots t^{2\ell_2+4\ell_3+6\ell_4+\cdots}. \end{aligned}$$

Finally, we obtain

$$[y^n]\mathcal{A}_o^{\to}(r,y) = \sum_{\ell_1,\ell_2,\dots} \binom{n}{\ell_1,\ell_2,\dots} r_1^{\ell_1} r_2^{\ell_2}\cdots,$$

the sum being taken over all vectors $(\ell_1,\ell_2,\dots)$ satisfying

$$\sum_i \ell_i = n \quad \text{and} \quad \sum_i 2(i-1)\ell_i = n-3.$$

We note that this condition is the same as in (28). Hence using (29) we have

$$\mathcal{A}_o^{\to}[\vec{n}] = 2n(n-2)!\binom{n}{n_1,n_2,\dots}. \tag{30}$$

For $\mathcal{A}_o^{\Delta}[\vec{n}]$, we have

$$\mathcal{A}_o^{\Delta}[\vec{n}] = n![y^n][r_1^{n_1}r_2^{n_2}\cdots]\mathcal{A}_{o,w}^{\Delta}(y).$$

But,

$$[y^n]\mathcal{A}_{o,w}^{\Delta}(y) \quad = \quad \frac{1}{n}\sum_{\ell_1,\ell_2,\dots} \binom{n}{\ell_1,\ell_2,\dots} r_1^{\ell_1} r_2^{\ell_2}\cdots,$$

the sum being taken on all vectors $(\ell_1,\ell_2,\dots)$ satisfying $\sum_i \ell_i = n$ and $\sum_i 2(i-1)\ell_i = n-3$, and we obtain

$$\mathcal{A}_o^{\Delta}[\vec{n}] = (n-1)!\binom{n}{n_1,n_2,\dots}. \tag{31}$$

It can be easily shown that $\mathcal{A}_o^{\underline{\Delta}}[\vec{n}] = 3\mathcal{A}_o^{\Delta}[\vec{n}]$, hence we have

$$\mathcal{A}_o^{\underline{\Delta}}[\vec{n}] = 3(n-1)!\binom{n}{n_1,n_2,\dots}. \tag{32}$$

Now using (30), (31), (32) and the dissymmetry theorem we find

$$\mathcal{A}_o[\vec{n}] = 2(n-2)!\binom{n}{n_1,n_2,\dots}. \tag{33}$$

- **Unlabelled case**

Let $\vec{n} = (n_1,n_2,\dots)$ be a coherent edge degree distribution. In order to compute the number $\widetilde{\mathcal{A}_o}^{\to}[\vec{n}]$ of unlabelled $\mathcal{A}_o^{\to}$-structures having $\vec{n}$ as edge degree distribution, we use the fact that given two weighted species $F_w$ and $G_v$, the generating series $\tilde{H}(y)$ of unlabelled $H$-structures, where $H = F_w(G_v)$, is given by

$$\tilde{H}(y) = Z_{F_w}(\tilde{G}_v(y), \tilde{G}_{v^2}(y^2), \tilde{G}_{v^3}(y^3),\dots). \tag{34}$$

In the present case, we have $\mathcal{A}_{o,w}^{\to} = YC_r(\mathcal{B}_r^2)$, and since the species $\mathcal{B}$ is asymmetric, $\tilde{\mathcal{B}}_r(y) = \mathcal{B}_r(y)$, hence

$$\widetilde{\mathcal{A}_o}^{\to}[\vec{n}] = [y^{n-1}][r_1^{n_1}r_2^{n_2}\cdots]Z_{C_r}(\mathcal{B}_r^2(y), \mathcal{B}_{r^2}^2(y^2), \mathcal{B}_{r^3}^2(y^3),\dots). \tag{35}$$

But $Z_{C_r}(y_1,y_2,\dots)$ can be expressed as the following sum:

$$Z_{C_r}(y_1,y_2,\dots) = \sum_{k\geq 1}\frac{r_k}{k}\sum_{d|k}\phi(d)y_d^{k/d}. \tag{36}$$

Combinatorially speaking, the integer $k$ represents the degree of the root edge. Hence, $k$ may only belong to $\mathrm{Supp}(\vec{n})$, the *support* of $\vec{n}$ which is the set of integers $i$ such that $n_i \neq 0$. Hence, we have

$$\widetilde{\mathcal{A}_o}^{\to}[\vec{n}] = [y^{n-1}][r_1^{n_1}r_2^{n_2}\cdots]\sum_{k\in\mathrm{Supp}(\vec{n})}\frac{r_k}{k}\sum_{d|k}\phi(d)\mathcal{B}_{r^d}^{2k/d}(y^d). \tag{37}$$

First, we compute

$$[y^{n-1}]\mathcal{B}_{r^d}^{2k/d}(y^d) = [y^{(n-1)/d}]\mathcal{B}_{r^d}^{2k/d}(y).$$

From Lagrange inversion, we have

$$
\begin{aligned}
[y^m]\mathcal{B}^{\ell}_{r^d}(y) &= \frac{1}{m}[t^{m-1}]\frac{d}{dt}\left(t^{\ell}\right) L^m_{r'^d}(t^2), \\
&= \frac{\ell}{m}\sum_{\ell_1,\ell_2,\ldots}\binom{m}{\ell_1,\ell_2,\ldots} r_1^{d\ell_1} r_2^{d\ell_2}\cdots, \qquad (38)
\end{aligned}
$$

where the $\ell_i$'s satisfy $\sum_i \ell_i = m$ and $\sum_i 2(i-1)\ell_i = m-\ell$. Now, letting $m=(n-1)/d$ and $\ell = 2k/d$, we find

$$
\widetilde{\mathcal{A}_o}^{\rightarrow}[\vec{n}] = [r_1^{n_1} r_2^{n_2}\cdots]\frac{2}{n-1}\sum_{k\in \mathrm{Supp}(\vec{n})}\sum_{d|k}\phi(d)\sum_{\ell_1,\ell_2,\ldots}\binom{(n-1)/d}{\ell_1,\ell_2,\ldots} r_1^{d\ell_1} r_2^{d\ell_2}\cdots r_k^{d\ell_k+1}\cdots. \qquad (39)
$$

Finally, we have

**Proposition 3.** Let $\vec{n}$ be a coherent edge degree distribution, then the number $\widetilde{\mathcal{A}_o}^{\rightarrow}[\vec{n}]$ of unlabelled oriented solid 2-trees pointed at an edge and having $\vec{n}$ as edge degree distribution is given by

$$
\widetilde{\mathcal{A}_o}^{\rightarrow}[\vec{n}] = \frac{2}{n-1}\sum_{k\in \mathrm{Supp}(\vec{n})}\sum_{d|\{k,\vec{n}-\delta_k\}}\phi(d)\binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}}, \qquad (40)
$$

where $\frac{\vec{n}-\delta_k}{d} = (\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_k-1}{d}, \ldots)$, for $d\geq 1$ and

$$
\binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}} = \binom{\frac{n-1}{d}}{n_1/d, n_2/d, \ldots, (n_k-1)/d, \ldots}.
$$

Let $\widetilde{\mathcal{A}}^{\Delta}_o[\vec{n}]$ and $\widetilde{\mathcal{A}}^{\underline{\Delta}}_o[n]$ be the numbers of unlabelled oriented solid 2-trees pointed respectively at a triangle and at a triangle pointed itself at one of its edge and having $\vec{n}$ as edge degree distribution. We have

**Proposition 4.** Let $\vec{n}$ be a coherent edge degree distribution, then the numbers $\widetilde{\mathcal{A}_o}^{\Delta}[\vec{n}]$ and $\widetilde{\mathcal{A}_o}^{\underline{\Delta}}[\vec{n}]$ are given by

$$
\widetilde{\mathcal{A}_o}^{\Delta}[\vec{n}] = \frac{1}{n}\binom{n}{n_1,n_2,\ldots} + \frac{\chi(3|\vec{n})}{n}\binom{n/3}{n_1/3, n_2/3,\ldots}, \qquad (41)
$$

$$
\widetilde{\mathcal{A}_o}^{\underline{\Delta}}[\vec{n}] = \frac{3}{n}\binom{n}{n_1,n_2,\ldots}, \qquad (42)
$$

where

$$
\chi(3|\vec{n}) = \begin{cases} 1, & \text{if all components of } \vec{n} \text{ are multiples of } 3 \\ 0, & \text{otherwise.}\end{cases}
$$

**Proof.** Let us start with $\widetilde{\mathcal{A}_o}^{\Delta}[\vec{n}]$. We have

$$
\begin{aligned}
\widetilde{\mathcal{A}_o}^{\Delta}[\vec{n}] &= [y^n][r_1^{n_1} r_2^{n_2}\cdots]\widetilde{\mathcal{A}}^{\Delta}_{o,w}(y), \\
&= [y^n][r_1^{n_1} r_2^{n_2}\cdots]Z_{C_3}(\tilde{\mathcal{B}}_r(y), \tilde{\mathcal{B}}_{r^2}(y^2),\ldots), \\
&= [y^n][r_1^{n_1} r_2^{n_2}\cdots]Z_{C_3}(\mathcal{B}_r(y), \mathcal{B}_{r^2}(y^2),\ldots).
\end{aligned}
$$

Since $Z_{C_3}(y_1,y_2,\ldots) = (y_1^3 + 2y_3)/3$,

$$
\widetilde{\mathcal{A}_o}^{\Delta}[\vec{n}] = \frac{1}{3}[y^n][r_1^{n_1} r_2^{n_2}\cdots]\left(\mathcal{B}_r^3(y) + 2\mathcal{B}_{r^3}(y^3)\right) \qquad (43)
$$

From equation (38) letting $m=n$, $\ell=3$ and $d=1$, we get

$$
[y^n]\mathcal{B}_r^3(y) = \frac{3}{n}\sum_{\ell_1,\ell_2,\ldots}\binom{n}{\ell_1,\ell_2,\ldots} r_1^{\ell_1} r_2^{\ell_2}\cdots, \qquad (44)
$$

where the $\ell_i$'s satisfy $\sum_i \ell_i = n$ and $\sum_i 2(i-1)\ell_i = n-3$. Now letting $m = n/3$, $\ell = 1$ and $d = 3$, we get

$$[y^n]\mathcal{B}_{r^3}(y^3) = [y^{n/3}]\mathcal{B}_{r^3}(y) = \frac{3}{n} \sum_{\ell_1,\ell_2,\ldots} \binom{n/3}{\ell_1, \ell_2, \ldots} r_1^{3\ell_1} r_2^{3\ell_2} \cdots, \tag{45}$$

where the $\ell_i$'s satisfy $\sum_i \ell_i = n$ and $\sum_i 2(i-1)\ell_i = n-1$. Now letting $\ell_i = n_i$ in (44) and $\ell_i = n_i/3$ in (45), we get equation (41). We obtain (42) in a very similar way. □

Finally, using the dissymmetry theorem, we obtain the final result of this section:

**Proposition 5.** Let $\vec{n}$ be a coherent edge degree distribution, then the number $\widetilde{\mathcal{A}_o}[\vec{n}]$ of unlabelled oriented solid 2-trees having $\vec{n}$ as edge degree distribution is given by

$$\widetilde{\mathcal{A}_o}[\vec{n}] = \frac{2}{n-1} \sum_{k \in \mathrm{Supp}(\vec{n})} \sum_{d \mid \{k, \vec{n}-\delta_k\}} \phi(d) \binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}} + \frac{\chi(3|\vec{n})}{n} \binom{\frac{n}{3}}{\frac{n_1}{3}, \frac{n_2}{3}, \ldots} - \frac{2}{3n} \binom{n}{n_1, n_2, \ldots}, \tag{46}$$

where

$$\chi(3|\vec{n}) = \begin{cases} 1, & \text{if all components of } \vec{n} \text{ are multiples of } 3, \\ 0, & \text{otherwise,} \end{cases}$$

$$\frac{\vec{n}-\delta_k}{d} = \left(\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_k - 1}{d}, \ldots\right) \text{ for } d \geq 1,$$

and

$$\binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}} = \binom{\frac{n-1}{d}}{n_1/d, n_2/d, \ldots, (n_k-1)/d, \ldots}.$$

## 3. Non-oriented solid 2-trees

In order to compute the numbers of labelled and unlabelled solid 2-trees, we use Burnside's Lemma with $\mathbb{Z}_2 = \{Id, \tau\}$, where the action of $\tau$ is to reverse the orientation of the structures.

### 3.1. **Enumeration according to the number of edges.**

• **Labelled case**

The labelled case is particularly simple since the only labelled oriented 2-tree which is left fixed under the action of $\tau$ is the structure consisting of a single oriented edge. Hence, we have

**Proposition 6.** The number $\mathcal{A}[n]$ of edge labelled solid 2-trees over $n$ edges is given by

$$\mathcal{A}[n] = \begin{cases} \frac{1}{2}\mathcal{A}_0[n] & \text{if } n > 1; \\ 1 & \text{if } n = 1. \end{cases} \tag{47}$$

Of course, the same argument will remain valid for all other pointed structures discussed in the previous section.

• **Unlabelled case**

In the unlabelled case, the action of $\tau$ is not so trivial. Figure 5 shows a structure which is left fixed under the action of $\tau$. Let $\mathcal{A}^-$ be the species of unoriented solid 2-trees rooted at an edge. This species can be expressed as the following quotient species (see [4]):

$$\mathcal{A}^- = \frac{\mathcal{A}_o^{\rightarrow}}{\mathbb{Z}_2} = \frac{YC(\mathcal{B}^2(Y))}{\mathbb{Z}_2}, \tag{48}$$

where $\mathbb{Z}_2 = \{\mathrm{Id}, \tau\}$ is the two element group consisting of the identity and $\tau$, whose action is to reverse the orientation of the edges. Hence, an unlabelled $\mathcal{A}^-$-structure is an orbit $\{a, \tau \cdot a\}$ under the action of $\mathbb{Z}_2$ where $a$ is any (oriented) unlabelled $\mathcal{A}_o^{\rightarrow}$-structure.

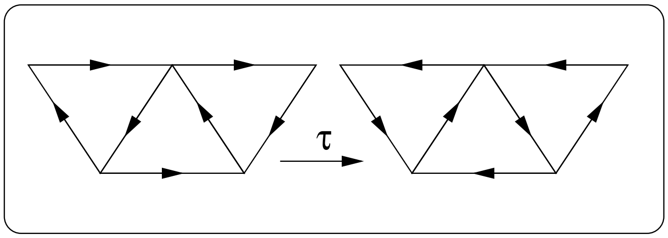


FIGURE 5. An unlabelled 2-tree invariant under the action of $\tau$.

Let us introduce the auxiliary species $\mathcal{B}_{\text{Sym}}$ of $\tau$-symmetric $\mathcal{B}$-structures, *i.e* the species of $\mathcal{B}$-structures left fixed under the edge orientation inversion. Denote by $\mathcal{B}_{\text{Sym}}(y)$ its ordinary generating series. Recall the functional equation verified by the species $\mathcal{B}$:

$$\mathcal{B} = YL(B^2).$$

In order to compute $\mathcal{B}_{\text{Sym}}(y)$, we have to distinguish two cases according to the parity of $k$, the length of the list of $\mathcal{B}^2$-structures attached to the rooted edge. First consider the case where $k$ is odd (Figure 6 shows an example where $k = 5$). A $\tau$-symmetric $\mathcal{B}$-structure must have a reflective symmetry plane. This plane contains the middle triangle of the list. When an inversion of the orientation of the rooted edge is applied, the two $\mathcal{B}$-structures glued on the two (non root) sides of the middle triangle (structures $\mathcal{B}_5$ and $\mathcal{B}_{5'}$ in Figure 6) are isomorphically exchange. The $k-1$ remaining triangles are exchanged pairwise carrying with them each of their attached $\mathcal{B}$-structures as shown in Figure 6. This gives a factor of $\mathcal{B}^k(y^2)$. We then have to sum the previous expression over all odd values of $k$. The case where $k$ is even, is very similar except that the symmetry plane must passes between two triangles as shown in Figure 7 and we get the same expression summed over all even values of $k$. Therefore, we have

$$\mathcal{B}_{\text{Sym}}(y) = y\sum_{k\geq 0}\mathcal{B}^k(y^2) = \frac{y}{1-\mathcal{B}(y^2)}. \tag{49}$$

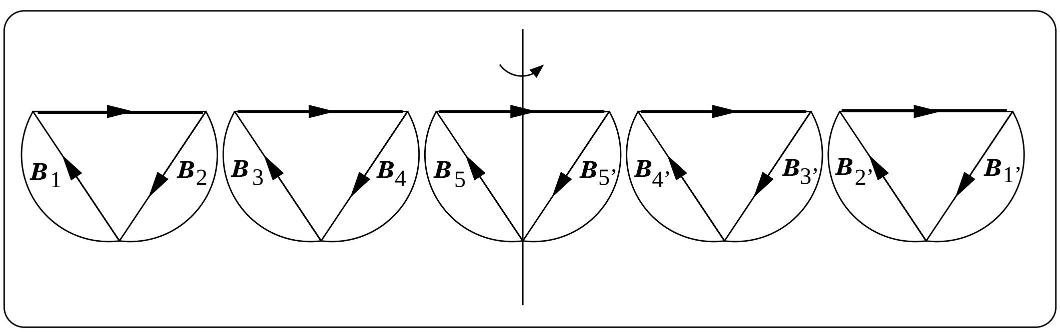


FIGURE 6. A $\mathcal{B}_{\text{Sym}}$-structure, $k$ odd.

From expression (49) and another use of Lagrange inversion, we easily obtain the following result.

**Proposition 7.** The number $\mathcal{B}_{\text{sym}}[m]$ of $\tau$-symmetric unlabelled oriented $\mathcal{B}$-structures is given by

$$\mathcal{B}_{\text{sym}}[m] = \begin{cases} \dfrac{1}{m+1}\dbinom{3m/2}{m} & \text{if } m \text{ is even,} \\[2ex] \dfrac{1}{m}\dbinom{(3m-1)/2}{m+1} + \dfrac{1}{3m}\dbinom{3(m+1)/2}{m+1} & \text{if } m \text{ is odd,} \end{cases} \tag{50}$$

where $m = (n-1)/2$ is the number of triangles and $n$, the number of edges.

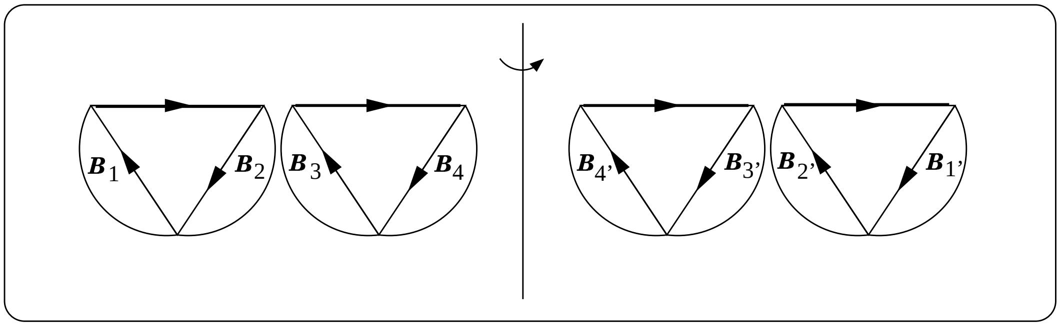


FIGURE 7. A $\mathcal{B}_{\mathrm{Sym}}$-structure, $k$ even.

We now give an expression for the generating function of unlabelled quotient structures, which will allow us to enumerate various kind of unlabelled solid 2-trees (see [4], proposition 2.2.4).

**Proposition 8.** Let $F$ be any (weighted) species and $G$, a group acting on $F$. Then the ordinary generating series of the quotient species $F/G$ is given by

$$(F/G)^{\sim}(y) = \frac{1}{|G|}\sum_{g\in G}\sum_{n\geq 0} |\mathrm{Fix}_{\tilde{F}_n}(g)|_w y^n, \tag{51}$$

where $\mathrm{Fix}_{\tilde{F}_n}(g)$ denotes the set of unlabelled $F$-structures left fixed under the action of $g \in G$ and $|\mathrm{Fix}_{\tilde{F}_n}(g)|_w$ represents the total weight of this set.

Using an unweighted version of Proposition 8 with $F = \mathcal{A}_o^{\to}$ and $G = \mathbb{Z}_2$, we obtain

$$\widetilde{\mathcal{A}}^{-}(y) = \frac{1}{2}\sum_{n\geq 0} |\mathrm{Fix}_{\tilde{\mathcal{A}}^{\to}_{o,n}}(\mathrm{Id})| y^n + \frac{1}{2}\sum_{n\geq 0} |\mathrm{Fix}_{\tilde{\mathcal{A}}^{\to}_{o,n}}(\tau)| y^n, \tag{52}$$

$$= \frac{1}{2}\widetilde{\mathcal{A}}_o^{\to}(y) + \frac{1}{2}\mathcal{B}_{\mathrm{sym}}(y), \tag{53}$$

since an oriented $\mathcal{A}^{-}$-structure left fixed under the action of $\tau$ is in fact a $\mathcal{B}_{\mathrm{Sym}}$-structure. Then, it becomes easy to extract the coefficient of $y^n$ in relation (53), and we get the number $\mathcal{A}^{-}[n]$ of edge pointed solid 2-trees over $n$ edges

$$\mathcal{A}^{-}[n] = \frac{1}{2}\widetilde{\mathcal{A}}_o^{\to}[n] + \frac{1}{2}\mathcal{B}_{\mathrm{Sym}}[n]. \tag{54}$$

We now consider the species $\mathcal{A}^{\vartriangle}$ of triangle rooted solid 2-trees. Since $\mathcal{A}^{\vartriangle} = \mathcal{A}_o^{\vartriangle}/\mathbb{Z}_2$, by virtue of Proposition 8, we have

$$\widetilde{\mathcal{A}}^{\vartriangle}(y) = \frac{1}{2}\sum_{n\geq 0} |\mathrm{Fix}_{\tilde{\mathcal{A}}^{\vartriangle}_{o,n}}(\mathrm{Id})| y^n + \frac{1}{2}\sum_{n\geq 0} |\mathrm{Fix}_{\tilde{\mathcal{A}}^{\vartriangle}_{o,n}}(\tau)| y^n, \tag{55}$$

where $|\mathrm{Fix}_{\tilde{\mathcal{A}}^{\vartriangle}_{o,n}}(\tau)|$, the number of $\tau$-symmetric $\mathcal{A}^{\vartriangle}$-structures over n edges has to be determined. As shown in Figure 8, such a structure must have an axis of symmetry which coincides with one of the root triangle's medians. Since the structure is already considered up to rotation around the root triangle, the choice among the three possible axes is arbitrary. The base side of the triangle must be a $\mathcal{B}_{\mathrm{Sym}}$-structure while the two other sides must be isomorphic copies of the same $\mathcal{B}$-structure. Therefore,

$$\widetilde{\mathcal{A}}^{\vartriangle}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_o^{\vartriangle}(y) + \frac{1}{2}\mathcal{B}_{\mathrm{Sym}}(y)\mathcal{B}(y^2). \tag{56}$$

In a very similar way, since $\mathcal{A}^{\underline{\vartriangle}} = \mathcal{A}_o^{\underline{\vartriangle}}/\mathbb{Z}_2$, we obtain

$$\widetilde{\mathcal{A}}^{\underline{\vartriangle}}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_o^{\underline{\vartriangle}}(y) + \frac{1}{2}\mathcal{B}_{\mathrm{Sym}}(y)\mathcal{B}(y^2). \tag{57}$$

Finally, using (53), (56) and (57) and using the dissymetry theorem, we get

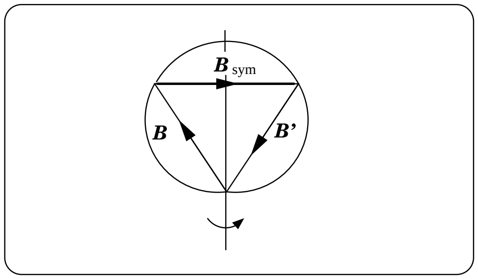


FIGURE 8. A $\tau$-symmetric $\mathcal{A}_o^{\Delta}$-structure.

**Proposition 9.** The ordinary generating function of solid 2-trees is given by

$$\mathcal{A}(y) = \frac{1}{2}(\mathcal{A}_o(y) + \mathcal{B}_{\text{Sym}}(y)), \tag{58}$$

where $\mathcal{B}_{\text{Sym}}(y)$ is the ordinary generating series of $\tau$-symmetric oriented $\mathcal{B}$-structures. Consequently, the number $\widetilde{\mathcal{A}}[m]$ of unoriented solid 2-trees over $m$ triangles is given by

$$\widetilde{\mathcal{A}}[m] = \frac{1}{2}(\widetilde{\mathcal{A}}_o[m] + \mathcal{B}_{\text{sym}}[m]), \tag{59}$$

where

$$\widetilde{\mathcal{A}_o}[m] = \frac{1}{3m}\sum_{d|m}\phi\left(\frac{m}{d}\right)\binom{3d}{d} + \chi(3|2m+1)\frac{2}{2m+1}\binom{m-1}{\frac{2m-2}{3}} - \frac{2}{3(2m+1)}\binom{3m}{m}.$$

and

$$\mathcal{B}_{\text{sym}}[m] = \begin{cases} \dfrac{1}{m+1}\dbinom{3m/2}{m} & \text{if } m \text{ is even,} \\[2ex] \dfrac{1}{m}\dbinom{(3m-1)/2}{m+1} + \dfrac{1}{3m}\dbinom{3(m+1)/2}{m+1} & \text{if } m \text{ is odd.} \end{cases} \tag{60}$$

To express $\widetilde{\mathcal{A}}[m]$ in term of $n$ the number of edges, we only have to set $m := \frac{n-1}{2}$.

### 3.2. Enumeration of non oriented solid 2-trees according to the edge degree distribution.

We consider again the weight function defined by

$$\begin{array}{rccc} w: & F[n] & \longrightarrow & Q[r_1, r_2, \ldots] \\ & s & \mapsto & w(s), \end{array} \tag{61}$$

where $r = (r_0, r_1, r_2, \ldots)$ is an infinite set of formal variables, $F$ is any species and $n$ is any positive integer.

**• Labelled case**

As mentioned in the previous section, the only labelled solid 2-tree left fixed under the action of $\tau$ consists in a single edge. Hence, given a valid edge degree distribution $\vec{n}$ we have

$$\mathcal{A}[\vec{n}] = \begin{cases} \frac{1}{2}\mathcal{A}_0[\vec{n}] & \text{if } n > 1; \\ 1 & \text{if } n = 1, \end{cases} \tag{62}$$

where $n$ is the number of edges and $\mathcal{A}[\vec{n}] = [y^n][r_1^{n_1} r_2^{n_2} \cdots]\mathcal{A}_w^-(y)$.

**• Unlabelled case**

Using the weighted versions of equations (53), (56) and (57), we get

$$\widetilde{\mathcal{A}}_w^{-}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_{o,w}^{\rightarrow}(y) + \frac{1}{2}\mathcal{B}_{\mathrm{sym},w}(y), \tag{63}$$

$$\widetilde{\mathcal{A}}_w^{\Delta}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_{o,w}^{\Delta}(y) + \frac{1}{2}\mathcal{B}_{\mathrm{sym},w}(y)\mathcal{B}_w(y^2), \tag{64}$$

$$\widetilde{\mathcal{A}}_w^{\underline{\Delta}}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_{o,w}^{\underline{\Delta}}(y) + \frac{1}{2}\mathcal{B}_{\mathrm{sym},w}(y)\mathcal{B}_w(y^2). \tag{65}$$

Now applying the dissymmetry theorem leads to

$$\widetilde{\mathcal{A}}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_{o,w}(y) + \frac{1}{2}\mathcal{B}_{\mathrm{sym},w}(y). \tag{66}$$

The only unknown term in the above equation is $\mathcal{B}_{\mathrm{sym},w}(y)$. We first establish an additional condition on the vertex degree distribution for an edge rooted oriented solid 2-tree to be $\tau$-symmetric. Since the root edge must remain fixed and all other edges are exchanged pairwise, the edge degree distribution vector $\vec{n}$ must have all its components even except one odd corresponding to the rooted edge.

For an edge degree distribution $\vec{n} = (n_1, n_2, \ldots)$ satisfying the previous condition, and using the fact that $\mathcal{B}_{\mathrm{sym,w}}(y) = yr_k\mathcal{B}^k(y^2)$, we have

$$\mathcal{B}_{\mathrm{sym,w}}[\vec{n}] = \frac{2k}{n-1}\binom{\frac{n-1}{2}}{\frac{\vec{n}-\delta_k}{2}}, \tag{67}$$

where $k$ is the root edge degree. We now present the final result of this paper.

**Proposition 10.** Let $\vec{n}$ be a vector satisfying

$$\sum_i n_i = n \quad \text{and} \quad \sum_i i n_i = 3m.$$

Then, the number $\widetilde{\mathcal{A}}[\vec{n}]$ of (non oriented) unlabelled solid 2-trees having $\vec{n}$ as edge degree distribution is given by

$$\widetilde{\mathcal{A}}[\vec{n}] = \frac{1}{2}\widetilde{\mathcal{A}}_o[\vec{n}] + \frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{sym}}[\vec{n}], \tag{68}$$

where

$$\widetilde{\mathcal{B}}_{\mathrm{sym}}[\vec{n}] = \begin{cases} \dfrac{2k}{n-1}\dbinom{\frac{n-1}{2}}{\frac{\vec{n}-\delta_k}{2}}, & \text{if } \vec{n} \text{ has a unique odd component,} \\ 0, & \text{otherwise,} \end{cases}$$

$\delta_k$ being the vector having 1 at the $k^{\mathrm{th}}$ component and 0 everywhere else, and

$$\widetilde{\mathcal{A}_o}[\vec{n}] = \frac{2}{n-1}\sum_{k\in\mathrm{Supp}(\vec{n})}\sum_{d|\{k,\vec{n}-\delta_k\}}\phi(d)\binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}} + \frac{\chi(3|\vec{n})}{n}\binom{n/3}{n_1/3, n_2/3, \ldots} - \frac{2}{3n}\binom{n}{n_1, n_2, \ldots}.$$

**Appendix.**
To conclude this paper, we give here two tables giving the numbers of unlabelled solid 2-trees oriented and unoriented as well as the number of unlabelled $\tau$-symmetric $\mathcal{B}$-structures. The first table gives these numbers according to the number $n$ of edges, and the second, according to edge degree distribution. We use the notation $1^{n_1}2^{n_2}\cdots$, where $i^{n_i}$ means $n_i$ edges of degree $i$.

| $n$ | $\widetilde{\mathcal{A}}_o[n]$ | $\mathcal{B}_{\text{sym}}[n]$ | $\widetilde{\mathcal{A}}[n]$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 |
| 7 | 2 | 2 | 2 |
| 9 | 7 | 3 | 5 |
| 11 | 19 | 7 | 13 |
| 13 | 86 | 12 | 49 |
| 15 | 372 | 30 | 201 |
| 17 | 1825 | 55 | 940 |
| 19 | 9143 | 143 | 4643 |
| 21 | 47801 | 273 | 24037 |

| $\vec{n}$ | $\widetilde{\mathcal{A}}_o[\vec{n}]$ | $\mathcal{B}_{\text{sym}}[\vec{n}]$ | $\widetilde{\mathcal{A}}[\vec{n}]$ |
|---|---|---|---|
| $1^7 2^1 3^1$ | 2 | 0 | 1 |
| $1^8 2^2 3^1$ | 9 | 3 | 6 |
| $1^{12} 2^1 3^1 4^1$ | 46 | 0 | 23 |
| $1^{10} 5^1$ | 3 | 1 | 2 |
| $1^{15} 4^1 5^1$ | 2 | 0 | 1 |
| $1^{16} 3^2 5^1$ | 17 | 5 | 11 |
| $1^{15} 2^2 7^1$ | 34 | 0 | 17 |

## References


[1] L. Beineke and R. Pippert, *The number of labeled k-dimensional trees*, Journal of Combinatorial Theory **6**, 200–205, (1969).

[2] F. Bergeron, G. Labelle, and P. Leroux, *Combinatorial Species and tree-like structures*, Encyclopedia of Mathematics and its Applications, vol. 67, Cambridge University Press, (1998).

[3] M. Bòna, M. Bousquet, G. Labelle and P. Leroux, *Enumeration of m-ary cacti*, Adv. in Appl. Math, **24**, 22–56, (2000).

[4] M. Bousquet, *Espèces de structures et applications au dénombrement de cartes et de cactus planaires*, Thèse de doctorat, UQÀM (1998). Publications du LaCIM, Vol. 24 (1999).

[5] T. Fowler, I. Gessel, G. Labelle, P. Leroux, *Specifying 2-trees*, Proceedings FPSAC'00, Moscow, 26-30 juin 2000, 202-213.

[6] T. Fowler, I. Gessel, G. Labelle, P. Leroux, *The Specification of 2-trees*, Advances in Applied Mathematics, to appear.

[7] F. Harary and E. Palmer, *Graphical Enumeration*, Academic Press, New York, (1973).

[8] G. Labelle, C. Lamathe and P. Leroux, *Développement moléculaire de l'espèce des 2-arbres planaires*, Proceedings GASCom 01, 41–46, (2001).

[9] G. Labelle and P. Leroux, *Enumeration of (uni- or bicolored) plane trees according to their degree distribution*, Discrete Math. **157**, 227–240, (1996).

[10] E. Palmer and R. Read, *On the Number of Plane 2-trees*, J. London Mathematical Society **6**, 583-592, (1973).

[11] N. J. A. Sloane and S. Plouffe, *The Encyclopedia of Integer Sequences*, Academic Press, San Diego, (1995). `http://www.research.att.com/~njas/sequences`


*E-mail address*: `[bousq2,lamathe]@math.uqam.ca`

LACIM, DÉPARTEMENT DE MATHÉMATIQUES,, UNIVERSITÉ DU QUÉBEC À MONTRÉAL.