ENUMERATION OF SOLID 2-TREES ACCORDING TO EDGE NUMBER AND EDGE DEGREE DISTRIBUTION

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ABSTRACT. The goal of this paper is to enumerate solid 2-trees according to the number of edges (or triangles) and also according to the edge degree distribution. We first enumerate oriented solid 2-trees using the general methods of the theory of species. In order to obtain non oriented enumeration formulas, we use quotient species which consists in a specialization of Pólya theory.

RÉSUMÉ. Le but de cet article est d'obtenir l'énumération des 2-arbres solides selon le nombre d'arêtes (ou de triangles) ainsi que selon la distribution des degrés des arêtes. Nous obtenons d'abord le dénombrement des 2-arbres solides orientés en utilisant les méthodes de la théorie des espèces. Pour obtenir le dénombrement des 2-arbres solides non orientés, nous utilisons la notion d'espèce quotient qui provient d'une spécialisation de la théorie de Pólya.

1. INTRODUCTION

Definition 1. Let $\mathcal{E}$ be a non-empty finite set of n elements called *edges*. A *2-tree* is either a single edge (if $n = 1$) or a non-empty subset $\mathcal{T} \subseteq \mathcal{P}_3(\mathcal{E})$ whose elements are called *triangles*, satisfying the following conditions:

1. For every pair $\{a, b\} = \{\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}\}$ of distinct elements of $\mathcal{T}$, we have $|a \cap b| \leq 1$, which means that two distinct triangles share at most one edge.
2. For every ordered pair $(a, b) = (\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\})$ of distinct elements of $\mathcal{T}$, there is a unique sequence $(t_0 = a, t_1, t_2, \ldots, t_k = b)$ such that for $i = 0, 1, \ldots, k-1$, we have $t_i \in \mathcal{T}$ and $|t_i \cap t_{i+1}| = 1$, which means that each pair of consecutive triangles in this sequence share exactly one edge.

An edge e and a triangle t are *incident* to each other if $e \in t$. The *degree* of an edge is the number of triangles which are incident to that edge. The *edge degree distribution* of a 2-tree is described by a vector $\vec{n} = (n_1, n_2, \ldots)$, where n_i is the number of edges of degree i. Since the case of a 2-tree reduced to a single edge (of degree 0) is obvious, we exclude it of this descritption. We denote by $\text{Supp}(\vec{n})$ the *support* of $\vec{n}$ which is the set of indices i such that $n_i \neq 0$. Figure 1 shows a 2-tree having 11 edges, 5 triangles and edge degree distribution given by $\vec{n} = (8, 2, 1)$.

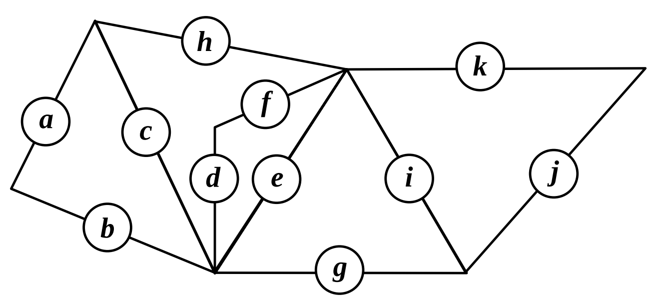


FIGURE 1. A 2-tree on $\mathcal{E} = \{a, b, c, d, e, f, g, h, i, j, k\}$.

Several classes of 2-trees have been studied before. Beineke and Pippert enumerate some k-dimensional trees in [1] labelled at vertices. In [9], Harary and Palmer count unlabelled 2-trees. For the enumeration of plane 2-trees, see [15], and for a classification according to symmetries of plane and planar 2-trees, see [12]. In [7, 8], Fowler et al. worked on general 2-trees and give asymptotical results. More recently, in [13], the authors generalize the results of Fowler et al. to the larger familly of k-gonal 2-trees. We also mention the works of Kloks in [10, 11] about partial biconnected 2-trees. Here, we consider a new class of 2-trees, that is, *solid* 2-trees, *i.e.*, 2-trees embedded in three-dimensional space.

The first result gives a sufficient and necessary condition on edges to ensure the existence of a 2-tree.

Lemma 1. Let m, n be two nonnegative integers and $\vec{n} = (n_1, n_2, \ldots)$, an infinite vector of nonnegative integers. Then:

1. There exists a 2-tree having m triangles and n edges if and only if $n = 2m + 1$.
2. There exists a 2-tree having n edges and $\vec{n}$ as edge degree distribution if and only if

$$\sum_i n_i = n \quad \text{and} \quad \sum_i i n_i = 3m. \tag{1}$$

Proof. Item 1 is quite obvious as the reader can check. For item 2, the condition $\sum_i n_i = n$ is straightforward. Concerning the relation $\sum_i i n_i = 3m$, it suffices to observe that the left-hand side counts the total degree of the structure, while, in the right-hand side, each triangle contributes for three units in the total degree. ■

We say that $\vec{n} = (n_1, n_2, \ldots)$ is a *coherent* (or *valid*) edge degree distribution if condition (1) is satisfied.

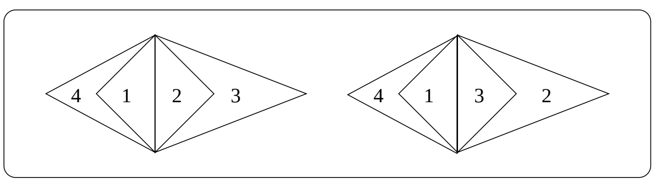


FIGURE 2. Two distinct solid 2-trees but the same 2-tree.

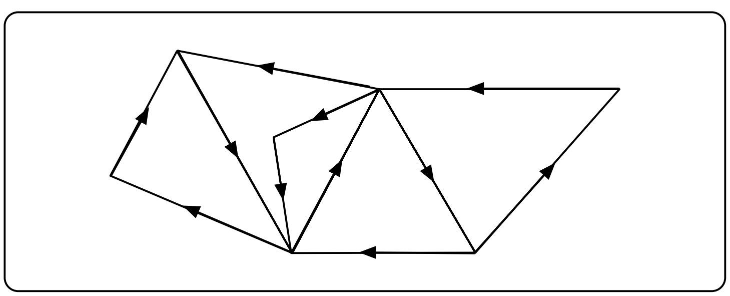


FIGURE 3. A well oriented 2-tree.

A *solid* 2-tree can be viewed topologically as a 2-tree in which the faces of the triangles cannot interpenetrate themselves. As a consequence, there is a cyclic configuration of triangles around each edge. Figure 2 shows an example of two different solid 2-trees which are in fact the same 2-tree. Indeed, the cyclic order on labels 1, 2, 3, 4 given to the triangles for the two 2-trees are different. A *well oriented* solid 2-tree is obtained from a solid 2-tree in the following way: first, pick any triangle and give a cyclic orientation on its edges; then each triangle adjacent to the first triangle inherits a cyclic orientation (see Figure 3). This process is repeated until all edges receive an orientation. By the arborescent nature of the structure, there will be no conflict (the orientation of each edge will always be well defined). Figure 3 shows an example of a well oriented 2-tree. The species of non-oriented and well oriented solid 2-trees will be denoted respectively by $\mathcal{A}$ and $\mathcal{A}_o$. For details about species, see [2]. In order to analyze these two species, the following auxiliary species will be used:

- The species of *triangles* X: a single triangle will be denoted by X;
- The species of *edges* Y: a single edge will be denoted by Y;
- The species L of *lists* or *linear orders*;
- The species C and C_3, respectively of oriented cycles and of oriented cycles of length 3;
- The species $\mathcal{A}^-$ and $\mathcal{A}_o^-$, respectively of non oriented and well oriented solid 2-trees *rooted at an edge*;
- The species $\mathcal{A}^\Delta$ and $\mathcal{A}_o^\Delta$, respectively of non oriented and well oriented solid 2-trees *rooted at a triangle*;
- The species $\mathcal{A}^{\underline{\Delta}}$ and $\mathcal{A}_o^{\underline{\Delta}}$, respectively of non oriented and well oriented solid 2-trees *rooted at a triangle having itself one of its edges distinguished*;
- Finally, the species $\mathcal{B}$ of *planted* oriented solid 2-trees which consists of an oriented root edge Y incident to a linear order (L-structure) of triangles X each of which having its two remaining sides being themselves $\mathcal{B}$-structures. Therefore, the species $\mathcal{B}$ satisfies the following combinatorial equation

$$\mathcal{B}(X,Y) = YL(X\mathcal{B}^2(X,Y)), \tag{2}$$

as illustrated by Figure 4.

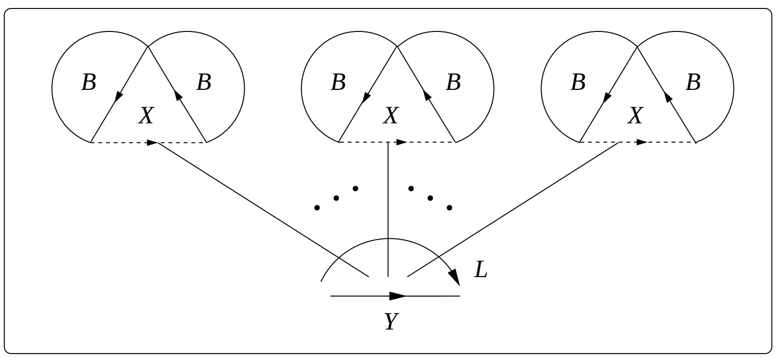


FIGURE 4. A $\mathcal{B}$-structure.

Note that $\mathcal{B}$ has been defined as a *two-sort* species where the sorts are X and Y. Since the numbers of edges n and of triangles m are linked by the relation $n = 2m + 1$, as stated in Lemma 1, equation (2) above can either be expressed as a one sort species in X alone by setting $Y := 1$, or in Y alone, by setting $X := 1$ respectively, giving the two following equations:

$$\mathcal{B}(X,1) = L(X\mathcal{B}^2(X,1)), \tag{3}$$

$$\mathcal{B}(1,Y) = YL(\mathcal{B}^2(1,Y)). \tag{4}$$

Recall that setting $X := 1$ in a two sort species $F(X,Y)$ essentially means unlabelling the elements of sort X. The second form in equation (4) is more suitable for the use of Lagrange inversion formula. Therefore, the species Y of edges will be used as the base singleton species to make our computations and we will rather use the shorter form $\mathcal{B}(Y) = YL(\mathcal{B}^2(Y))$ for (4). Hence, the structures are labelled at edges. However, some results will be more concise when expressed as a function of the number m of triangles.

In this paper, we make an extensive use of Lagrange inversion formula (see [2]). Let $A(y)$ and $R(y)$ be formal series satisfying $A(y) = yR(A(y))$ and $R(0) = 0$. If F is another formal series, then

$$[y^n]F(A(y)) = \frac{1}{n}[t^{n-1}]F'(t)R^n(t), \tag{5}$$

where $[y^n]F(A(y))$ denotes the coefficient of y^n in $F(A(y))$.

Another main tool used in this paper is the following dissymmetry theorem which has been proved in [7, 8]. Note that in their paper, the authors made a proof for general 2-trees but obviously, the proof is also valid for both well oriented and non oriented solid 2-trees.

Theorem 1. The species $\mathcal{A}_o$ and $\mathcal{A}$, respectively of well oriented and (non oriented) solid 2-trees, satisfy the following relations:

$$\mathcal{A}_o^{-} + \mathcal{A}_o^{\Delta} = \mathcal{A}_o + \mathcal{A}_o^{\underline{\Delta}}, \tag{6}$$

and

$$\mathcal{A}^{-} + \mathcal{A}^{\Delta} = \mathcal{A} + \mathcal{A}^{\underline{\Delta}}. \tag{7}$$

□

To each species F, we associate two series: the exponential generating series of labelled structures $F(x)$ and the ordinary generating series of unlabelled structures $\widetilde{F}(x)$, as follows:

$$F(x) = \sum_{n\geq 0} |F[n]| \frac{x^n}{n!}, \tag{8}$$

$$\widetilde{F}(x) = \sum_{n\geq 0} |\widetilde{F}[n]|\, x^n, \tag{9}$$

where $|F[n]|$ and $|\widetilde{F}[n]|$ are respectively the numbers of labelled and unlabelled F-structures over n elements.

2. Well oriented solid 2-trees

We begin this section by expressing the species appearing in the dissymmetry theorem (oriented case) in terms of the species $\mathcal{B}$.

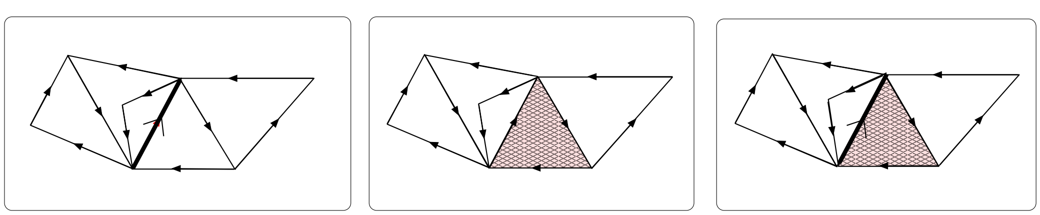
Proposition 1. The species $\mathcal{A}_o^{-}$, $\mathcal{A}_o^{\Delta}$ and $\mathcal{A}_o^{\underline{\Delta}}$ satisfy the following isomorphisms of species:

$$\mathcal{A}_o^{-}(Y) = Y + YC(\mathcal{B}^2(Y)), \tag{10}$$

$$\mathcal{A}_o^{\Delta}(Y) = C_3(\mathcal{B}(Y)), \tag{11}$$

$$\mathcal{A}_o^{\underline{\Delta}}(Y) = \mathcal{B}(Y)^3. \tag{12}$$

Proof. Let us begin with relation (10). The term Y corresponds to the case of a single rooted edge. In the general case, as illustrated by Figure 5 a), by convention with the right-hand rule, we define a cyclic order over the triangles glued around the oriented root-edge. Next, each triangle in this cyclic configuration, possesses, on its two remaining oriented edges, two $\mathcal{B}$-structures, leading to the expression $YC(\mathcal{B}^2(Y))$. For (11), it suffices to remark that, since the structures are (well) oriented, there is a cyclic order of length three around the edges of the root triangle (see Figure 5 b)). These edges being oriented, we can attach $\mathcal{B}$-structures on them, giving quite directly (11). We obtain (12) in a very similar way (see Figure 5)). ■



a) b) c)

Figure 5. Illustration of equations (10), (11) and (12).

2.1. **Enumeration according to the number of edges.**

In this section, we obtain the labelled and unlabelled enumeration of oriented solid 2-trees according to the number n of edges. We also give formulas in terms of the number m of triangles.

• Labelled case

Let $\mathcal{A}_o[n]$ be the set of edge-labelled solid 2-trees over n edges. We similarly define $\mathcal{A}_o^-[n]$, $\mathcal{A}_o^\Delta[n]$ and $\mathcal{A}_o^{\underline{\Delta}}[n]$. Our first task is to determine $|\mathcal{A}_o^-[n]|$, the cardinality of the set $\mathcal{A}_o^-[n]$. By applying Lagrange inversion with $F(t) = C(t^2) = -\ln(1-t^2)$ and $R(t) = L(t^2) = (1-t^2)^{-1}$, we find, for $n > 1$,

$$\begin{aligned}[y^n]\mathcal{A}_o^-(y) &= [y^{n-1}]C(\mathcal{B}^2(y)), \\ &= \frac{2}{3(n-1)}\binom{3(n-1)/2}{n-1}.\end{aligned}$$

Hence, we have

$$|\mathcal{A}_o^-[n]| = n![y^n]\mathcal{A}_o^-(y) = \frac{2}{3}n(n-2)!\binom{\frac{3(n-1)}{2}}{n-1}. \tag{13}$$

Note that, when a solid 2-tree over n edges is labelled, we have n different choices for the root edge. Therefore

$$n|\mathcal{A}_o[n]| = |\mathcal{A}_o^-[n]|,$$

and the next proposition follows.

Proposition 2. The number $|\mathcal{A}_o[n]|$ of well oriented edge-labelled solid 2-trees over n edges is given by

$$|\mathcal{A}_o[n]| = \frac{2}{3}(n-2)!\binom{\frac{3(n-1)}{2}}{n-1}, \quad n > 1. \tag{14}$$

□

Note that if we express equation (14) as a function of m, the number of triangles, we obtain

$$|\mathcal{A}_{o,t}[m]| = \frac{(m-1)!}{3}\frac{1}{2m+1}\binom{3m}{m}, \quad m \geq 2, \tag{15}$$

where the index t in $|\mathcal{A}_{o,t}[m]|$ means that the structures are labelled at triangles instead of edges.

• Unlabelled case

We first need to compute the ordinary generating series $\widetilde{\mathcal{A}}_o^-(y)$ of unlabelled $\mathcal{A}_o^-$-structures. In order to accomplish this, we use the following property.

Theorem 2. ([2]) Let F and G be two species. Then, we have

$$(F(G))^\sim(x) = Z_F(\widetilde{G}(x), \widetilde{G}(x^2), \widetilde{G}(x^3), \ldots), \tag{16}$$

where the *cycle index series* Z_F of a species is defined by

$$Z_F(x_1, x_2, \ldots) = \sum_{k\geq 0}\frac{1}{k!}\sum_{\sigma\in\mathcal{S}_k}\mathrm{fix}F[\sigma]x_1^{\sigma_1}x_2^{\sigma_2}x_3^{\sigma_3}\cdots, \tag{17}$$

where $\mathcal{S}_k$ is the symmetric group of order k, σ_i, the number of cycles of length i in the permutation $\sigma \in \mathcal{S}_k$ and $\mathrm{fix}F[\sigma]$, the number of F-structures left fixed under the relabelling induced by σ. □

For example, if $F = C$, the species of oriented cycles, we have

$$Z_C(x_1, x_2, \ldots) = \sum_{k\geq 1}\frac{\phi(k)}{k}\ln\left(\frac{1}{1-x_k}\right), \tag{18}$$

where ϕ is the Euler function. Now, applying this to the species $\mathcal{A}_o^- = Y + YC(\mathcal{B}^2)$, we get

$$\begin{aligned}\widetilde{\mathcal{A}}_o^-(y) &= y + yZ_C(\widetilde{\mathcal{B}}^2(y), \widetilde{\mathcal{B}}^2(y^2), \widetilde{\mathcal{B}}^2(y^3), \dots) \\ &= y + y\sum_{k\geq 1} \frac{\phi(k)}{k} \ln\left(\frac{1}{1-\widetilde{\mathcal{B}}^2(y^k)}\right).\end{aligned}$$

We note that since $\mathcal{B}$ is asymmetric (there are exactly $n!$ labelled structures for each unlabelled structures or equivalently, the stabilizer of each $\mathcal{B}$-structure is trivial), we have $\widetilde{\mathcal{B}}(y) = \mathcal{B}(y)$. Hence, for $n > 1$,

$$\begin{aligned}|\widetilde{\mathcal{A}}_o^-[n]| &= [y^n]\widetilde{\mathcal{A}}_o^-(y), \\ &= [y^{n-1}]\sum_{k\geq 1} \frac{\phi(k)}{k} \ln\left(\frac{1}{1-\mathcal{B}^2(y^k)}\right).\end{aligned}$$

But, using the fact that $[y^n]H(y^k) = [y^{n/k}]H(y)$ and Lagrange inversion,

$$\begin{aligned}[y^{n-1}]\ln\left(\frac{1}{1-\mathcal{B}^2(y^k)}\right) &= \frac{2k}{n-1}[t^{\frac{n-1}{k}-2}](1-t^2)^{-\frac{n-1}{k}-1} \\ &= \frac{2k}{3(n-1)}\binom{3(n-1)/2k}{(n-1)/k}.\end{aligned}$$

Obviously, k must divide $n-1$ and $(n-1)/k$ must be even. Letting $d = (n-1)/k$, we finally get

$$|\widetilde{\mathcal{A}}_o^-[n]| = \frac{2}{3(n-1)} \sum_d \phi(\tfrac{n-1}{d}) \binom{3d/2}{d}, \tag{19}$$

the sum being taken over all even divisors d of $n-1$. To compute $|\widetilde{\mathcal{A}}_o^{\triangle}[n]|$, we use equation (11) and the fact that

$$Z_{C_3}(y_1, y_2, \dots) = \frac{1}{3}(y_1^3 + 2y_3).$$

We have

$$[y^n]\mathcal{B}^3(y) = \frac{1}{n}\binom{3(n-1)/2}{n-1},$$

and

$$[y^n]\mathcal{B}(y^3) = [y^{n/3}]\mathcal{B}(y) = \frac{3}{n}\binom{(n-3)/2}{n/3-1},$$

so that,

$$|\widetilde{\mathcal{A}}_o^{\triangle}[n]| = \frac{1}{3n}\binom{\frac{3(n-1)}{2}}{n-1} + \frac{2}{n}\chi(3|n)\binom{\frac{(n-3)}{2}}{\frac{n}{3}-1}, \tag{20}$$

where $\chi(3|n) = 1$ if 3 divides n and 0 otherwise. It can be easily shown, by a very similar way, that

$$|\widetilde{\mathcal{A}}_o^{\underline{\triangle}}[n]| = \frac{1}{n}\binom{\frac{3(n-1)}{2}}{n-1}. \tag{21}$$

So, by virtue of the dissymmetry theorem (6), we get the following result:

Proposition 3. The number of unlabelled well oriented solid 2-trees over n edges is given by

$$|\widetilde{\mathcal{A}}_o[n]| = \frac{2}{3(n-1)} \sum_d \phi\left(\frac{n-1}{d}\right)\binom{3d/2}{d} + \chi(3|n)\frac{2}{n}\binom{\frac{n-3}{2}}{\frac{n}{3}-1} - \frac{2}{3n}\binom{\frac{3(n-1)}{2}}{n-1}, \tag{22}$$

the sum being taken over all even divisors d of $n-1$. □

We can also write $|\widetilde{\mathcal{A}}_{o,t}[m]|$, in function of the number m of triangles, as follows

$$|\widetilde{\mathcal{A}}_{o,t}[m]| = \frac{1}{3m}\sum_{d|m}\phi\left(\frac{m}{d}\right)\binom{3d}{d} + \chi(3|2m+1)\frac{2}{2m+1}\binom{m-1}{\frac{2m-2}{3}} - \frac{2}{3(2m+1)}\binom{3m}{m}.$$

Note that this expression also counts the number of unlabelled 3-gonal cacti on m triangles (see [3]). There is an quite direct bijection between these objects and solid 2-trees. The sequence of these numbers is known as sequence A054423 in the on-line encyclopedia of integers sequences ([16]). To

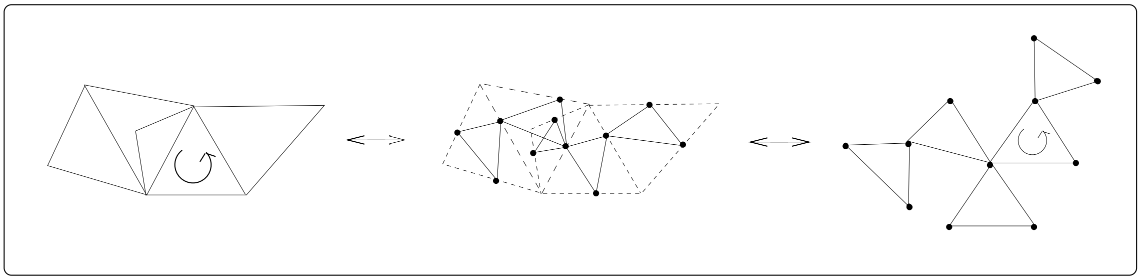


FIGURE 6. Bijection between solid 2-trees and cacti

obtain a (uncolored) 3-gonal cactus from a solid 2-tree, construct the dual of each triangle by putting vertices on edges of each triangle, and join vertices belonging to the same triangle (see Figure 6). Preserving the cyclic order gives a 3-gonal cactus. This construction closely resembles the one of the edge-graph of a solid 2-tree.

2.2. Enumeration according to edge degree distribution.

For enumeration according to edge degree distribution, we follow the approach of Labelle and Leroux [14] for plane trees. Consider $r = (r_1, r_2, r_3, \ldots)$ an infinite vector of formal variables. Recall that $\mathcal{A}[n]$ is the set of solid 2-trees over n edges. In order to keep track of the edge degree distribution, we introduce, for a given integer n, the following weight function (see [14]):

$$(23) \qquad \begin{array}{rccc} w: & \mathcal{A}[n] & \longrightarrow & \mathbb{Q}[r_1, r_2, \ldots] \\ & s & \longmapsto & w(s) \end{array}$$

where $\mathbb{Q}[r_1, r_2, \ldots]$ is the ring of polynomials over the field of rational numbers $\mathbb{Q}$ in the variables $r_1, r_2, \ldots$, and where the weight of a given $\mathcal{A}$-structure s is defined by $w(s) = r_1^{n_1} r_2^{n_2} \cdots$, where n_i is the number of edges of degree i in the structure s. Equations (2), (10), (11) and (12) have the following weighted versions:

$$(24) \qquad \mathcal{B}_r = Y L_{r'}(\mathcal{B}_r^2),$$

and

$$(25) \qquad \mathcal{A}^{-}_{o,w}(Y) = Y + Y C_r(\mathcal{B}_r^2),$$

$$(26) \qquad \mathcal{A}^{\Delta}_{o,w}(Y) = C_3(\mathcal{B}_r),$$

$$(27) \qquad \mathcal{A}^{\underline{\Delta}}_{o,w}(Y) = \mathcal{B}_r^3,$$

where C_r is the weighted species of cycles such that a cycle of length i has the weight r_i, and its derivative $L_{r'}$ which is the species of lists where a list of length i has the weight r_{i+1}. It is well known that these species have the following generating series of labelled structures (see [2, 14]):

$$C_r(y) = r_1 y + \frac{r_2}{2}y^2 + \frac{r_3}{3}y^3 + \cdots$$

and

$$L_{r'}(y) = (C_r(y))' = r_1 + r_2 y + r_3 y^2 + \cdots.$$

Let $\vec{n} = (n_1, n_2, n_3, \ldots)$ be a vector of nonnegative integers. Recall that, from Lemma 1, there exists a 2-tree having a total of n edges and n_i edges of degree i, $i \geq 1$, if and only if the following relations are satisfied:

$$\sum_i n_i = n \quad \text{and} \quad \sum_i i n_i = 3\binom{n-1}{2}. \tag{28}$$

Let us begin the weighted enumeration by the labelled case.

- **Labelled case**

Let $\vec{n}$ be a coherent vector in the sense of Lemma 1 (satisfying (28)). Then, the number $|\mathcal{A}_o^-[\vec{n}]|$ of well oriented edge-rooted labelled solid 2-trees having $\vec{n}$ as edge degree distribution, is given by

$$|\mathcal{A}_o^-[\vec{n}]| = n![r_1^{n_1} r_2^{n_2} \cdots][y^n]\mathcal{A}_{o,w}^-(y). \tag{29}$$

We have

$$\begin{aligned}
[y^n]\mathcal{A}_{o,w}^-(y) &= \frac{1}{n-1}[t^{n-2}]\frac{d}{dt}(C_r(t^2)) \cdot L_{r'}^{n-1}(t^2) \\
&= \frac{2}{n-1}[t^{n-3}](r_1 + r_2 t^2 + r_3 t^4 + \cdots)^n \\
&= \frac{2}{n-1}[t^{n-3}] \sum_{\ell_1+\ell_2+\cdots=n} \binom{n}{\ell_1, \ell_2, \ldots} r_1^{\ell_1} r_2^{\ell_2} \cdots t^{2\ell_2+4\ell_3+6\ell_4+\cdots}.
\end{aligned}$$

Finally, we obtain

$$[y^n]\mathcal{A}_{o,w}^-(y) = \sum_{\ell_1,\ell_2,\ldots} \binom{n}{\ell_1, \ell_2, \ldots} r_1^{\ell_1} r_2^{\ell_2} \cdots,$$

the sum being taken over all vectors $(\ell_1, \ell_2, \ldots)$ satisfying

$$\sum_i \ell_i = n \quad \text{and} \quad \sum_i 2(i-1)\ell_i = n-3.$$

We note that this condition is equivalent to relation (28). Hence, using (29), we have

$$|\mathcal{A}_o^-[\vec{n}]| = 2n(n-2)!\binom{n}{n_1, n_2, \ldots}. \tag{30}$$

As in the unweighted case, we have

$$|\mathcal{A}_o^-[\vec{n}]| = n|\mathcal{A}_o[\vec{n}]|,$$

and we get the following result.

Proposition 4. Let $\vec{n}$ be a coherent edge degree distribution. Then, the number of oriented solid 2-trees having $\vec{n}$ as edge degree distribution, $|\mathcal{A}_o[\vec{n}]|$, is given by

$$|\mathcal{A}_o[\vec{n}]| = 2(n-2)!\binom{n}{n_1, n_2, \ldots}. \tag{31}$$

□

We now give the unlabelled weighted enumeration.

- **Unlabelled case**

Let $\vec{n} = (n_1, n_2, \ldots)$ be a coherent edge degree distribution. In order to compute the number $|\widetilde{\mathcal{A}}_o^-[\vec{n}]|$ of unlabelled $\mathcal{A}_o^-$-structures having $\vec{n}$ as edge degree distribution, we use the weighted version of Theorem 2.

Theorem 3. ([2]) Given two weighted species F_w and G_v, the generating series $\widetilde{H}(y)$ of unlabelled H-structures, where $H = F_w(G_v)$, is given by

$$\widetilde{H}(y) = Z_{F_w}(\widetilde{G}_v(y), \widetilde{G}_{v^2}(y^2), \widetilde{G}_{v^3}(y^3), \ldots), \tag{32}$$

with $G_{v^k}(y^k) = p_k \circ G_v(y)$ where p_k denotes the k^{th} power sum and for all structure s, $v^k(s) = (v(s))^k$. □

In the present case, we have $\mathcal{A}^-_{o,w} = Y + YC_r(\mathcal{B}^2_r)$, and since the species $\mathcal{B}$ is asymmetric, that is $\widetilde{\mathcal{B}}_r(y) = \mathcal{B}_r(y)$,

$$|\widetilde{\mathcal{A}}^-_o[\vec{n}]| = [r_1^{n_1} r_2^{n_2} \cdots][y^{n-1}]\, Z_{C_r}(\mathcal{B}^2_r(y), \mathcal{B}^2_{r^2}(y^2), \mathcal{B}^2_{r^3}(y^3), \ldots). \tag{33}$$

But, the cycle index series of the weighted species C_r, $Z_{C_r}(y_1, y_2, \ldots)$, can be expressed as the following sum:

$$Z_{C_r}(y_1, y_2, \ldots) = \sum_{k\geq 1} \frac{r_k}{k} \sum_{d|k} \phi(d) y_d^{k/d}. \tag{34}$$

Roughly speaking, the integer k represents the degree of the root edge in the $\mathcal{A}^-_o$-structure. Hence, k may only belong to $\mathrm{Supp}(\vec{n})$, the *support* of $\vec{n}$, which consists in the set of integers $i \geq 1$ such that $n_i \neq 0$. So, we have

$$|\widetilde{\mathcal{A}}^-_o[\vec{n}]| = [r_1^{n_1} r_2^{n_2} \cdots][y^{n-1}] \sum_{k\in \mathrm{Supp}(\vec{n})} \frac{r_k}{k} \sum_{d|k} \phi(d) \mathcal{B}^{2k/d}_{r^d}(y^d). \tag{35}$$

First, we compute

$$[y^{n-1}]\mathcal{B}^{2k/d}_{r^d}(y^d) = [y^{(n-1)/d}]\mathcal{B}^{2k/d}_{r^d}(y).$$

Using Lagrange inversion, we get the following result, which will be usefull during computations:

Lemma 2. We have,

$$[y^m]\mathcal{B}^{\ell}_{r^d}(y) = \frac{\ell}{m} \sum_{\ell_1, \ell_2, \ldots} \binom{m}{\ell_1, \ell_2, \ldots} r_1^{d\ell_1} r_2^{d\ell_2} \cdots, \tag{36}$$

where the ℓ_i's satisfy $\sum_i \ell_i = m$ and $\sum_i 2(i-1)\ell_i = m - \ell$. □

Now, letting $m = (n-1)/d$ and $\ell = 2k/d$ in the previous lemma, we find

$$|\widetilde{\mathcal{A}}^-_o[\vec{n}]| = [r_1^{n_1} r_2^{n_2} \cdots] \frac{2}{n-1} \sum_{k\in \mathrm{Supp}(\vec{n})} \sum_{d|k} \phi(d) \sum_{\ell_1, \ell_2, \ldots} \binom{(n-1)/d}{\ell_1, \ell_2, \ldots} r_1^{d\ell_1} r_2^{d\ell_2} \cdots r_k^{d\ell_k + 1} \cdots. \tag{37}$$

Finally, we have:

Proposition 5. Let $\vec{n}$ be a coherent edge degree distribution. Then, the number $|\widetilde{\mathcal{A}}^-_o[\vec{n}]|$ of unlabelled oriented solid 2-trees pointed at an edge and having $\vec{n}$ as edge degree distribution is given by

$$|\widetilde{\mathcal{A}}^-_o[\vec{n}]| = \frac{2}{n-1} \sum_{k\in \mathrm{Supp}(\vec{n})} \sum_{d|\{k, \vec{n}-\delta_k\}} \phi(d) \binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}}, \tag{38}$$

where $\frac{\vec{n}-\delta_k}{d} = (\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_k - 1}{d}, \ldots)$, for $d \geq 1$,

$$\binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}} = \binom{\frac{n-1}{d}}{n_1/d, n_2/d, \ldots, (n_k-1)/d, \ldots},$$

and $d|\{k, \vec{n}-\delta_k\}$ means that the integer d must divide k and all components of the vector $\vec{n}-\delta_k$. □

Let $|\widetilde{\mathcal{A}}^{\Delta}_o[\vec{n}]|$ and $|\widetilde{\mathcal{A}}^{\underline{\Delta}}_{\vec{o}}[\vec{n}]|$ be the numbers of unlabelled oriented solid 2-trees pointed respectively at a triangle and at a triangle rooted itself at one of its edges and having $\vec{n}$ as edge degree distribution. Next proposition gives explicit formulas for these numbers.

Proposition 6. Let $\vec{n}$ be a coherent edge degree distribution, then the numbers $|\widetilde{\mathcal{A}}_o^{\triangle}[\vec{n}]|$ and $|\widetilde{\mathcal{A}}_o^{\blacktriangle}[\vec{n}]|$ are given by

$$|\widetilde{\mathcal{A}}_o^{\triangle}[\vec{n}]| = \frac{1}{n}\binom{n}{n_1, n_2, \ldots} + \frac{\chi(3|\vec{n})}{n}\binom{n/3}{n_1/3, n_2/3, \ldots}, \tag{39}$$

$$|\widetilde{\mathcal{A}}_o^{\blacktriangle}[\vec{n}]| = \frac{3}{n}\binom{n}{n_1, n_2, \ldots}, \tag{40}$$

where

$$\chi(3|\vec{n}) = \begin{cases} 1, & \text{if all components of } \vec{n} \text{ are multiples of 3,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let us start with $|\widetilde{\mathcal{A}}_o^{\triangle}[\vec{n}]|$. We have

$$\begin{aligned} |\widetilde{\mathcal{A}}_o^{\triangle}[\vec{n}]| &= [r_1^{n_1} r_2^{n_2}\cdots][y^n]\widetilde{\mathcal{A}}_{o,w}^{\triangle}(y) \\ &= [r_1^{n_1} r_2^{n_2}\cdots][y^n] Z_{C_3}(\widetilde{\mathcal{B}}_r(y)\widetilde{\mathcal{B}}_{r^2}(y^2), \ldots). \end{aligned}$$

Since $Z_{C_3}(y_1, y_2, \ldots) = (y_1^3 + 2y_3)/3$, and $\widetilde{\mathcal{B}}_r(y) = \mathcal{B}_r(y)$,

$$|\widetilde{\mathcal{A}}_o^{\triangle}[\vec{n}]| = \frac{1}{3}[r_1^{n_1} r_2^{n_2}\cdots][y^n]\left(\mathcal{B}_r^3(y) + 2\mathcal{B}_{r^3}(y^3)\right). \tag{41}$$

From equation (36) in Lemma 2, letting $m = n$, $\ell = 3$ and $d = 1$, we get

$$[y^n]\mathcal{B}_r^3(y) = \frac{3}{n}\sum_{\ell_1, \ell_2, \ldots}\binom{n}{\ell_1, \ell_2, \ldots} r_1^{\ell_1} r_2^{\ell_2}\cdots, \tag{42}$$

where the ℓ_i's satisfy $\sum_i \ell_i = n$ and $\sum_i 2(i-1)\ell_i = n-3$. Now letting $m = n/3$, $\ell = 1$ and $d = 3$ in (36), we obtain

$$[y^n]\mathcal{B}_{r^3}(y^3) = [y^{n/3}]\mathcal{B}_{r^3}(y) = \frac{3}{n}\sum_{\ell_1, \ell_2, \ldots}\binom{n/3}{\ell_1, \ell_2, \ldots} r_1^{3\ell_1} r_2^{3\ell_2}\cdots, \tag{43}$$

where the ℓ_i's satisfy $\sum_i \ell_i = n$ and $\sum_i 2(i-1)\ell_i = n-1$. Now letting $\ell_i = n_i$ in (42) and $\ell_i = n_i/3$ in (43), we get equation (39). We obtain (40) in a very similar way, details are left to the reader. ■

Finally, using the dissymmetry theorem (6), we obtain the final result of this section:

Proposition 7. Let $\vec{n}$ be a coherent edge degree distribution. Then the number $|\widetilde{\mathcal{A}}_o[\vec{n}]|$ of unlabelled oriented solid 2-trees having $\vec{n}$ as edge degree distribution is given by

$$|\widetilde{\mathcal{A}}_o[\vec{n}]| = \frac{2}{n-1}\sum_{k\in\mathrm{Supp}(\vec{n})}\sum_{d|\{k,\vec{n}-\delta_k\}}\phi(d)\binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}} + \frac{\chi(3|\vec{n})}{n}\binom{\frac{n}{3}}{\frac{n_1}{3}, \frac{n_2}{3}, \ldots} - \frac{2}{3n}\binom{n}{n_1, n_2, \ldots}, \tag{44}$$

where

$$\chi(3|\vec{n}) = \begin{cases} 1, & \text{if all components of } \vec{n} \text{ are multiples of 3,} \\ 0, & \text{otherwise,} \end{cases}$$

$$\frac{\vec{n}-\delta_k}{d} = \left(\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_k - 1}{d}, \ldots\right) \text{ for } d \geq 1,$$

and

$$\binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}} = \binom{\frac{n-1}{d}}{\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_k-1}{d}, \ldots}.$$

□

3. Non-oriented solid 2-trees

In order to compute the numbers of labelled and unlabelled solid 2-trees, we use Burnside's Lemma with the group $\mathbb{Z}_2 = \{Id, \tau\}$, where the action of τ is to reverse the orientation of the structures. This involves the notion of quotient species (see [4]).

3.1. **Enumeration according to the number of edges.**

As in the unweighted case, we begin with the labelled and unlabelled enumeration according to the number of edges.

• **Labelled case**

The labelled case is particularly simple since every labelled oriented 2-tree has exactly two possible orientations except the structure consisting of a single oriented edge. Hence, we have:

Proposition 8. The number $|\mathcal{A}[n]|$ of edge-labelled solid 2-trees over n edges is given by

$$|\mathcal{A}[n]| = \begin{cases} \frac{1}{2}|\mathcal{A}_o[n]|, & \text{if } n > 1, \\ 1, & \text{if } n = 1. \end{cases} \tag{45}$$

□

Of course, the same argument remains valid for all other pointed structures discussed in the previous section.

• **Unlabelled case**

In the unlabelled case, the action of τ is not so trivial. Figure 7 shows an oriented 2-tree which is left fixed under the action of τ. Let $\mathcal{A}^-$ be the species of (unoriented) solid 2-trees rooted at an edge. This species can be expressed as the following quotient species (see [7, 8, 13] for quotient species related to 2-trees):

$$\mathcal{A}^- = \frac{\mathcal{A}_o^-}{\mathbb{Z}_2} = \frac{Y + YC(\mathcal{B}^2(Y))}{\mathbb{Z}_2}, \tag{46}$$

where $\mathbb{Z}_2 = \{\mathrm{Id}, \tau\}$ is the two-element group consisting of the identity and τ, whose action is to reverse the orientation of the edges. Hence, an unlabelled $\mathcal{A}^-$-structure is an orbit $\{a, \tau \cdot a\}$ under the action of $\mathbb{Z}_2$, where a is any (oriented) unlabelled $\mathcal{A}_o^-$-structure. Roughly speaking, quotient by $\mathbb{Z}_2$ corresponds to forget the orientation in the structures.

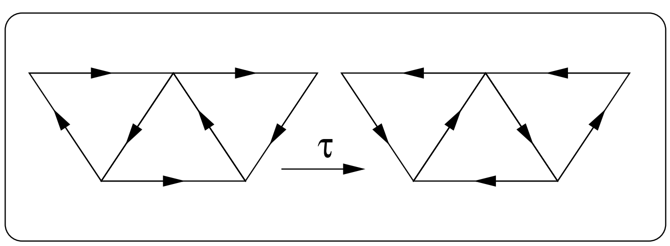


Figure 7. An unlabelled 2-tree invariant under the action of τ.

Let us introduce the auxiliary species $\mathcal{B}_{\mathrm{Sym}}$ of τ-symmetric $\mathcal{B}$-structures, *i.e*, the species of $\mathcal{B}$-structures left fixed under the edge orientation inversion. Denote by $\widetilde{\mathcal{B}}_{\mathrm{Sym}}(y)$ its ordinary generating series of unlabelled structures. Recall the functional equation verified by the species $\mathcal{B}$:

$$\mathcal{B} = YL(\mathcal{B}^2).$$

In order to compute $\widetilde{\mathcal{B}}_{\mathrm{Sym}}(y)$, we have to distinguish two cases according to the parity of k, the length of the list of $\mathcal{B}^2$-structures attached to the rooted edge. First consider the case where k is odd (Figure 8 shows an example where $k = 5$). A $\mathcal{B}$-structure is τ-symmetric if it can be embedded in space in such a way that the action of reversing the orientation of all edges corresponds to flip the whole structure back to itself by reversing the end points of the root edge. When an inversion of the orientation of the rooted edge is applied, the two $\mathcal{B}$-structures glued on the two (non root) sides of the middle triangle (structures $\mathcal{B}_5$ and $\mathcal{B}_{5'}$ in Figure 8) are isomorphically exchanged. The $k-1$ remaining triangles are exchanged pairwise carrying with them each of their attached $\mathcal{B}$-structures as shown in Figure 8, where $\mathcal{B}_i \cong \mathcal{B}_{i'}$. This gives a factor of $\mathcal{B}^k(y^2)$. We then have to sum the previous expression over all odd values of k. The case where k is even is very similar except that there is no middle triangle, as shown in Figure 9 and we get the same expression summed over all even values of k. It leads us to

$$\widetilde{\mathcal{B}}_{\mathrm{Sym}}(y) = y\sum_{k\geq 0}\mathcal{B}^k(y^2) = \frac{y}{1-\mathcal{B}(y^2)}. \tag{47}$$

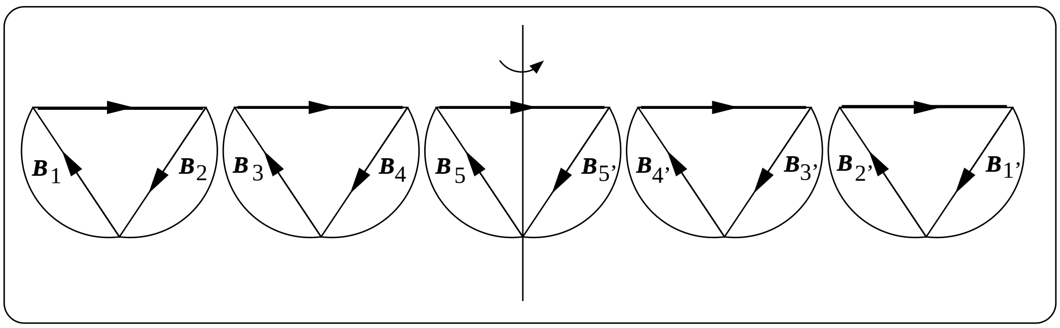


Figure 8. A $\mathcal{B}_{\mathrm{Sym}}$-structure, k odd.

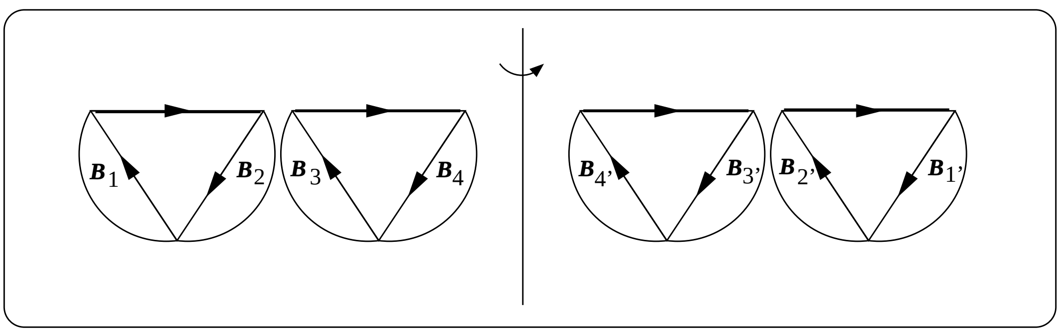


Figure 9. A $\mathcal{B}_{\mathrm{Sym}}$-structure, k even.

From expression (47) and another use of Lagrange inversion, we easily obtain the following result.

Proposition 9. The number $|\widetilde{\mathcal{B}}_{\mathrm{sym}}[m]|$ of τ-symmetric unlabelled oriented $\mathcal{B}$-structures over m triangles is given by

$$|\widetilde{\mathcal{B}}_{\mathrm{sym}}[m]| = \begin{cases} \dfrac{1}{m+1}\dbinom{3m/2}{m}, & \text{if } m \text{ is even,} \\[2ex] \dfrac{1}{m}\dbinom{(3m-1)/2}{m+1} + \dfrac{1}{3m}\dbinom{3(m+1)/2}{m+1}, & \text{if } m \text{ is odd.} \end{cases} \tag{48}$$

□

We can also express $|\widetilde{\mathcal{B}}_{\mathrm{sym}}[m]|$ as follows:

$$|\widetilde{\mathcal{B}}_{\mathrm{sym}}[m]| = \begin{cases} \frac{1}{2k+1}\binom{3k}{k}, & \text{if } m = 2k, \\ \frac{1}{2k+1}\binom{3k+1}{k+1}, & \text{if } m = 2k+1. \end{cases} \tag{49}$$

Note that, the numbers $|\widetilde{\mathcal{B}}_{\mathrm{sym}}[m]|$ also enumerate several classes of symmetric objects (in some sense), in particular symmetric diagonally convex directed polyominoes, or symmetric non-crossing trees, ... (see [5, 6]). These numbers are indexed in the on-line Encyclopedia of integer sequences [16] as the sequence A047749.

We now give an expression for the generating function of unlabelled quotient structures, which will allow us to enumerate various kind of unlabelled solid 2-trees.

Proposition 10. ([4]) Let F be any (weighted) species and G, a group acting on F. Then the ordinary generating series of the quotient species F/G is given by

$$(F/G)^{\sim}(y) = \frac{1}{|G|}\sum_{g\in G}\sum_{n\geq 0} |\mathrm{Fix}_{\widetilde{F}_n}(g)|_w \, y^n, \tag{50}$$

where $\mathrm{Fix}_{\widetilde{F}_n}(g)$ denotes the set of unlabelled F-structures over n edges left fixed under the action of the element $g \in G$ and $|\mathrm{Fix}_{\widetilde{F}_n}(g)|_w$ represents the total weight of this set. □

Using an unweighted version of Proposition 10, with $F = \mathcal{A}_o^-$ and $G = \mathbb{Z}_2 = \{\mathrm{Id}, \tau\}$, we obtain

$$\widetilde{\mathcal{A}}^-(y) = \frac{1}{2}\sum_{n\geq 0} |\mathrm{Fix}_{\widetilde{\mathcal{A}}^-_{o,n}}(\mathrm{Id})| y^n + \frac{1}{2}\sum_{n\geq 0} |\mathrm{Fix}_{\widetilde{\mathcal{A}}^-_{o,n}}(\tau)| y^n, \tag{51}$$

$$= \frac{1}{2}\widetilde{\mathcal{A}}_o^-(y) + \frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{sym}}(y), \tag{52}$$

since the oriented $\mathcal{A}^-$-structures left fixed under the action of τ have the same generating series as the $\mathcal{B}_{\mathrm{Sym}}$-structures. Hence, it becomes easy to extract the coefficient of y^n in relation (52), and we get the number $|\mathcal{A}^-[n]|$ of edge-pointed solid 2-trees over n edges,

$$|\mathcal{A}^-[n]| = \frac{1}{2}|\widetilde{\mathcal{A}}_o^-[n]| + \frac{1}{2}|\widetilde{\mathcal{B}}_{\mathrm{Sym}}[n]|. \tag{53}$$

We now consider the species $\mathcal{A}^{\Delta}$ of triangle rooted solid 2-trees. Since $\mathcal{A}^{\Delta} = \mathcal{A}_o^{\Delta}/\mathbb{Z}_2$, by virtue of Proposition 10, we have

$$\widetilde{\mathcal{A}}^{\Delta}(y) = \frac{1}{2}\sum_{n\geq 0} |\mathrm{Fix}_{\widetilde{\mathcal{A}}^{\Delta}_{o,n}}(\mathrm{Id})| y^n + \frac{1}{2}\sum_{n\geq 0} |\mathrm{Fix}_{\widetilde{\mathcal{A}}^{\Delta}_{o,n}}(\tau)| y^n, \tag{54}$$

where $|\mathrm{Fix}_{\widetilde{\mathcal{A}}^{\Delta}_{o,n}}(\tau)|$, the number of τ-symmetric $\mathcal{A}^{\Delta}$-structures over n edges has to be determined. As shown in Figure 10, such a structure must have an axis of symmetry which coincides with one of the root triangle's medians. Since the structure is already considered up to rotation around the root triangle, the choice among the three possible axes is arbitrary. The base side of the triangle must be a $\mathcal{B}_{\mathrm{Sym}}$-structure while the two other sides must be isomorphic copies of the same $\mathcal{B}$-structure ($\mathcal{B} \cong \mathcal{B}'$). Therefore,

$$\widetilde{\mathcal{A}}^{\Delta}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_o^{\Delta}(y) + \frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{Sym}}(y)\mathcal{B}(y^2). \tag{55}$$

In a very similar way, since $\mathcal{A}^{\underline{\Delta}} = \mathcal{A}_o^{\underline{\Delta}}/\mathbb{Z}_2$, we obtain

$$\widetilde{\mathcal{A}}^{\underline{\Delta}}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_o^{\underline{\Delta}}(y) + \frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{Sym}}(y)\mathcal{B}(y^2). \tag{56}$$

Finally, combining equations (52), (55), (56) and the dissymmetry theorem, we get:

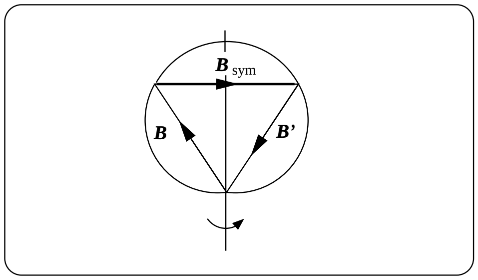


FIGURE 10. A τ-symmetric $\mathcal{A}_o^{\Delta}$-structure.

Proposition 11. The ordinary generating function of unlabelled solid 2-trees is given by

$$\widetilde{\mathcal{A}}(y) = \frac{1}{2}(\widetilde{\mathcal{A}}_o(y) + \widetilde{\mathcal{B}}_{\mathrm{Sym}}(y)), \tag{57}$$

where $\widetilde{\mathcal{B}}_{\mathrm{Sym}}(y)$ is the ordinary generating series of τ-symmetric $\mathcal{B}$-structures. Consequently, the number $|\widetilde{\mathcal{A}}_t[m]|$ of unoriented solid 2-trees over m triangles is given by

$$|\widetilde{\mathcal{A}}_t[m]| = \frac{1}{2}(|\widetilde{\mathcal{A}}_{o,t}[m]| + |\widetilde{\mathcal{B}}_{\mathrm{sym}}[m]|), \tag{58}$$

where

$$|\widetilde{\mathcal{A}}_{o,t}[m]| = \frac{1}{3m}\sum_{d|m}\phi\left(\frac{m}{d}\right)\binom{3d}{d} + \chi(3|2m+1)\frac{2}{2m+1}\binom{m-1}{\frac{2m-2}{3}} - \frac{2}{3(2m+1)}\binom{3m}{m},$$

and

$$|\widetilde{\mathcal{B}}_{\mathrm{sym}}[m]| = \begin{cases} \dfrac{1}{m+1}\dbinom{3m/2}{m}, & \text{if } m \text{ is even}, \\[2ex] \dfrac{1}{m}\dbinom{(3m-1)/2}{m+1} + \dfrac{1}{3m}\dbinom{3(m+1)/2}{m+1}, & \text{if } m \text{ is odd}. \end{cases} \tag{59}$$

□

Note that, to express $|\widetilde{\mathcal{A}}_t[m]|$ in terms of n the number of edges, we only have to set $n := 2m+1$ in the previous expressions.

3.2. Enumeration of solid 2-trees according to the edge degree distribution.

We consider again the weight function defined by

$$\begin{array}{rccc} w: & \mathcal{A}[n] & \longrightarrow & \mathbb{Q}[r_1, r_2, \ldots] \\ & s & \mapsto & w(s), \end{array} \tag{60}$$

where $r = (r_1, r_2, r_3, \ldots)$ is an infinite set of formal variables and n is any positive integer.

• **Labelled case**

Using the same argument as in the unweighted case, we have

$$|\mathcal{A}[\vec{n}]| = \begin{cases} \frac{1}{2}|\mathcal{A}_o[\vec{n}]|, & \text{if } n > 1, \\ 1, & \text{if } n = 1, \end{cases} \tag{61}$$

where $\vec{n}$ is a valid edge degree distribution, n is the number of edges and $|\mathcal{A}[\vec{n}]| = [r_1^{n_1} r_2^{n_2}\cdots][y^n]\mathcal{A}_w(y)$.

• **Unlabelled case**

Using the weighted versions of equations (52), (55) and (56), we easily get

$$\widetilde{\mathcal{A}}_w^-(y) = \frac{1}{2}\widetilde{\mathcal{A}}_{o,w}^-(y) + \frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{sym},w}(y), \tag{62}$$

$$\widetilde{\mathcal{A}}_w^{\vartriangle}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_{o,w}^{\vartriangle}(y) + \frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{sym},w}(y)\mathcal{B}_w(y^2), \tag{63}$$

$$\widetilde{\mathcal{A}}_w^{\underline{\vartriangle}}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_{o,w}^{\underline{\vartriangle}}(y) + \frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{sym},w}(y)\mathcal{B}_w(y^2), \tag{64}$$

where $\widetilde{\mathcal{B}}_{\mathrm{sym},w}(y)$ is the ordinary generating series of unlabelled weighted τ-symmetric $\mathcal{B}$-structures. Now, applying the dissymmetry theorem, leads to

$$\widetilde{\mathcal{A}}(y) = \frac{1}{2}\widetilde{\mathcal{A}}_{o,w}(y) + \frac{1}{2}\widetilde{\mathcal{B}}_{\mathrm{sym},w}(y). \tag{65}$$

The only unknown term in the above equation is $\widetilde{\mathcal{B}}_{\mathrm{sym},w}(y)$. We first establish an additional condition on the edge degree distribution for an edge-rooted oriented solid 2-tree to be τ-symmetric. Since the root edge must remain fixed and all other edges are exchanged pairwise, the edge degree distribution vector $\vec{n}$ must have all its components even except one odd corresponding to the degree of the rooted edge.

For an edge degree distribution $\vec{n} = (n_1, n_2, \ldots)$ satisfying the previous condition, and using the fact that $\widetilde{\mathcal{B}}_{\mathrm{sym},w}(y) = yr_k\mathcal{B}^k(y^2)$, we have

$$|\widetilde{\mathcal{B}}_{\mathrm{sym}}[\vec{n}]| = \frac{2k}{n-1}\binom{\frac{n-1}{2}}{\frac{\vec{n}-\delta_k}{2}}, \tag{66}$$

where k corresponds to the root edge degree. We now present the final result of this paper.

Proposition 12. Let $\vec{n}$ be a vector satisfying

$$\sum_i n_i = n \quad \text{and} \quad \sum_i i n_i = 3m.$$

Then, the number $|\widetilde{\mathcal{A}}[\vec{n}]|$ of (non oriented) unlabelled solid 2-trees having $\vec{n}$ as edge degree distribution is given by

$$|\widetilde{\mathcal{A}}[\vec{n}]| = \frac{1}{2}|\widetilde{\mathcal{A}}_o[\vec{n}]| + \frac{1}{2}|\widetilde{\mathcal{B}}_{\mathrm{sym}}[\vec{n}]|, \tag{67}$$

where

$$|\widetilde{\mathcal{B}}_{\mathrm{sym}}[\vec{n}]| = \begin{cases} \dfrac{2k}{n-1}\dbinom{\frac{n-1}{2}}{\frac{\vec{n}-\delta_k}{2}}, & \text{if } \vec{n} \textit{ has a unique odd component}, \\ 0, & \text{otherwise}, \end{cases}$$

δ_k being the vector having 1 at the k^{th} component and 0 everywhere else, and

$$|\widetilde{\mathcal{A}}_o[\vec{n}]| = \frac{2}{n-1}\sum_{k\in \mathrm{Supp}(\vec{n})}\sum_{d|\{k,\vec{n}-\delta_k\}}\phi(d)\binom{\frac{n-1}{d}}{\frac{\vec{n}-\delta_k}{d}} + \frac{\chi(3|\vec{n})}{n}\binom{n/3}{n_1/3, n_2/3, \ldots} - \frac{2}{3n}\binom{n}{n_1, n_2, \ldots}.$$

Appendix.

To conclude this paper, we give here two tables giving the numbers of unlabelled solid 2-trees oriented and unoriented as well as the number of unlabelled τ-symmetric $\mathcal{B}$-structures. The first table gives these numbers according to the number n of edges for odd values of n from 1 up to 21, and the second, according to edge degree distribution for a few vectors $\vec{n}$. We use the notation $1^{n_1}2^{n_2}\cdots$, where i^{n_i} means n_i edges of degree i.

n	$\lvert\widetilde{\mathcal{A}}_o[n]\rvert$	$\lvert\widetilde{\mathcal{B}}_{\text{sym}}[n]\rvert$	$\lvert\widetilde{\mathcal{A}}[n]\rvert$
1	1	1	1
3	1	1	1
5	1	1	1
7	2	2	2
9	7	3	5
11	19	7	13
13	86	12	49
15	372	30	201
17	1825	55	940
19	9143	143	4643
21	47801	273	24037

TABLE 1. Number of solid 2-trees according to the number of edges

$\vec{n}$	$\lvert\widetilde{\mathcal{A}}_o[\vec{n}]\rvert$	$\lvert\widetilde{\mathcal{B}}_{\text{sym}}[\vec{n}]\rvert$	$\lvert\widetilde{\mathcal{A}}[\vec{n}]\rvert$
$1^7 2^1 3^1$	2	0	1
$1^8 2^2 3^1$	9	3	6
$1^{12} 2^1 3^1 4^1$	46	0	23
$1^{10} 5^1$	3	1	2
$1^{15} 4^1 5^1$	2	0	1
$1^{16} 3^2 5^1$	17	5	11
$1^{15} 2^2 7^1$	34	0	17

TABLE 2. Number of solid 2-trees according to edge degree distribution

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