Stirling Numbers Interpolation using Permutations with Forbidden Subsequences

G. Labelle *, P. Leroux *, E. Pergola †, R. Pinzani †

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Abstract. We present a family of number sequences which interpolates between the sequences B_n, of Bell numbers, and $n!$. It is defined in terms of permutations with forbidden patterns or subsequences. The introduction, as a parameter, of the number m of right–to–left minima yields an interpolation between Stirling numbers of the second kind $S(n,m)$ and of the first kind (signless) $c(n,m)$. Moreover, q-counting the restricted permutations by special inversions gives an interpolation between variants of the usual q-analogues of these numbers.

Résumé. Nous présentons une famille de suites de nombres qui interpole entre la suite B_n des nombres de Bell et la suite $n!$. Cette famille est définie en termes de permutations à motifs interdits. L'introduction comme paramètre du nombre d'éléments saillants minimums de gauche à droite donne une interpolation plus fine entre les nombres de Stirling de deuxième espèce $S(n,m)$ et de première espèce (sans signe) $c(n,m)$. De plus, un q-comptage de ces permutations selon des inversions particulières donne une interpolation entre des variantes des q-analogues habituels de ces nombres.

1 Introduction

The study of Stirling numbers and their q–analogues has a long history; in the last twenty years mathematicians have been interested in models giving combinatorial interpretations of classical relations involving the q–analogues of Stirling numbers. In 1961, Gould [13] gives expressions in terms of symmetric functions. A combinatorial treatment of q–Stirling numbers of second kind, involving finite dimensional vector spaces over a field $\mathcal{K}_q$ of cardinality q and inversions of restricted growth functions corresponding to set partitions is due

*LaCIM, Département de mathématiques, Université du Québec à Montréal, C.P. 8888, Succ. Centre–Ville, Montréal (Québec), Canada, H3C 3P8. e–mail: labelle.gilbert@uqam.ca, leroux.pierre@uqam.ca

†Dipartimento di Sistemi e Informatica, Universitá di Firenze, Via Lombroso 6/17, 50134 Firenze, Italy, e-mail: elisa@dsi.unifi.it, pinzani@dsi.unifi.it

to Milne [18, 19, 20]. In [11], Garsia and Remmel introduce particular rook placements in Ferrers boards. Later, Leroux [15] introduces 0–1 tableaux to prove the conjecture of Butler [8] concerning the q–log concavity for q–Stirling numbers, and De Médicis and Leroux [16, 17] study and generalize q–Stirling numbers of both kinds, using this interpretation. See also Wachs and White [25].

On the other hand, the study of permutations with forbidden subsequences has made meaningful progresses in the last thirty years: Simion and Schmidt have showed that the n–th Catalan number is the common value for the number of permutations with a single forbidden subsequence of length three [23]; Bóna in [5, 6] and Gessel in [12] provide some other results for permutations avoiding a single forbidden subsequence of length four. Concerning permutations avoiding a single subsequence of length greater than four, Regev [21] obtained an interesting result, that is: the number of permutations of length n avoiding the pattern $1\dots(k+1)$ is asymptotically equal to $c(k-1)^{2n}n^{(2k-k^2)/2}$, where c is a constant. Pell, Fibonacci, Motzkin and Schröder numbers are sequences which count permutations avoiding more than one forbidden subsequence. We refer to Guibert [14] and West [26] for an exhaustive survey on the results and on the tools used to study permutations with forbidden subsequences and to Bóna [7] for recent results.

In this paper we put these two research areas together. In particular, we give combinatorial interpretations of q–analogues of Stirling numbers of both kinds in terms of permutations with forbidden subsequences. More precisely, in the spirit of two previous works of Barcucci, Del Lungo, Pergola, Pinzani [3, 4], we introduce an infinite family $\{\mathcal{B}_n^j\}_{j\geq 1}$ of permutations with forbidden subsequences whose cardinalities interpolate between the Bell number B_n and $n!$. By considering right–to–left minima and j^{th}-kind inversions (see the definition in Section 2), this specializes to an interpolation between Stirling numbers of the second kind $S(n,m)$ and of the first kind (signless) $c(n,m)$ and their q–analogues. In fact, for j large $\mathcal{B}_n^j$ is the set of all permutations. For $j=1$, there is a simple bijection between $\mathcal{B}_n^1$ and set partitions of $\{1,2,\dots,n\}$ for which right–to–left minima of permutations correspond to blocks, and first-kind inversions, essentially to usual inversions in partitions.

In Section 2, we recall the concept of permutation with forbidden subsequences and generalize some classical definitions about permutations. We also recall the classical definitions of the q-analogues $S_q[n,m]$ and $c_q[n,m]$ of the Stirling numbers. In Section 3, we introduce a class of permutations with one forbidden subsequence, counted by the Bell numbers and we call them Bell permutations for this reason. This is the case $j=1$. These permutations avoid the subsequence $4\bar{1}32$; this is a natural extension of the forbidden pattern which consists of three decreasing elements in a permutation [23]. A bijection with set partitions is established and also the connection with the classical q–analogue. In Section 4, the forbidden subsequence characterizing Bell permutations is generalized, and we obtain an infinite family B^j of classes of permutations. The n–th term of each number sequence associated to the class lies between the

n–th Bell number and $n!$. An evaluation of Bell polynomials is obtained in the particular case $j = 2$, and the q–analogue is given a combinatorial interpretation. The permutations of length n counted by the n–th term of this sequence are in bijection with bicolored set partition on a $(n-1)$–element set, and both a recursive and a direct bijection is presented in Section 5. Section 6 contains enumerative results on the classes of permutations $\mathcal{B}^j = \bigcup_{n\geq 1} \mathcal{B}^j_n$, $j \geq 1$, and a combinatorial interpretation of polynomials $a^{(k,j)}_{n,m}(q)$ such that $a^{(m+1,1)}_{n,m}(q) = q^{n-m} S_q[n,m]$ and $a^{(n+1,\infty)}_{n,m}(q) = q^{n-m} c_q[n,m]$.

2 Notations and Definitions

In this section we recall the concepts of permutations with forbidden subsequences and generalize some classical definitions about permutations. In particular, the concept of j^{th}–kind inversion is introduced. We also recall the classical q-analogues of Stirling numbers and the concept of generating tree.

A permutation $\pi = \pi(1)\pi(2)\ldots\pi(n)$ on $[n] = \{1, 2, \ldots, n\}$ is a bijection from $[n]$ to $[n]$. Let S_n be the set of permutations on $[n]$. A permutation $\pi \in S_n$ *contains a subsequence of type* $\tau \in S_k$ if and only if a sequence of indices $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ exists such that $\pi(i_1)\pi(i_2)\ldots\pi(i_k)$ is ordered as τ. We denote the set of permutations of S_n *avoiding* subsequences of type τ by $S_n(\tau)$. The concept of permutation avoiding a subsequence of type τ can be extended to any totally ordered set ℓ, for example, for ℓ an l–element subset of $[n]$, we can use the notation $S_\ell(\tau)$ in this case.

Example 2.1 The permutation 58132674 belongs to $S_8(4321)$ because none of its subsequences of length 4 are of type 4321. This permutation does not belong to $S_8(4132)$ because there exist some subsequences of type 4132 like, for example, $\pi(2)\pi(3)\pi(6)\pi(8) = 8164$.

A *barred* subsequence $\bar{\tau}$ on $[k]$ is a permutation of S_k having a bar over one of its elements. Let τ be a permutation on $[k]$ identical to $\bar{\tau}$ but unbarred and $\hat{\tau}$ be the permutation on $[k-1]$ made up of the $(k-1)$ unbarred elements of $\bar{\tau}$, rewritten to be a permutation on $[k-1]$. A permutation $\pi \in S_n$ *contains a type* $\bar{\tau}$ *subsequence* if π contains a type $\hat{\tau}$ subsequence that, in turn, does not expand to a type τ subsequence. We denote the set of permutations of S_n not containing type $\bar{\tau}$ subsequences by $S_n(\bar{\tau})$ and we set $S(\bar{\tau}) = \bigcup_{n\geq 1} S_n(\bar{\tau})$ In words, $\pi \in S_n(\bar{\tau})$ if and only if any subsequence of type $\hat{\tau}$ of π can be extended to a subsequence of type τ.

Example 2.2 If $\bar{\tau} = 4\bar{1}32$ then $\tau = 4132$ and $\hat{\tau} = 321$. The permutation $\pi = 58132674$ belongs to $S_8(\bar{\tau})$ because all its subsequences of type $\hat{\tau}$:
$\pi(1)\pi(4)\pi(5) = 532$, $\pi(2)\pi(4)\pi5) = 832$, $\pi(2)\pi(6)\pi(8) = 864$ and
$\pi(2)\pi(7)\pi(8) = 874$ are subsequences of a sequence of type τ, because
$\pi(1)\pi(3)\pi(4)\pi(5) = 5132$, $\pi(2)\pi(3)\pi(4)\pi(5) = 8132$,
$\pi(2)\pi(4)\pi(6)\pi(8) = 8364$ and $\pi(2)\pi(5)\pi(7)\pi(8) = 8274$ are of type τ.

If we have the set $\tau_1 \in S_{k_1}, \ldots, \tau_p \in S_{k_p}$ of barred or unbarred permutations, we denote the set $S_n(\tau_1) \cap \ldots \cap S_n(\tau_p)$ by $S_n(\tau_1, \ldots, \tau_p)$ or by $S_n(F)$, if $F = \{\tau_1, \ldots, \tau_p\}$. For $\pi \in S_n$, we call *insertion sites* the $n+1$ positions lying on the left of $\pi(i)$, $1 \leq i \leq n$, and on the right of $\pi(n)$; the site i is the one on the left of $\pi(i)$ and the site $(n+1)$ is on the right of $\pi(n)$. The site i of $\pi \in S_n(F)$ is called *active* if the insertion of $n+1$ into the position between $\pi(i-1)$ and $\pi(i)$ gives a permutation belonging to the set $S_{n+1}(F)$; otherwise it is said to be *inactive*.

Example 2.3 The permutation $\pi = 58132674 \in S_8(4\bar{1}32)$ has 4 active sites that is the sites: 3, 5, 8 and 9. Indeed, the permutations: 589132674, 581392674, 581326794 and 581326749 belong to $S_9(4\bar{1}32)$, while the remaining sites are inactive, for example 581326974 has the subsequence 974 of type 321 but it is not a subsequence of a sequence of type 4132.

Let π be a permutation on $[n]$. The element $\pi(i)$, $1 \leq i \leq n$, is a *right–to–left minimum* if $\pi(i) < \pi(t)$, for all $t \in [i+1, n]$. This means that an index i_1, $i+1 \leq i_1 \leq n$, such that $\pi(i) > \pi(i_1)$ does not exist. We propose to generalize the concept of right–to–left minimum as follows: let π be a permutation on $[n]$; the element $\pi(i)$, $1 \leq i \leq n$, is a *j–th kind right–to–left minimum* if and only if a sequence of indices of length j: $i_1, \ldots, i_j$, $i+1 \leq i_1 < \ldots < i_j \leq n$, such that $\pi(i) > \pi(i_l)$, $1 \leq l \leq j$ does not exist. This implies that the j rightmost elements of π are trivially j–th kind right–to–left minima. Of course a right–to–left minimum is the same as a first kind right–to–left minimum while each element of the permutation is an ∞–kind right–to–left minimum. Hence the number of ∞–kind right–to–left minima is the length of the permutation.

Example 2.4 The permutation $\pi = 58132764$ has:

- 3 right–to–left minima: $\pi(3) = 1$, $\pi(5) = 2$ and $\pi(8) = 4$;
- 5 second kind right–to–left minima: $\pi(3) = 1$, $\pi(4) = 3$, $\pi(5) = 2$, $\pi(7) = 6$ and $\pi(8) = 4$;
- 6 third kind right–to–left minima: $\pi(3) = 1$, $\pi(4) = 3$ $\pi(5) = 2$, $\pi(6) = 7$, $\pi(7) = 6$ and $\pi(8) = 4$;
- 8 ∞–kind right–to–left minima.

Let π be a permutation on $[n]$. An *inversion* is a pair of indices, (s, t), $1 \leq s < t \leq n$, such that $\pi(s) > \pi(t)$; furthermore, we say that it is a *j–th kind inversion* if $\pi(t)$ is a j–th kind right–to–left minimum. Following this definition the classical concept of an inversion becomes an ∞–kind inversion, while the number of inversions with respect to the right–to–left minima are first kind inversions. We use the notation $\mathrm{inv}_j(\pi)$ to denote the number of j–th kind inversions of π.

Example 2.5 The permutation $\pi = 58132764$ of Example 2.4 has:

- 9 first kind inversions: $(1,3)$,$(1,5)$,$(1,8)$,$(2,3)$,$(2,5)$,$(2,8)$,$(4,5)$,$(6,8)$,$(7,8)$;
- 12 second kind inversions: $(1,3)$,$(1,4)$,$(1,5)$,$(1,8)$,$(2,3)$,$(2,4)$,$(2,5)$,$(2,7)$, $(2,8)$,$(4,5)$,$(6,8)$,$(7,8)$;
- 13 third kind inversions: $(1,3)$,$(1,4)$,$(1,5)$,$(1,8)$,$(2,3)$,$(2,4)$,$(2,5)$,$(2,6)$, $(2,7)$,$(2,8)$,$(4,5)$,$(6,8)$,$(7,8)$;
- 13 ∞–kind inversions: $(1,3)$,$(1,4)$,$(1,5)$,$(1,8)$,$(2,3)$,$(2,4)$,$(2,5)$,$(2,6)$, $(2,7)$,$(2,8)$,$(4,5)$,$(6,8)$,$(7,8)$.

Hence we have $\mathrm{inv}_1(\pi) = 9$, $\mathrm{inv}_2(\pi) = 12$, $\mathrm{inv}_3(\pi) = 13$, $\mathrm{inv}_\infty(\pi) = 13$.

The classical q-analogues of the Stirling numbers of the first kind (signless) $c_q[n,m]$ and of the second kind $S_q[n,m]$, as defined by Gould [13] in 1961, are characterized by the generating functions

$$\sum_{m=0}^{n} c_q[n,m] z^{n-m} y^m = \prod_{i=0}^{n-1} (y + [i]_q z), \tag{2.1}$$

$$\sum_{n=m}^{\infty} S_q[n,m] z^{n-m} = \prod_{i=1}^{m} \frac{1}{1-[i]_q z}, \tag{2.2}$$

where $[i]_q := 1 + q + \ldots + q^{i-1} = (q^i - 1)/(q-1)$ denotes the usual q-analogues of i. They satisfy the recurrences

$$c_q[n+1,m] = c_q[n,m-1] + [n]_q c_q[n,m], \tag{2.3}$$

$$S_q[n+1,m] = S_q[n,m-1] + [m]_q S_q[n,m]. \tag{2.4}$$

A combinatorial interpretation of the polynomial $S_q[n,m]$ as the generating function of the partitions of an n-element set into m blocks, where the variable q marks the "inversions", has been given by Milne [19, formula (4.9)], Leroux [15, formula (2.1)] and Wachs and White [25, the statistics **lb**]. The definition is as follows. Let p be a partition of the set $[n] = \{1, 2, \ldots, n\}$, written in standard form (see example 3.1). An *inversion* of p is a pair (α, β) of elements of $[n]$ such that $\alpha > \beta$, α appears to the left of β, and β is a block minimum. Let $\mathrm{inv}(p)$ denote the number of inversions of p. Then we have:

$$S_q[n,m] = \sum_{p \in \mathrm{Par}(n,m)} q^{\mathrm{inv}(p)},$$

where $\mathrm{Par}(n,m)$ denotes the set of partitions of $[n]$ into m blocks.

A combinatorial interpretation of the polynomials $c_q[n,m]$ is given by Leroux in [15].

The concept of generating tree was introduced by Chung, Graham, Hoggatt and Kleiman in [9] for the study of Baxter permutations. Later West applied it to the study of various permutations with forbidden subsequences [27]. Generating trees and succession rules can be used in combinatorics to deduce enumerative results about various combinatorial objects [1], permitting also their random generation [2].

The *generating trees* used in this paper to study permutations are rooted trees, labelled in $\mathbb{N}$, having the property that the labels of the set of children of each node x can be determined from the label of x itself. More precisely, such a generating tree is specified by a recursive definition consisting of:

1. the basis: the label of the root,

2. the inductive step: a set of succession rules that yields a multiset of labelled children which depends solely on the label of the parent. Moreover, the number of labelled children produced by a parent with label k, is exactly k; so the label size gives the degree of the node itself.

A succession rule can be used to describe the growth of the objects to which it is related and also to obtain the number sequence counting the objects themselves. The introduction of a parameter, say j, in a succession rule allows us to obtain a denumerable family of number sequences. In [3] the introduction of such a parameter into the classical succession rule for the Motzkin numbers allowed the authors to define number sequences such that the n–th number of each of them is lying between the n–th Motzkin and Catalan numbers. Moreover, the permutations enumerated by each number sequence are identified: they are permutations with two forbidden subsequences; the first, of length three, is fixed and the second has a length which increases with j. In [4] the introduction of the parameter j in the classical succession rule for the Catalan numbers defines number sequences such that the n–th term interpolates between the n–th Catalan number and $n!$. The objects that each sequence counts are permutations with $j!$ forbidden subsequences of length $(j+2)$.

3 Bell permutations and set partitions

The Stirling numbers of the second kind, denoted by $S(n,m)$, for $n \geq m \geq 0$, count the ways of partitioning a set of n objects into m nonempty subsets, called blocks. The number of partitions of an n–element set is given by the sum over m, $0 \leq m \leq n$, of $S(n,m)$; this defines the n–th Bell number, denoted by B_n [22]. For example, there are 7 ways of partitioning a 4–element set into two blocks:

$$\{1,2,3\}\ \{4\};\ \{1,2,4\}\ \{3\};\ \{1,3,4\}\ \{2\};\ \{1,2\}\ \{3,4\};\ \{1,3\}\ \{2,4\};\ \{1,4\}\ \{2,3\};\ \{1\}\ \{2,3,4\},$$

and the total number of partitions is

$$B_4 = \sum_{m=0}^{4} S(4,m) = 0+1+7+6+1 = 15. \text{ Note that } S(0,0) = B(0) = 1.$$

The standard representation of a given set partition consists in using the increasing order within each block and, in listing the blocks according to the increasing order of their minimum elements. We consider a new representation of the partition by moving the minimum element from the first to the last position in each block and then erasing the curly braces. The sequence of elements thus obtained is a permutation such that its (first kind) right–to–left minima are exactly the minimum elements of the blocks in the partition.

Example 3.1 Let us consider the following partition of an 8–element set into three blocks, written in standard form:

$$p = \{1,5,8\}\ \{2,3\}\ \{4,6,7\}.$$

The new representation described above is the permutation:

$$\pi(p) = 5\ 8\ \underline{1}\ 3\ \underline{2}\ 6\ 7\ \underline{4}$$

which has exactly three (underlined) right–to–left minima.

We observe that the permutation $\pi = \pi(p)$ obtained from a partition p of an n–element set contains a subsequence of type $\hat{\tau} = 321$ if and only if it is a subsequence of some sequence of type $\tau = 4132$. In other words, three indices i_1, i_2, i_3, $i_1 < i_2 < i_3$, such that $\pi(i_1) > \pi(i_2) > \pi(i_3)$ can be found in π if and only if it exists an index j, $i_1 < j < i_2 < i_3$, such that $\pi(i_1)\,\pi(j)\pi(i_2)\,\pi(i_3)$ is of type 4132. Such a condition is described by the forbidden subsequence $4\bar{1}32$. Let $\pi(i_1) < \cdots < \pi(i_m)$ be the m right–to–left minima of π; then $\pi(i_l)$, $1 \leq l \leq m$, is the first element of the l^{th} block in the corresponding partition, while the elements to the left of $\pi(i_l)$ and to the right of $\pi(i_{l-1})$ (if $l > 1$) are all the elements belonging to the l^{th} block of the partition. Permutations in $S_n(4\bar{1}32)$ with m right–to–left minima are counted by the Stirling numbers of the second kind, and $S_n(4\bar{1}32)$ is enumerated by the Bell numbers, B_n. Moreover, the first-kind inversions of π, i.e. the inversions with respect to the right–to–left minima, are essentially the same as the classical inversions of the original partition p. The difference here is that each non minimum element contributes one more to the inversions since in $\pi(p)$ it is written to the left of the minimum element of its block. Hence we have

$$\mathrm{inv}_1(\pi(p)) = \mathrm{inv}(p) + n - m, \tag{3.1}$$

if the partition has m blocks, and we see that the q-counting of permutations in $S_n(4\bar{1}32)$ with m right–to–left minima, according to the number of first-kind inversions, is given by $q^{n-m}S_q[n,m]$.

Proposition 3.1 *We have*

$$\sum_{\pi \in S_n(4\bar{1}32)} q^{\mathrm{inv}_1(\pi)} \quad = \quad q^{n-m}S_q[n,m]. \tag{3.2}$$

The first construction we take into consideration for the class $S(4\bar{1}32)$ is a recursive construction which allows to obtain $S_{n+1}(4\bar{1}32)$, starting with $S_n(4\bar{1}32)$. It uses the concept of active site of a permutation introduced in Section 2. Let $\pi \in S_n(4\bar{1}32)$ and $i_1 < i_2 < ... < i_{m-2} < n$ be the indices of its $(m-1)$ right–to–left minima, namely $\pi(i_1), \pi(i_2), ..., \pi(i_{m-2}), \pi(n)$. The active sites of π are the sites on the immediate left of each right–to–left minimum and on the right of the last element, that is, sites of π are $i_1, i_2, \ldots, i_{m-2}$, n and $n+1$. Indeed, the insertion of $n+1$ into the site $(n+1)$ does not cause any occurrence of the forbidden subsequence 321; by inserting $n+1$ into the site l, $l = i_1, ..., i_{m-2}, n$ we can obtain the forbidden subsequences 321 if and only if there exist two indices t_1, t_2 such that $l < t_1 < t_2$ and $n+1 > \pi(t_1) > \pi(t_2)$, but in this case $n+1\pi(l)\pi(t_1)\pi(t_2)$ is of type 4132. Each other site is inactive: take a site lying on the left of $\pi(i)$ that is not a right–to–left minimum. This means that there exists $i_1 > i:\ \pi(i) > \pi(i_1)$, and the insertion of $n+1$ on the left of $\pi(i)$ gives $n+1\pi(i)\pi(i_1)$, that is a decreasing sequence of length three, with $n+1$ and $\pi(i)$ adjacent elements and we get a forbidden subsequence 321. Observe that the insertion of $n+1$ into the site $n+1$ increases the number of right–to–left minima of π while each other insertion does not change this number in the permutation. The above arguments prove the following proposition:

Proposition 3.2 *Let $\pi \in S_n(4\bar{1}32)$ be a permutation with $k \geq 2$ active sites, that is the sites i_1, i_2, $\ldots$, i_{k-2}, n and $(n+1)$. Then the number of active sites is still k in the permutation obtained by inserting $n+1$ into any active site different from the rightmost one; the permutation obtained from π by inserting $n+1$ into the site $n+1$ has $(k+1)$ active sites: i_1, i_2, $\ldots$, i_{k-2}, n, $(n+1)$ and $(n+2)$.*

If we classify the permutations of $S_n(4\bar{1}32)$, $n \geq 1$, according to their number of active sites then we can synthetically describe this recursive construction by the succession rule:

$$\begin{cases} basis: & (2) \\ inductive\ \ step: & (k) \to (k)^{k-1}(k+1), \end{cases} \tag{3.3}$$

since the only permutation of $S_1(4\bar{1}32)$, that is 1, has two active sites.

The expansion of this succession rule gives the generating tree of Fig. 1, where the active sites are denoted by "_". Consequently if $p_{n,k} = |\{\pi \in S_n(4\bar{1}32) :$ π has k active sites$\}|$ then

$$\begin{cases} p_{1,2} & = & 1, \\ p_{n+1,k} & = & p_{n,k-1} + (k-1)p_{n,k}, \quad 2 \leq k \leq n+2, \end{cases} \tag{3.4}$$

which is the recursive relation of the Stirling numbers of the second kind (see Comtet [10]), replacing $p_{n,k}$ by $S(n, k-1)$. Moreover the number of new first-kind inversions which are created by inserting $n+1$ into the k active sites $i_1, i_2, \ldots, i_{k-2}, n$, and $n+1$ is respectively $k-1, k-2, \ldots, 2, 1$, and 0. Hence the polynomial $p_{n,k}(q)$ which q-counts these permutations according to first-kind inversions satisfies the recurrence

$$p_{n,k}(q) \quad = \quad p_{n,k-1}(q) + q[k-1]_q p_{n,k}(q). \tag{3.5}$$

This is coherent with our previous observation that $p_{n,m+1}(q) = q^{n-m} S_q[n,m]$ using (2.4).

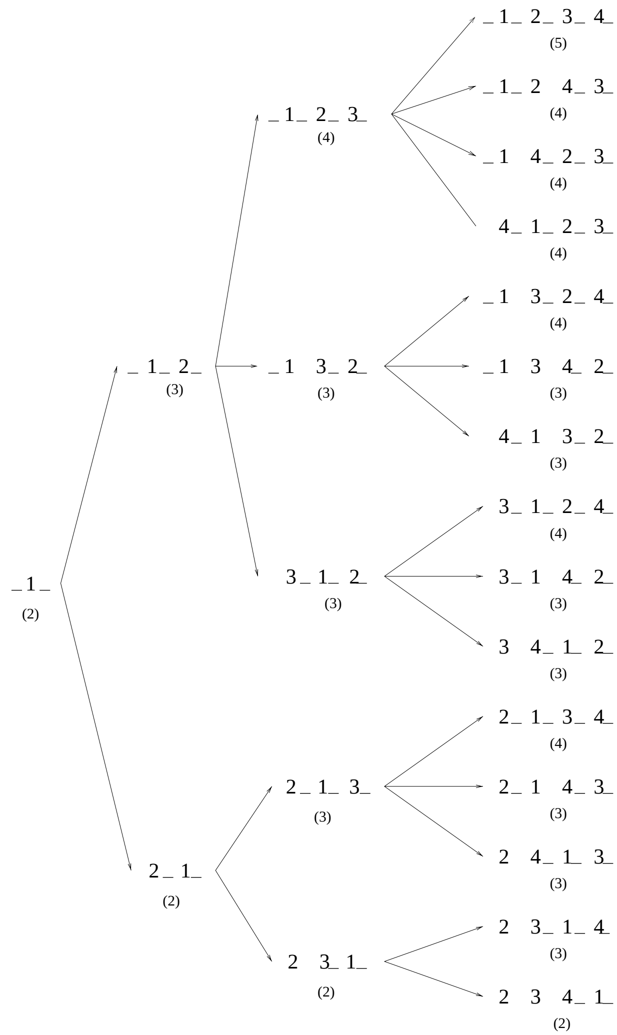

Figure 1: The generating tree for $4\bar{1}32$–avoiding permutations.

Proposition 3.3 *Permutations in $S_n(4\bar{1}32)$ are counted by the n–th Bell number, $S(n,m)$ counts the number of permutations in $S_n(4\bar{1}32)$ with m right–to–*

left minima or equivalently with $(m+1)$ active sites, and $q^{n-m}S_q[n,m]$ q-counts these permutations according to first-kind inversions.

Definition 3.4 *Permutations in $S_n(4\bar{1}32)$ are called Bell permutations.*

Another approach in order to generate $S(4\bar{1}32)$ permutations, is to construct $S_{n+1}(4\bar{1}32)$, using $S_1(4\bar{1}32), S_2(4\bar{1}32), ..., S_n(4\bar{1}32)$. Indeed, the permutations in $S_{n+1}(4\bar{1}32)$ with m right–to–left minima can be obtained in the following way. For each l such that $0 \leq l \leq n$:

- extract an l–element subset ℓ from the set $\{2, ..., n+1\}$,
- construct the permutations in $S_\ell(4\bar{1}32)$ with $(m-1)$ right–to–left minima,
- add the element 1 on its left,
- place on the left of 1 the remaining $(n-l)$ elements in an increasing order.

Here the notation $S_\ell(\tau)$ refers to the permutations of the totally ordered set ℓ which avoid the pattern τ. The principle of this approach is represented by Fig. 2.

This implies that:

$$p_{n+1,k+1} = \sum_{l=m}^{n} \binom{n}{l} p_{l,k}. \tag{3.6}$$

As $p_{n,m+1} = S(n,m)$ we obtain a combinatorial interpretation of the well known relation involving the second kind Stirling numbers (see Comtet, [10]), by means of Bell permutations. There is a q-analogue of (3.6) due to Mercier and Sundaram (see [16, formula (2.12)]), whose combinatorial proof uses the concept of non-inversions of a partition p. We can define the concept of (first-kind) non-inversions for permutations as follows: a *non-inversion* of π is a pair (i,j), with $i<j$ such that $\pi(i)$ is a right–to–left minimum and $\pi(i) < \pi(j)$. For $\pi = \pi(p)$, this corresponds to the statistics **ls** of [6]. Let us denote by $\overline{S}_q[n,m]$ the polynomial which q-counts the permutations in $S_n(4\bar{1}32)$ having m right–to–left minima, according to non-inversions. Our first recursive construction of permutations in $S_{n+1}(4\bar{1}32)$, which inserts the element $n+1$ in permutations of $S_n(4\bar{1}32)$ shows that

$$\overline{S}_q[n+1,m] \quad = \quad q^{m-1}\overline{S}_q[n,m-1] + [m]_q\overline{S}_q[n,m]. \tag{3.7}$$

This implies that in fact

$$\overline{S}_q[n,m] \quad = \quad q^{\binom{m}{2}} S_q[n,m]. \tag{3.8}$$

The q-analogue of (3.6) is then given by

$$\overline{S}_q[n+1,m] \quad = \quad \sum_{l=m-1}^{n} \binom{n}{l} q^l \overline{S}_q[l,m-1], \tag{3.9}$$

since, in the second construction of $S_{n+1}(4\bar{1}32)$ described above, and summarized by Figure 2, the only new non-inversions that are created come from the l elements which are to the right of 1.

By summing over m, we obtain the q-analogue $B_n(q)$ of the Bell numbers as defined by Milne in [15],

$$B_n(q) \quad = \quad \sum_{m=0}^{n} \overline{S}_q[n,m] \tag{3.10}$$

with the combinatorial interpretation

$$B_n(q) \quad = \quad \sum_{\pi \in S_n(4\bar{1}32)} q^{\mathrm{nin}(\pi)} \tag{3.11}$$

where $\mathrm{nin}(\pi)$ denotes the number of non-inversions of π. Moreover, the second construction of the permutations in $S_n(4\bar{1}32)$, giving rise to (3.9) also yields the reccurence formula

$$B_{n+1}(q) \quad = \quad \sum_{l=0}^{n} \binom{n}{l} q^l B_l(q), \tag{3.12}$$

which is due to Milne [18] and extends the classical recursion for Bell numbers (see [10]).

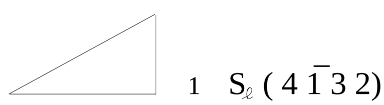

Figure 2: The second construction for $S(4\bar{1}32)$ permutations.

4 Generalized Bell permutations

In this section we introduce a parameter j in the succession rule (3.3) giving the Bell numbers. Each value of j yields a number sequence such that the n–th term lies between B_n and $n!$. We are interested in characterizing the permutations enumerated by these number sequences.

Let us carefully examine the succession rule (3.3): the "exponents" of the terms on the right hand side of the inductive step are $k-1$ for the label (k) and 1 for the label $(k+1)$. We can make these "exponents" depend on a parameter j, thus giving the "exponent" $k-j$ to the label (k) and j to the label $(k+1)$; moreover if $k \leq j$ then only the label $(k+1)$ is obtained exactly k times. The exact form of the succession rule we obtain is

$$\begin{cases} basis: & (2) \\ inductive \;\; step: & (k) \rightarrow (k+1)^k, \qquad\qquad k \leq j \\ inductive \;\; step: & (k) \rightarrow (k)^{k-j}(k+1)^j, \quad k > j. \end{cases} \tag{4.1}$$

It is easy to verify that if $j = 1$, then the succession rule (4.1) reduces to (3.3).

In (4.1) the "exponent" of a label means the number of times the label must be repeated and the number of terms on the right hand side of the inductive step is exactly k. The idea is to perform (4.1) on permutations and try to characterize the class we obtain. The first step is to give an interpretation of (4.1) in terms of active sites in a permutation; we have to decide how the active sites are modified when a new element is added into a permutation with a fixed number of active sites. The second step is to describe the resulting permutations in terms of forbidden subsequences. We refer to the first active site as the leftmost active site in the permutation and so on, and we make the following choices:

- if a new element is inserted in the l^{th} active site, $l \leq k-j$, then the site on the left of the inserted element is inactive and the number of active sites do not change in the new permutation (see Fig 3, (case $\bullet$)),
- if a new element is inserted in the l^{th} active site, $l \geq k-j+1$, then the site on the left of the inserted element is also active and the number of active sites grows by one (see Fig. 3, (case $\circ$)).

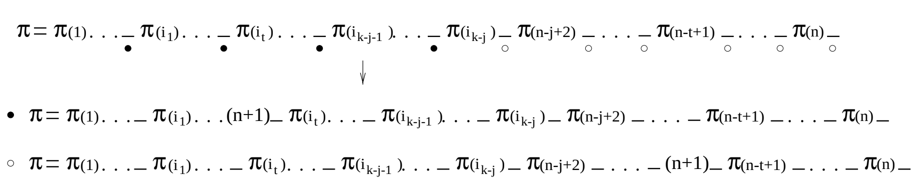

Figure 3: The active sites in the permutations obtained from a permutation of length n with $(k-1)$ right–to–left minima of j–th kind, by inserting $n+1$.

The permutations we obtain avoid the subsequences $(j+2)(j+1)\sigma$ where $\sigma \in S_j$ and the elements corresponding to $(j+2)$ and $(j+1)$ are consecutive. In terms of permutations with forbidden subsequences such a condition is given by the union of j sets of permutations with forbidden subsequences: $\bigcup_{i=1}^{j} S\left(\bar{F}_i^j\right)$ where $\bar{F}_i^j$ is a set of barred subsequences $\bar{\tau} = (j+3)\bar{i}(j+2)\sigma_i$ with σ_i a permutation on the set $\{(j+1), ..., (i+1), (i-1), ..., 1\}$; so $\left|\bar{F}_i^j\right| = j!$ and $|\bar{\tau}| = j+3$.

Example 4.1 Let $j = 2$ then $1 \leq i \leq 2$. The set $\bar{F}_2^2$ obtained for $i = 2$ is $\{5\bar{2}431, 5\bar{2}413\}$.

Let us note that in the union i can assume all values between 1 and j. This means that we are not interested in the value of the element lying between $(j+3)$ and $(j+2)$, but at least one element must exist between $(j+3)$ and $(j+2)$. Such a condition avoids subpatterns of two adjacent decreasing elements having at least j smaller elements on their right. Moreover, i cannot be equal to $(j+1)$ because the subsequence $(j+3)(j+1)\sigma$ (σ being a permutation of length j) is of the forbidden type. Let $\mathcal{B}^j$ be the class of permutations defined by $\mathcal{B}^j = \bigcup_{n\geq 1} \mathcal{B}^j_n$ where $\mathcal{B}^j_n = \bigcup_{i=1}^{j} S_n(\bar{F}^j_i)$.

Proposition 4.1 *For $j \geq 1$, let π be a permutation in $\mathcal{B}^j_n$ having $k \geq 2$ active sites: $i_1, \ldots, i_{k-j}, n-j+2, n-j+1, \ldots, n+1$. Then the number of active sites does not change in the permutation obtained by inserting $n+1$ into the site i_t, $t = 1, ..., k-j$; the permutation obtained from π by inserting $n+1$ into the site $(n+1-t)$, $0 \leq t \leq j-1$, has $(k+1)$ active sites: $i_1, \ldots, i_{k-j}, n-j+2, \ldots, n+1, n+2$.*

Corollary 4.2 *The class $\mathcal{B}^j$ has a recursive construction described by the succession rule* (4.1)*.*

5 Bicolored set partitions and permutations

In Section 3 we illustrated the case $j = 1$, that is we showed that $4\bar{1}32$–avoiding permutations are counted by the Bell numbers and gave a bijection with set partitions. We also presented q-analogues. For $j = 2$ we now show that the number of $\mathcal{B}^2$-permutations, that is of $(5\bar{1}432, 5\bar{1}423)$ or $(5\bar{2}431, 5\bar{2}413)$–avoiding permutations are values of Bell polynomials whose n–th term is defined by $\sum_{m\geq 0} 2^m S(n-1, m)$ (see [24], sequence M1662). These numbers count bicolored set partitions (that is to say each block can be red or black) and there is a bijection between these two classes of structures. This correspondence can be easily obtained by applying the succession rules:

$$\begin{cases} basis: & (2) \\ inductive \;\; step: & (k) \rightarrow (k)^{k-2}(k+1)^2, \quad k \geq 2, \end{cases} \tag{5.1}$$

to the bicolored set partitions, obtaining a recursive bijection. In bicolored set partitions the label k represents the number of blocks plus two. Given an n–element set bicolored partition with $k-2$ blocks, labeled by (k), we can add on its right the block $\{(n+1)\}$ that can be red or black and in this case the number of blocks becomes $k-1$, so the label of these new partitions is $(k+1)$; or we can insert $n+1$ into any of the blocks of the partition, the color remaining the same. This bijection is represented in Fig. 4, where the red blocks are those with the underlined elements.

Under this bijection $p \longmapsto \pi(p)$ between bicolored set partitions and $\mathcal{B}^2$-permutations, we have the following parameter correspondences:

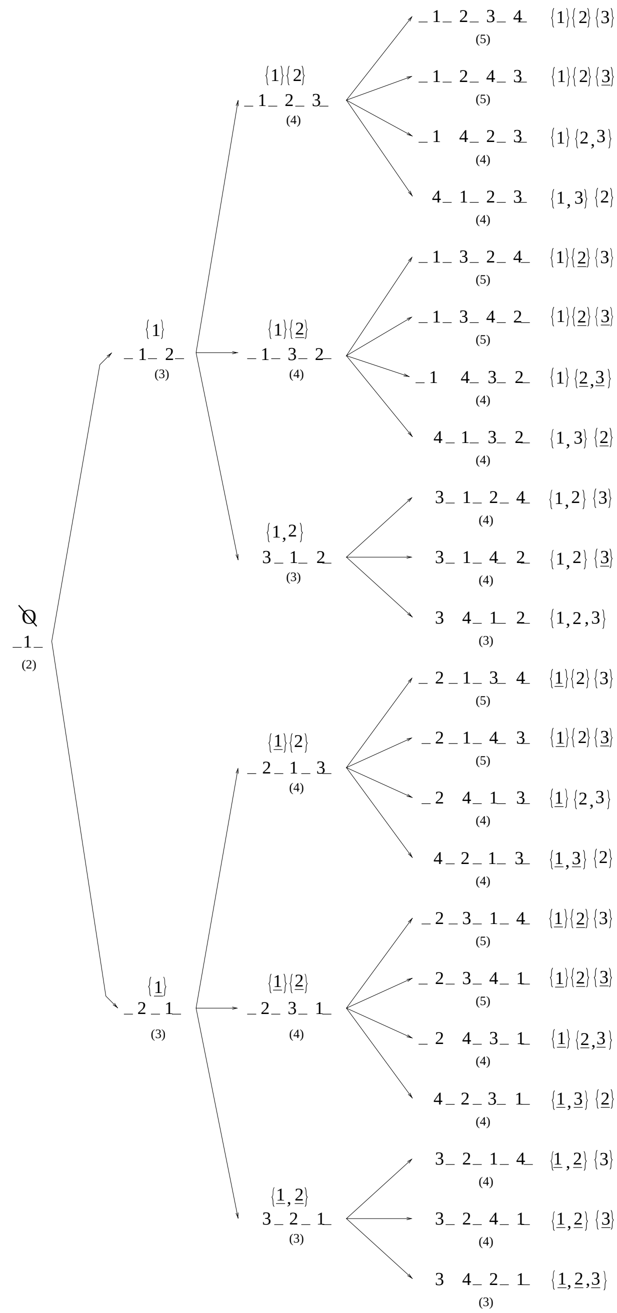

Figure 4: The first four levels of the generating tree for permutations in $\mathcal{B}^2$ and the constructive bijection with the bicolored set partitions.

bicolored set partitions	$\mathcal{B}^2$-permutations
cardinality of the partitioned set $+1$	$n =$ length of the permutations
nb of black blocks	nb of right–to–left minima -1
$m =$ nb of blocks	nb of second kind right–to–left minima -1
nb of red blocks + nb of inversions $+2(n-m-1)$	nb of second–kind inversions

In particular, if p is a colored partition of $[n-1]$ having m blocks, r of which are red, then we have

$$\mathrm{inv}_2(\pi(p)) \quad = \quad \mathrm{inv}(p) + 2(n-m-1) + r \tag{5.2}$$

and it follows that the q-counting, with respect to second-kind inversions, of the set $\mathcal{B}^2_{n,m}$ of permutations in $\mathcal{B}^2_n$ having $m+1$ second-kind left–to–right minima is given by

$$\sum_{\pi \in \mathcal{B}^2_{n,m}} q^{\mathrm{inv}_2(\pi)} \quad = \quad q^{2(n-m-1)} S_q[n-1,m](1+q)^m. \tag{5.3}$$

The standard representation for bicolored set partitions is the same as for the set partition, but, in this case, the blocks can be red or black. In order to directly obtain a permutation in $S_n(5\bar{1}432, 5\bar{1}423) \bigcup S_n(5\bar{2}431, 5\bar{2}413)$ from a $(n-1)$–element set bicolored partition we consider a new representation of the partition. It is obtained by performing the following steps:

1. shift each number in the bicolored set partition of one unit obtaining a $(n-1)$–element set bicolored partition on $\{2, \ldots, n\}$;
2. move the minimum element from the first to the last position into each block;
3. add on the left of the resulting partition the black block $\{1\}$;
4. erase the curly braces but maintain the color of numbers;
5. starting from the left to the right, place each black number which is a right–to–left minimum in the position on the left of the position occupied by the nearest black number on its right which is a right–to–left minimum; the last (right-most) black left–to–right minimum should be at the extreme right, jumping over red elements if necessary;
6. use only the black color for the numbers in the obtained sequence.

Example 5.1 The following bicolored set partition p, where the red elements are underlined,

$$p = \{1,4\}\{\underline{2},\underline{3},\underline{7}\}\{\underline{5},\underline{8},\underline{9}\}\{6,11\}\{\underline{10}\}, \tag{5.4}$$

bijectively corresponds to the permutation:

$$\pi(p) = 5\ 1\ 4\ 8\ 3\ 9\ 10\ 6\ 12\ 2\ 11\ 7. \tag{5.5}$$

As a matter of fact, this is the final result obtained by performing the above described 6 steps on (5.4) as follows:

1. $\{2, 5\}\{\underline{3}, \underline{4}, \underline{8}\}\{\underline{6}, \underline{9}, \underline{10}\}\{7, 12\}\{\underline{11}\}$;
2. $\{5, 2\}\{\underline{4}, \underline{8}, \underline{3}\}\{\underline{9}, \underline{10}, \underline{6}\}\{12, 7\}\{\underline{11}\}$;
3. $\{1\}\{5, 2\}\{\underline{4}, \underline{8}, \underline{3}\}\{\underline{9}, \underline{10}, \underline{6}\}\{12, 7\}\{\underline{11}\}$;
4. $1\ 5\ 2\ \underline{4}\ \underline{8}\ \underline{3}\ \underline{9}\ \underline{10}\ \underline{6}\ 12\ 7\ \underline{11}$;
5. $5\ 1\ \underline{4}\ \underline{8}\ \underline{3}\ \underline{9}\ \underline{10}\ \underline{6}\ 12\ 2\ \underline{11}\ 7$;
6. $5\ 1\ 4\ 8\ 3\ 9\ 10\ 6\ 12\ 2\ 11\ 7$.

These 6 steps can be performed in an inverse order allowing us to pass from a particular number sequence in $S_n(5\bar{1}432, 5\bar{1}423) \bigcup S_n(5\bar{2}431, 5\bar{2}413)$ to a bicolored set partition in one and only one way. In particular it suffices to search second kind right–to–left minima from the permutation in order to perform steps 6 and 5 in reverse order.

Moreover, the permutation $\pi(p)$ that we obtain belongs to $S_n(5\bar{1}432, 5\bar{1}423) \bigcup S_n(5\bar{2}431, 5\bar{2}413)$ because the sequence of numbers does not contain two consecutive decreasing elements having on their right two smaller elements.

If $j = \infty$, then we obtain all permutations and $n!$ appears. Moreover the ∞-kind inversions are simply the usual inversions in a permutation. Let $c_{n,m}(q)$ denote the polynomial obtained by q-counting by inversions the permutations of $[n]$ having m right–to–left minima. The recursive construction of these permutations, inserting the element $n+1$ into one of the $n+1$ active sites, shows that

$$c_{n+1,m}(q) = c_{n,m-1}(q) + q[n]_q c_{n,m}(q). \tag{5.6}$$

It follows, using (2.3), that

$$c_{n,m}(q) = q^{n-m} c_q[n, m] \tag{5.7}$$

and, summing over m, we find that

$$[n]!_q = \sum_{m=0}^{n} q^{n-m} c_q[n, m], \tag{5.8}$$

where $[n]!_q = \prod_{i=1}^{n} [i]_q$ is the q-analogues of $n!$.

For each other value of $j \geq 3$ we obtain sequences of numbers such the n–th term of each of them is between B_n and $n!$ (see Fig. 5). These sequences do not appear in the Sloane–Plouffe book [24]: "The Encyclopedia of Integer Sequences".

Index	Family of permutations	Numbers
j = 1	$S_n(4\bar{1}32)$	1 2 5 15 52 203 877 4140 21147 115975 678570 ...
j = 2	$S_n(5\bar{1}432, 5\bar{1}423) \cup S_n(5\bar{2}431, 5\bar{2}413)$	1 2 6 22 94 454 2430 14214 89918 610182 4412798 ...
j = 3	$S_n(6\bar{1}5432, 6\bar{1}5423, 6\bar{1}5342, 6\bar{1}5324, 6\bar{1}5243, 6\bar{1}5234) \cup$ $S_n(6\bar{2}5431, 6\bar{2}5413, 6\bar{2}5341, 6\bar{2}5314, 6\bar{2}5143, 6\bar{2}5134) \cup$ $S_n(6\bar{3}5421, 6\bar{3}5412, 6\bar{3}5241, 6\bar{3}5214, 6\bar{3}5142, 6\bar{3}5124)$	1 2 6 24 114 618 3732 24702 177126 1363740 11195286 ...
j = 4	⋮	1 2 6 24 120 696 4536 32568 254136 2133816 19130040 ...
j = 5	⋮	1 2 6 24 120 720 4920 37320 309120 2763720 26440920 ...
.	⋮	⋮
.	⋮	
$j = \infty$	S_n	1 2 6 24 120 720 5040 40320 362880 3628800 39916800 ...

Figure 5: Table of permutations.

6 Enumerative results for $\mathcal{B}^j$-permutations

For each j, we are interested in the enumeration of the permutations in $\mathcal{B}^j$ according to the length, number of right–to–left minima and the number of j–th kind inversions. The reason we introduce this last parameter is to give a combinatorial interpretation of the q–analogue that we obtain in a natural way from (4.1) by giving a "weight" to the label on the right–hand side of each inductive step in (4.1). More precisely the i–th child of a label (k) is given the weight q^{k-i}.

Let $a_k^j(x,y,q)$ be the generating function of $\mathcal{B}^j$–permutations with k active sites, according to their length (variable x), the number of right–to–left minima (y) and the number of j–th kind inversions (q). A detailed analysis of the parameter modifications in the recursive construction of the permutations yield the following recursive relations for $a_k^j(x,y,q)$:

$$
\begin{aligned}
a_2^j(x,y,q) &= xy, \\
a_k^j(x,y,q) &= xya_{k-1}^j(x,y,q) + xq[k-2]_q a_{k-1}^j(x,y,q), 3 \le k \le j, \qquad (6.1) \\
a_k^j(x,y,q) &= xya_{k-1}^j(x,y,q) + xq[j-1]_q a_{k-1}^j(x,y,q) + xq^j[k-j]_q a_k^j(x,y,q), \\
&\qquad k \ge j+1;
\end{aligned}
$$

Solving the recursions, we obtain the following:

Proposition 6.1 *The generating function $a_k^j(x,y,q)$ for $\mathcal{B}^j$–permutations verify:*

$$
\begin{aligned}
a_k^j(x,y,q) &= x^{k-1} \prod_{i=0}^{k-2} (y + q[i]_q), & 2 \le k \le j; \\
a_k^j(x,y,q) &= x^{k-1} (y + q[j-1]_q)^{k-j} \frac{\prod_{i=0}^{j-2} (y+q[i]_q)}{\prod_{i=1}^{k-j} (1-xq^j[i]_q)}, & k \ge j+1.
\end{aligned}
$$

The coefficient $[x^n y^m]\, a_k^j(x,y,q)$ gives a polynomial in q-counting the $\mathcal{B}^j$–permutations of length n, having m right–to–left minima and k active sites, according to their number of j–th kind inversions.

Corollary 6.2 *Let $a_{n,m}^{(k,j)}(q) = [x^n y^m]\, a_k^j(x,y,q)$, $m \leq k-1$; then we have*

$$a_{n,m}^{(k,j)}(q) = \delta_{n,k-1} c_q[k-1,m] q^{k-1-m}, \quad 2 \leq k \leq j; \tag{6.2}$$

$$a_{n,m}^{(k,j)}(q) = q^{j(n+1-k)+(k-m-1)} S_q[n+1-j,k-j]\left([j-1]_q\right)^{k-j-m} \sum_{i=0}^{j-1} \binom{k-j}{m-i} c_q[j-1,i]\left([j-1]_q\right)^i, \quad k \geq j+1; \tag{6.3}$$

where $\delta_{i,j}$ is the Kronecker delta.

Let us now examine the polynomials $a_{n,m}^{(k,j)}(q)$ for some particular values of j.

- If $j=1$, then equation (6.3) should be used and the result is different from 0 if and only if the exponent of $([j-1]_q) = 0$ is zero, that is $k = m+1$. Once n and m are fixed the only possibility is $a_{n,m}^{(m+1,1)}(q) = S_q[n,m]q^{n-m}$ which confirms the results of Section 3.
- If $j=2$ then equations (6.2) and (6.3) give:

  $$\begin{array}{rcl} a_{1,1}^{(2,2)}(q) & = & 1, \\ a_{n,m}^{(k,2)}(q) & = & \binom{k-2}{m-1} S_q[n-1,k-2] q^{2n+1-k-m}, \quad k \geq 3. \end{array}$$

  By summing over m we obtain the polynomials for the permutations with forbidden subsequences $(5\bar{1}432, 5\bar{1}423)$ or $(5\bar{2}431, 5\bar{2}413)$ of length n having k active sites according to the number of their second kind inversions:

  $$\sum_{1 \leq m \leq n} a_{n,m}^{(k,2)}(q) = q^{2(n+1-k)} S_q[n-1,k-2](1+q)^{k-2}, \quad n \geq 2. \tag{6.4}$$

  This is coherent with (5.3) since k active sites in $\pi(p)$ corresponds to $k-2$ blocks in p.
- If $j=\infty$ then equation (6.2) gives:

  $$a_{n,m}^{(n+1,\infty)}(q) = c_q[n,m]q^{n-m}, \qquad n \geq 1,$$

  as expected.

The meaning of "Stirling numbers interpolation" lies in the observation that the permutations of length n having m right–to–left minima are counted by the second kind Stirling numbers for $j=1$ and by the first kind Stirling numbers $c(n,m)$ for $j=\infty$. In the intermediate cases this number, $p_{n,m}^{(j)}$, is such that $S(n,m) \leq p_{n,m}^{(j)} \leq c(n,m)$, and it is given by

$$p_{n,m}^{(j)} = \sum_{k=2}^{n} a_{n,m}^{(k,j)}(1), \tag{6.5}$$

where $a_{n,m}^{(k,j)}(1)$ can be computed by setting $q = 1$ in (6.3).

References

[1] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, ECO: A Methodology for the Enumeration of Combinatorial Objects, *Journal of Difference Equations and Applications*, Vol. 5 (1999) 435–490.

[2] E. Barcucci, A. Del Lungo, E. Pergola, Random generation of tree and other combinatorial objects, *Theoretical Computer Science*, 218 (1999) 219-232.

[3] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, From Motzkin to Catalan permutations, *Discrete Mathematics*, 217 (2000) 33–49.

[4] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, Permutation avoiding an increasing number of length–increasing forbidden subsequences, *Discrete Mathematics and Theoretical Computer Science*, 4 (2000) 31-44.

[5] M. Bóna, Permutations avoiding certain patterns. The case of length 4 and some generalizations, *Discrete Mathematics*, 175 (1997) 55–67.

[6] M. Bóna, Exact enumeration of 1342–avoiding permutations; a close link with labelled trees and planar maps, *Journal of Combinatorial Theory Series A*, 80 (1997) 257–272.

[7] M. Bóna, The solution of a Conjecture of Stanley and Wilf for all layered patterns, *Proceedings of 11th FPSAC*, Barcelone (1999) 62–71.

[8] L. M. Butler, The q–log concavity of q–binomial coefficients, *Journal of Combinatorial Theory Series A*, 54 (1990) 53–62.

[9] F.R.K. Chung, R.L. Graham, V.E. Hoggatt, M. Kleiman, The number of Baxter permutations, *Journal of Combinatorial Theory, Series A*, 24 (1978) 382–394.

[10] L. Comtet, *Advanced Combinatorics*, Reidel (1979).

[11] A. M. Garsia, J. B. Remmel, q–Counting rook configurations and a formula of Frobenius, *Journal of Combinatorial Theory Series A*, 41 (1986) 246–275.

[12] I. M. Gessel, Symmetric functions and P–recursiveness, *Journal of Combinatorial Theory Series A*, 53 (1990) 257–285.

[13] H. W. Gould, The q–Stirling numbers of first and second kinds, *Duke Math. Journal*, 28 (1961) 281–289.

[14] O. Guibert, Combinatoire des permutations à motifs exclus en liaison avec les mots, les cartes planaires et les tableaux de Young, *Thèse de l'Université Bordeaux 1* (1996).

[15] P. Leroux, Reduced matrices and q–log concavity properties of q–Stirling numbers, *Journal of Combinatorial Theory Series A*, 54 (1990) 64–84.

[16] A. De Médicis, P. Leroux, A unified combinatorial approach for q-(and p, q–) Stirling numbers, *Journal of Statistical Planning and Inference*, 34 (1993) 89–105.

[17] A. De Médicis, P. Leroux, Generalized Stirling numbers, convolution formulae and p, q–analogues, *Canadian Journal of Mathematics*, 47 (1995) 474–499.

[18] S.C. Milne, A q–analog of restricted growth functions, Dobinski's equality, and Charlier polynomials, *Trans. Amer. Math. Soc.*, 245 (1978) 89–118.

[19] S.C. Milne, Restricted growth functions, rank row matchings of partition lattices, and q–Stirling numbers, *Advances in Mathematics*, 43 (1982) 173–196.

[20] S.C. Milne, Mapping of subspaces into subsets, *Journal of Combinatorial Theory Series A*, 33 (1982) 36–47.

[21] A. Regev, Asymptotic values for degrees associated with strips of Young diagrams, *Advances in Mathematics*, 41 (1981) 115–136.

[22] J. Riordan, *An introduction to combinatorial analysis*, Wiley (1958).

[23] R. Simion, F. W. Schmidt, Restricted permutations, *European Journal of Combinatorics*, 6 (1985) 383–406.

[24] N. Sloane, S. Plouffe, *Encyclopedia of Integer Sequences*, Academic Press, New York (1995).

[25] M. Wachs, D. White, p, q–Stirling numbers and set partition statistics, *Journal of Combinatorial Theory Series A*, 56 (1991), 27–46.

[26] J. West, Permutations with forbidden subsequences and stack–sortable permutations, *PhD thesis, Massachusetts Institute of Technology*, Cambridge (1990).

[27] J. West, Generating trees and forbidden subsequences, *Discrete Mathematics*, 157 (1996) 363–374.