Generalized binomial coefficients underlying generalized Fibonacci sequences

R. L. Ollerton
University of Western Sydney, Penrith Campus DC1797, Australia

A. G. Shannon
KvB Institute of Technology, North Sydney 2060, and
Warrane College, The University of NSW, Kensington 1465, Australia

1. Introduction

The Fibonacci sequence arises naturally as the diagonal sums of the binomial coefficient array with terms ${}^nC_m$ (where n gives rows, m gives columns). In order to generalize, the following well-known equations are restated in a suitable form. The Fibonacci sequence satisfies

$$F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 1 \text{ with } F_0 = 0,\ F_1 = 1,$$

while the binomial coefficients satisfy the partial recurrence relation

$${}^nC_m = {}^{n-1}C_m + {}^{n-1}C_{m-1} \text{ for } n \geq 0 \text{ and } m \geq 1$$

with ${}^nC_m = 0$ elsewhere except for ${}^nC_0 = 1$, $n \geq 0$. The relationship between them is

$$\begin{aligned} F_{n+1} &= \sum_{m=0}^{n} {}^{n-m}C_m \\ &= \sum_{m=0}^{\lfloor n/2 \rfloor} {}^{n-m}C_m \quad \text{for } n \geq 0. \end{aligned}$$

Given some generalization of the binomial coefficients, a straight-forward generalization of the Fibonacci sequence is obtained by taking diagonal sums of the generalized binomial array (as in [12] for example). The reverse procedure, that of finding generalized binomial coefficients which underlie a generalized Fibonacci sequence, is not so clear cut. The uniqueness of the standard Fibonacci-binomial relationship and other examples will be considered as particular cases.

2. Generalized binomial coefficients

An obvious and reasonably broad generalization of the Fibonacci sequence is provided by the recurrence relation

$$F(n+1) = g(n) + \sum_{i=0}^{b} f(i)F(n-i) \text{ for } b \geq 0 \text{ and } n \geq 1, \tag{1}$$

with initial conditions given for $n = -b$ to 0. Also, $g(n)$ is given for $n \geq 1$ and $f(i)$ is given for $0 \leq i \leq b$ with $f(b) \neq 0$. It is of course a simple matter to set up an array of coefficients $C(n,m)$ with diagonal sums equal to a given sequence. For instance, let $C(n,0) = F(n+1)$ for $n \geq 0$ and $C(n,m) = 0$ elsewhere. To avoid such trivial solutions, we shall require the underlying generalized binomial coefficients to satisfy a recurrence relation of first order in the *n* dimension and order *b* in the *m* dimension, and to have only the same number of free boundary values as the generalized Fibonacci sequence has initial values, viz. $C(n,0)$ for $0 \leq n \leq b$. With this in mind, let

$$C(n,m) = \sum_{i=0}^{b} c(m,i)C(n-1,m-i) \text{ for } n \geq 0 \text{ and } m \geq 1 \tag{2}$$

with $C(n,m) = 0$ elsewhere except for $C(n,0)$, $n \geq 0$. The functions $c(m,i)$ and $C(n,0)$ are to be determined with $c(m,b)$ not equivalent to zero. Although $c(0,0)$ does not appear in (2), it will also be needed. Motivated by the foregoing, we form the weighted diagonal sum

$$F(n+1) = \sum_{m=0}^{n} w(m)C(n-m,m), \text{ for given non-zero } w(m) \text{ and } n \geq 0. \tag{3}$$

The boundary conditions on *C* imply that the upper summation limit in (3) may be reduced to $\lfloor bn/(b+1) \rfloor$. Substituting (2) into (3) for $n \geq 1$ and $m \geq 1$,

$$
\begin{aligned}
F(n+1) &= w(0)C(n,0) + \sum_{m=1}^{n} w(m) \sum_{i=0}^{b} c(m,i)C(n-1-m,m-i) \\
&= w(0)\left(C(n,0) - \sum_{i=0}^{b} c(0,i)C(n-1,-i) \right) + \sum_{i=0}^{b} \sum_{m=0}^{n} w(m)c(m,i)C(n-1-m,m-i) \\
&= g(n) + \sum_{i=0}^{b} \sum_{m=0}^{n} f(i)w(m-i)C(n-1-m,m-i) \qquad \text{(see identities below)} \\
&= g(n) + \sum_{i=0}^{b} \sum_{m=-i}^{n-i} f(i)w(m)C(n-i-1-m,m) \qquad \text{(changing the summation index to } j=m-i \text{ then reverting to } m\text{)} \\
&= g(n) + \sum_{i=0}^{b} f(i) \sum_{m=0}^{n-i-1} w(m)C(n-i-1-m,m) \qquad \text{(using the boundary conditions)} \\
&= g(n) + \sum_{i=0}^{b} f(i)F(n-i) \qquad \text{(valid for } n-b-1 \geq 0 \text{, i.e. } n \geq b+1\text{)} \qquad (4)
\end{aligned}
$$

with the required identities

$$
\begin{aligned}
g(n) &= w(0)\left(C(n,0) - \sum_{i=0}^{b} c(0,i)C(n-1,-i) \right) \\
&= w(0)\big(C(n,0) - c(0,0)C(n-1,0)\big) \qquad (n \geq b+1) \qquad (5)
\end{aligned}
$$

and

$$
w(m)c(m,i) = f(i)w(m-i) \quad \text{for } 0 \leq i \leq b,\ m \geq i. \qquad (6)
$$

(6) may be solved to determine a condition for the form of the generalized binomial recurrence relation weight in terms of given functions:

$$
c(m,i) = \frac{f(i)w(m-i)}{w(m)}. \qquad (7)
$$

In particular, $c(0,0) = f(0)$.

We now turn to the boundary values $C(n,0)$, $n \geq 0$. $C(0,0) = F(1)/w(0)$ from (3). The derivation leading to (4) can be made valid for $1 \leq n \leq b$ by suitable choice of $C(n,0)$. From (3),

$$
C(n,0) = F(n+1) - \sum_{m=1}^{n} w(m)C(n-m,m), \quad 1 \leq n \leq b. \qquad (8)
$$

This is well-defined as it involves only terms which may be calculated at earlier stages. The remaining boundary values for $n \ge b+1$ may then be determined from (5), so that

$$C(n,0) = g(n)/w(0) + f(0)C(n-1,0), \tag{9}$$

with initial value $C(b,0)$ given by (8) for $b \ge 1$ or by $C(0,0)$ when $b = 0$. Thus, the only "free" boundary values, i.e. those which must be determined directly from the F sequence, are $C(n,0)$, $0 \le n \le b$.

3. Examples

Example 1

The Fibonacci sequence has $g(n) = 0$, $b = 1$, $f(0) = f(1) = 1$ and $F(0) = 0$, $F(1) = 1$. Choosing $w(m) = 1$, $F(n+1) = \sum_{m=0}^{n} C(n-m, m)$ with $c(m,i) = 1$ from (7) with boundary values $C(0,0) = 1$, $C(1,0) = F(2) - 0 = 1$ from (8) and $C(n,0) = C(n-1,0) = 1$ for $n \ge 2$ from (9). We thus recover

$$C(n,m) = \sum_{i=0}^{1} C(n-1, m-i) \text{ for } n \ge 0 \text{ and } m \ge 1$$

with $C(n,m) = 0$ elsewhere except for $C(n,0) = 1$, $n \ge 0$. The relationship between the Fibonacci numbers and the binomial coefficients is therefore unique (for the form of the partial recurrence relation) up to the weight function $w(m)$.

Example 2

The sequence $V(n+1) = 1 + \sum_{i=1}^{q} V(n-i)$ with $V(-n) = 0$ for $0 \le n \le q-1$ and $V(1) = 1$ was considered in [12]. Taking $q = 2$ gives $g(n) = 1$, $b = 2$, $f(0) = 0$, $f(1) = f(2) = 1$ and $V(-1) = V(0) = 0$, $V(1) = 1$. Thus, choosing $w(m) = 1$, the corresponding generalized binomial coefficients satisfy $V(n+1) = \sum_{m=0}^{n} C(n-m, m)$ for $n \ge 0$ as well as (2) with

$$c(m,i)=\begin{cases}0 & i=0\\ 1 & i=1,2\end{cases} \text{ from (7),}$$

with $C(0,0)=V(1)=1$ and $C(1,0)=V(2)-0=1$, $C(2,0)=V(3)-C(1,1)=2-1=1$ from (8) while $C(n,0)=g(n)/w(0)=1$ for $n\geq 3$ from (9). Table 1 shows some of these coefficients.

Example 3

While the underlying generalized binomial coefficients are unique up to the weight function for the given form of the recurrence relation, it is interesting to note that different though equivalent generalized Fibonacci recurrence relations lead to different underlying coefficients. For instance, in the previous example an equivalent recurrence relation is given by

$$V(n+1)=V(n)+V(n-1)-V(n-1-q)\,.$$

Taking $q=2$ now gives $g(n)=0$, $b=3$, $f(0)=f(1)=1$, $f(2)=0$, $f(3)=-1$ and $V(-2)=V(-1)=V(0)=0$, $V(1)=1$. Again choosing $w(m)=1$, the corresponding generalized binomial coefficients satisfy $V(n+1)=\sum_{m=0}^{n}C(n-m,m)$ for $n\geq 0$ as well as (2) now with

$$c(m,i)=\begin{cases}-1 & i=3\\ 0 & i=2\\ 1 & i=0,1\end{cases}$$

and $C(n,0)=1$ for $n\geq 0$. Table 2 shows these coefficients. It may be verified that both recurrence relations given above for $V(n+1)$ as well as the diagonal sums of the arrays in both Tables 1 and 2 do give the sequence $V(n+1)=\{1,1,2,3,4,6,8,11,15,...\}$ for $n\geq 0$.

Further, the Fibonacci sequence is also generated by

$$F(n+1)=1+\sum_{i=1}^{\infty}F(n-i)$$

which leads to $C(n,m)=\sum_{i=1}^{\infty}C(n-1,m-i)$ for $n\geq 0$ and $m\geq 1$, with $C(n,m)=0$ elsewhere except for $C(n,0)=1$, $n\geq 0$. These coefficients are shown in Table 3 from which it may be seen that diagonal sums give the Fibonacci sequence. The resulting coefficients have combinatorial interpretations [12].

Example 4

As an example of the use of the weight function, taking $w(m)=m!^{-1}$ in Example 1 gives

$$c(m,i)=\frac{f(i)w(m-i)}{w(m)}=\frac{m!}{(m-i)!}={}^{m}C_{i}i! \text{ so that}$$

$$C(n,m)=\sum_{i=0}^{1}{}^{m}C_{i}i!C(n-1,m-i) \text{ and } F(n+1)=\sum_{m=0}^{n}m!^{-1}C(n-m,m)$$

with suitable boundary conditions. This example can clearly be extended to larger values of b. The resulting coefficients also have combinatorial interpretations [12].

4. Conclusion

This paper explores ways of generalizing binomial coefficients and their relation to Fibonacci numbers. Other ways of generalizing the binomial coefficients in this context have included that of Jarden [9] who replaced the natural numbers in the binomial coefficients by the generalized Fibonacci numbers of Lucas [11]. Hoggatt [7] continued this line of thought by developing analogues of Pascal's triangle for the generalized "Fibonomial" coefficients. These triangles were further developed by Gould [6] who generalized all of the corresponding results previously found for ordinary and q-binomial coefficients [3] in a remarkable sequence of seven theorems. Gould built on the work of Fontené [5] who suggested a generalization of the binomial coefficients by replacing the natural numbers by an arbitrary sequence of real or complex numbers. Such a sequence is called a Raney sequence [2], the Fibonacci sequence being a Raney sequence in the Fibonomial coefficients.

These Fontené coefficients were rediscovered by Ward [14] in a paper where he developed a calculus of sequences which generalized the calculus of Jackson [8].

A parallel development was the consideration of the properties of Gaussian binomial coefficients by Carlitz [4], Polya and Alexanderson [13] and Alexanderson and Klosinski [1]. Lee and Phillips [10] went a step further by considering Gaussian multinomial coefficients.

In the present paper, conditions for non-trivial generalized binomial coefficients which correspond to a generalized Fibonacci-type sequence have been derived. As well as answering the fundamental question of the uniqueness of the standard Fibonacci-binomial relationship, other generalized relationships have been investigated. Further exploration in this area may be worthwhile.

References

1. G.L. Alexanderson & L.F. Klosinski. "A Fibonacci analogue of Gaussian binomial coefficients." *The Fibonacci Quarterly* **12.2** (1974):129-132.
2. S. Ando & D. Sato. "On the generalized binomial coefficients defined by strong divisibility sequences." In *Applications of Fibonacci Numbers*, vol. 8, pp. 1-10. Ed. F. Howard. Dordrecht: Kluwer, 1999.
3. L. Carlitz. "Some determinants of *q*-binomial coefficients." *Journal für die Reine und Angewandte Mathematik* **226** (1967):216-220.
4. L. Carlitz. "Sequences and inversions." *Duke Mathematical Journal* **37.1** (1970):193-198.
5. G. Fontené. "Généralisation d'une formula connue." *Nouv. Ann. Math.* **4.15** (1915):112.

6. H.W. Gould. "The bracket function and Fontené-Ward generalized binomial coefficients with application to Fibonomial coefficients." *The Fibonacci Quarterly* **7.1** (1969):23-40, 55.

7. V.E. Hoggatt, Jr. "Fibonacci numbers and generalized binomial coefficients." *The Fibonacci Quarterly* **5.4** (1967):383-400.

8. F.H. Jackson. "q-difference equations." *American Journal of Mathematics* **32** (1910):305-314.

9. D. Jarden. *Recurring sequences*. Jerusalem: Riveon Lematimatika, 1958.

10. S.L. Lee & G.M. Phillips. "A recurrence relation for Gaussian multinomial coefficients." In *Applications of Fibonacci Numbers*, vol. 3, pp. 239-240. Ed. G.E. Bergman, A.N. Philippou & A.F. Horadam. Dordrecht: Kluwer, 1990.

11. E. Lucas. "Théorie des functions numériques simplement périodique." *American Journal of Mathematics* **1** (1878):184-240, 289-321.

12. R.L. Ollerton & A.G. Shannon. "Extensions of generalized binomial coefficients." *The Fibonacci Quarterly* (forthcoming).

13. G. Polya & G.L. Alexanderson. "Gaussian binomial coefficients." *Elemente der Mathematik* **26.5** (1971):102-109.

14. M. Ward. "A calculus of sequences." *American Journal of Mathematics* **58** (1936):255-266.

AMS classification numbers: 11B39, 11B65, 15A36

Table 1. The generalized binomial coefficient array of Example 2 for $n = 0$ to 8 ($n \geq 0$ gives rows, $m \geq 0$ gives columns).

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	2	3	4	3	1	0	0	0	0	0	0	0	0	0	0
1	1	2	3	5	7	7	4	1	0	0	0	0	0	0	0	0
1	1	2	3	5	8	12	14	11	5	1	0	0	0	0	0	0
1	1	2	3	5	8	13	20	26	25	16	6	1	0	0	0	0
1	1	2	3	5	8	13	21	33	46	51	41	22	7	1	0	0
1	1	2	3	5	8	13	21	34	54	79	97	92	63	29	8	1

Table 2. Different generalized binomial coefficients which give the same diagonal sums as in Table 1.

1	0	0	0	0	0	0	0	0
1	1	0	−1	0	0	0	0	0
1	2	1	−2	−2	0	1	0	0
1	3	3	−2	−6	−3	3	3	0
1	4	6	0	−11	−12	2	12	6
1	5	10	5	−15	−29	−10	25	30
1	6	15	14	−15	−54	−44	30	84
1	7	21	28	−7	−84	−112	1	168
1	8	28	48	14	−112	−224	−104	253

Table 3. Alternative generalized binomial coefficients which give the Fibonacci numbers as diagonal sums.

1	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8
1	1	2	4	7	11	16	22	29
1	1	2	4	8	15	26	42	64
1	1	2	4	8	16	31	57	99
1	1	2	4	8	16	32	63	120
1	1	2	4	8	16	32	64	127
1	1	2	4	8	16	32	64	128