Errata and Addenda to *Mathematical Constants*

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At this point, there are more additions than errors to report...

1.2. The Golden Mean. The cubic irrational $\chi = 1.8392867552...$ is mentioned elsewhere in the literature with regard to iterative functions [1, 2, 3] (the four-numbers game is a special case of what are known as Ducci sequences) and geometric constructions [4, 5].

1.3. The Natural Logarithmic Base. A proof of the formula

$$\frac{e}{2} = \left(\frac{2}{1}\right)^{\frac{1}{2}} \cdot \left(\frac{2 \cdot 4}{3 \cdot 3}\right)^{\frac{1}{4}} \cdot \left(\frac{4 \cdot 6 \cdot 6 \cdot 8}{5 \cdot 5 \cdot 7 \cdot 7}\right)^{\frac{1}{8}} \cdots$$

appears in [6]; Hurwitzian continued fractions for $e^{1/q}$ and $e^{2/q}$ appear in [7, 8, 9, 10]. Define the following set of integer k-tuples

$$N_k = \left\{ (n_1, n_2, \ldots, n_k) : \sum_{j=1}^{k} \frac{1}{n_j} = 1 \text{ and } 1 \le n_1 < n_2 < \ldots < n_k \right\}.$$

Martin [11] proved that

$$\min_{(n_1,n_2,\ldots,n_k)\in N_k} n_k \sim \frac{e}{e-1} k$$

as $k \to \infty$, but it remains open whether

$$\max_{(n_1,n_2,\ldots,n_k)\in N_k} n_1 \sim \frac{1}{e-1} k.$$

Croot [12] made some progress on the latter: He proved that $n_1 \ge (1 + o(1))k/(e-1)$ for infinitely many values of k, and this bound is best possible. Also, define $f_0(x) = x$ and, for each $n > 0$,

$$f_n(x) = (1 + f_{n-1}(x) - f_{n-1}(0))^{\frac{1}{x}}.$$

This imitates the definition of e, in the sense that the exponent $\to \infty$ and the base $\to 1$ as $x \to 0$. We have $f_1(0) = e = 2.718...$,

$$f_2(0) = \exp\left(-\tfrac{e}{2}\right) = 0.257..., \qquad f_3(0) = \exp\left(\tfrac{11-3e}{24} \exp\left(1 - \tfrac{e}{2}\right)\right) = 1.086...$$

and $f_4(0) = 0.921...$ (too complicated an expression to include here). Does a pattern develop here?

1.5. Euler-Mascheroni Constant. Vacca's series was, in fact, anticipated by Nielsen [13]. The following series [14] suggest that $\ln(4/\pi)$ is an "alternating Euler constant":

$$\gamma = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right) = -\int_0^1\int_0^1 \frac{1-x}{(1-xy)\ln(xy)}dx\,dy,$$

$$\ln\left(\frac{4}{\pi}\right) = \sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right) = -\int_0^1\int_0^1 \frac{1-x}{(1+xy)\ln(xy)}dx\,dy.$$

Sample criteria for the irrationality of γ appear in [15, 16, 17]. Long ago, Mahler attempted to prove that γ is transcendental; the closest he came to this was to prove the transcendentality of the constant [18, 19]

$$\frac{\pi Y_0(2)}{2J_0(2)} - \gamma$$

where $J_0(x)$ and $Y_0(x)$ are the zeroth Bessel functions of the first and second kinds. (Unfortunately the conclusion cannot be applied to the terms separately!)

1.8. Khintchine-Lévy Constants. If x is a quadratic irrational, then its continued fraction expansion is periodic; hence $\lim_{n\to\infty} M(n,x)$ is easily found and is algebraic. For example, $\lim_{n\to\infty} M(n,\varphi) = 1$, where φ is the Golden mean. Golubeva [20, 21] studied the set S of values $\lim_{n\to\infty}\ln(Q_n)/n$ taken over all quadratic irrationals x. She proved that $S \subseteq [\ln(\varphi),\infty)$ and that $\pi^2/(12\ln(2))$ is an accumulation point of S. It is likely that S has a structure similar to the Markov spectrum (section 2.31) in the sense that a left hand portion of S probably consists only of isolated points and a right hand portion of S is much denser.

1.11. Chaitin's Constant. Calude & Stay [22] suggested that the uncomputability of bits of Ω can be recast as a uncertainty principle.

2.1. Hardy-Littlewood Constants. Fix $\varepsilon > 0$. Let $N(x,k)$ denote the number of positive integers $n \le x$ with $\Omega(n) = k$, where k is allowed to grow with x. Nicolas [23] proved that

$$\lim_{x\to\infty}\frac{N(x,k)}{(x/2^k)\ln(x/2^k)} = \frac{1}{4C_{\text{twin}}} = \frac{1}{4}\prod_{p>2}\left(1+\frac{1}{p(p-2)}\right) = 0.3786950320....$$

under the assumption that $(2+\varepsilon)\ln(\ln(x)) \le k \le \ln(x)/\ln(2)$. More relevant results appear in [24].

2.2. Meissel-Mertens Constants. See [25] for more occurrences of the constants M and M'. Higher-order asymptotic series for $\mathrm{E}_n(\omega)$, $\mathrm{Var}_n(\omega)$, $\mathrm{E}_n(\Omega)$ and $\mathrm{Var}_n(\Omega)$ are given in [26].

2.4. Artin's Constant. Let $\iota(n) = 1$ if n is square-free and $\iota(n) = 0$ otherwise. Then [27, 28]

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \iota(n)\iota(n+1) = \prod_{p}\left(1-\frac{2}{p^2}\right) = 0.3226340989... = -1 + 2(0.6613170494...),$$

that is, the Feller-Tornier constant also arises with regard to consecutive square-free numbers.

2.5. Hafner-Sarnak-McCurley Constant. In the "Added In Press" section (pages 601–602), the asymptotics of coprimality and of square-freeness are discussed for the Gaussian integers and for the Eisenstein-Jacobi integers. More about sums involving $2^{\omega(n)}$ and $2^{-\omega(n)}$ appears in [73]. Also, the asymptotics of $\sum_{n=1}^{N} 3^{\Omega(n)}$, due to Tenenbaum, are mentioned in [26].

2.7. Euler Totient Constants. Let $f(n) = n\varphi(n)^{-1} - e^{\gamma}\ln(\ln(n))$. Nicolas [30] proved that $f(n) > 0$ for infinitely many integers n by the following reasoning. Let P_k denote the product of the first k prime numbers. If the Riemann hypothesis is true, then $f(P_k) > 0$ for all k. If the Riemann hypothesis is false, then $f(P_k) > 0$ for infinitely many k and $f(P_l) \leq 0$ for infinitely many l.

Let $U(n)$ denote the set of values $\leq n$ taken by φ and v_n denote its cardinality; for example [31], $U(15) = \{1, 2, 4, 6, 8, 10, 12\}$ and $v(15) = 7$. Let $\ln_3(x) = \ln(\ln(\ln(n)))$ and $\ln_4(x) = \ln(\ln_3(x))$ for convenience. Ford [32] proved that

$$v(n) = \tfrac{n}{\ln(n)} \exp\left\{C[\ln_3(n) - \ln_4(n)]^2 + D\ln_3(n) - [D + \tfrac{1}{2} - 2C]\ln_4(n) + O(1)\right\}$$

as $n \to \infty$, where

$$C = -\tfrac{1}{2\ln(\rho)} = 0.8178146464...,$$

$$D = 2C\left(1 + \ln(F'(\rho)) - \ln(2C)\right) - \tfrac{3}{2} = 2.1769687435...$$

$$F(x) = \sum_{k=1}^{\infty}\left((k+1)\ln(k+1) - k\ln(k) - 1\right)x^k$$

and $\rho = 0.5425985860...$ is the unique solution on $[0, 1)$ of the equation $F(\rho) = 1$. Also,

$$\lim_{n\to\infty} \frac{1}{v(n)\ln(\ln(n))}\sum_{m\in U(n)} \omega(m) = \frac{1}{1-\rho} = 2.1862634648...$$

which contrasts with a related result of Erdös & Pomerance [33]:

$$\lim_{n\to\infty} \frac{1}{n\ln(\ln(n))^2}\sum_{m=1}^{n} \omega(\varphi(n)) = \frac{1}{2}.$$

These two latter formulas hold as well if ω is replaced by Ω. See [34] for more on Euler's totient.

2.8. Pell-Stevenhagen Constants. The constant P is transcendental via a general theorem on values of modular forms due to Nesterenko [35, 36].

2.10. Sierpinski's Constant. Sierpinski's formulas for $\hat{S}$ and $\tilde{S}$ contained a few errors: they should be [37, 38, 39, 40, 41, 42]

$$\hat{S} = \gamma + S - \frac{12}{\pi^2}\zeta'(2) + \frac{\ln(2)}{3} - 1 = 1.7710119609... = \frac{\pi}{4}(2.2549224628...),$$

$$\tilde{S} = 2S - \frac{12}{\pi^2}\zeta'(2) + \frac{\ln(2)}{3} - 1 = 2.0166215457... = \frac{1}{4}(8.0664861829...).$$

Also, in the summation formula at the top of page 125, D_n should be D_k.

2.13. Mills' Constant. Let $q_1 < q_2 < \ldots < q_k$ denote the consecutive prime factors of an integer $n > 1$. Define

$$F(n) = \sum_{j=1}^{k-1}\left(1 - \frac{q_j}{q_{j+1}}\right) = \omega(n) - 1 - \sum_{j=1}^{k-1}\frac{q_j}{q_{j+1}}$$

if $k > 1$ and $F(n) = 0$ if $k = 1$. Erdös & Nicolas [43] demonstrated that there exists a constant $C' = 1.70654185...$ such that, as $n \to \infty$, $F(n) \leq \sqrt{\ln(n)} - C' + o(1)$, with equality holding for infinitely many n. Further, $C' = C + \ln(2) + 1/2$, where [43, 44]

$$C = \sum_{i=1}^{\infty}\left\{\ln\left(\frac{p_{i+1}}{p_i}\right) - \left(1 - \frac{p_i}{p_{i+1}}\right)\right\} = 0.51339467..., \qquad \sum_{i=1}^{\infty}\left(\frac{p_{i+1}}{p_i} - 1\right)^2 = 1.65310351...,$$

and $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ... is the sequence of all primes.

2.15. Glaisher-Kinkelin Constant. Ehrhardt [45] proved Dyson's conjecture regarding the asymptotic expansion of $E(s)$ as $s \to \infty$. In the last paragraph on page 141, the polynomial $q(x)$ should be assumed to have degree n.

2.16. Stolarsky-Harboth Constant. Given a positive integer n, define s_1^2 to be the largest square not exceeding n. Then define s_2^2 to be the largest square not exceeding $n - s_1^2$, and so forth. Hence $n = \sum_{j=1}^{r} s_j^2$ for some r. We say that n is a *greedy sum of distinct squares* if $s_1 > s_2 > \ldots > s_r$. Let $A(N)$ be the number of such integers $n < N$, plus one. Montgomery & Vorhauer [46] proved that $A(N)/N$ does not tend to a constant, but instead that there is a continuous function $f(x)$ of period 1 for which

$$\lim_{k\to\infty}\frac{A(4\exp(2^{k+x}))}{4\exp(2^{k+x})} = f(x), \qquad \min_{0\leq x\leq 1} f(x) = 0.50307... < \max_{0\leq x\leq 1} f(x) = 0.50964...$$

where k takes on only integer values. This is reminiscent of the behavior discussed for digital sums.

2.18. Porter-Hensley Constants. Lhote [47] developed rigorous techniques for computing certain variances to high precision, for example, $4\lambda_1''(2)$.

2.20. Erdös' Reciprocal Sum Constants. A sequence of positive integers $b_1 < b_2 < \ldots < b_m$ is a B_h-sequence if all h-fold sums $b_{i_1} + b_{i_2} + \cdots + b_{i_h}$, $i_1 \leq i_2 \leq \ldots \leq i_h$, are distinct. Given n, choose a B_h-sequence $\{b_i\}$ so that $b_m \leq n$ and m is maximal; let $F_h(n)$ be this value of m. It is known that $C_h = \limsup_{n\to\infty} n^{-1/h} F_h(n)$ is finite; we further have [48, 49, 50, 51, 52, 53]

$$C_2 = 1, \qquad 1 \leq C_3 \leq (7/2)^{1/3}, \qquad 1 \leq C_4 \leq 7^{1/4}.$$

More generally, a sequence of positive integers $b_1 < b_2 < \ldots < b_m$ is a $B_{h,g}$-sequence if, for every positive integer k, the equation $x_1 + x_2 + \cdots + x_h = k$, $x_1 \leq x_2 \leq \ldots \leq x_h$, has at most g solutions with $x_j = b_{i_j}$ for all j. Defining $F_{h,g}(n)$ and $C_{h,g}$ analogously, we have [53, 54, 55, 56, 57]

$$\tfrac{4\sqrt{7}}{7} \leq C_{2,2} \leq \tfrac{\sqrt{21}}{2}, \qquad \tfrac{3\sqrt{2}}{4} g^{1/2} + o(g^{1/2}) \leq C_{2,g} \leq \min\left\{\tfrac{7}{2}g - \tfrac{7}{4}, \tfrac{17g}{5}\right\}^{1/2}$$

as $g \to \infty$.

2.21. Stieltjes Constants. If $d_k(n)$ denotes the number of sequences x_1, x_2, ..., x_k of positive integers such that $n = x_1 x_2 \cdots x_k$, then [58, 59, 60]

$$\sum_{n=1}^{N} d_2(n) \sim N\ln(N) + (2\gamma_0 - 1)N \qquad (d_2 \text{ is the divisor function}),$$

$$\sum_{n=1}^{N} d_3(n) \sim \frac{1}{2} N\ln(N)^2 + (3\gamma_0 - 1)N\ln(N) + (-3\gamma_1 + 3\gamma_0^2 - 3\gamma_0 + 1)N,$$

$$\begin{aligned}\sum_{n=1}^{N} d_4(n) \quad \sim \quad & \frac{1}{6} N\ln(N)^3 + \frac{4\gamma_0 - 1}{2} N\ln(N)^2 + (-4\gamma_1 + 6\gamma_0^2 - 4\gamma_0 + 1)N\ln(N) \\ & + (2\gamma_2 - 12\gamma_1\gamma_0 + 4\gamma_1 + 4\gamma_0^3 - 6\gamma_0^2 + 4\gamma_0 - 1)N\end{aligned}$$

as $N \to \infty$. More generally, $\sum_{n=1}^{N} d_k(n)$ can be asymptotically expressed as N times a polynomial of degree $k-1$ in $\ln(N)$, which in turn can be described as the residue at $z = 1$ of $z^{-1}\zeta(z)^k N^z$. See also [26] for an application of $\{\gamma_j\}_{j=0}^{\infty}$ to asymptotic series for $\mathrm{E}_n(\omega)$ and $\mathrm{E}_n(\Omega)$.

2.25. Cameron's Sum-Free Set Constants. Erdös [61] and Alon & Kleitman [62] showed that any finite set B of positive integers must contain a sum-free subset A such that $|A| > \frac{1}{3}|B|$. See also [63, 64, 65]. The largest constant c such that

$|A| > c|B|$ must satisfy $1/3 \le c < 12/29$, but its exact value is unknown. Using harmonic analysis, Bourgain [66] improved the original inequality to $|A| > \frac{1}{3}(|B|+2)$. Green [67] demonstrated that $s_n = O(2^{n/2})$, but the values $c_o = 6.8...$ and $c_e = 6.0...$ await more precise computation.

2.30. Pisot-Vijayaraghavan-Salem Constants. Compare the sequence $\{(3/2)^n\}$, for which little is known, with the recursion $x_0 = 0$, $x_n = \{x_{n-1} + \ln(3/2)/\ln(2)\}$, for which a musical interpretation exists. If a guitar player touches a vibrating string at a point two-thirds from the end of the string, its fundamental frequency is dampened and a higher overtone is heard instead. This new pitch is a perfect fifth above the original note. It is well-known that the "circle of fifths" never closes, in the sense that 2^{x_n} is never an integer for $n > 0$. Further, the "circle of fifths", in the limit as $n \to \infty$, fills the continuum of pitches spanning the octave [68, 69].

2.32. De Bruijn-Newman Constant. Further work regarding Li's criterion, which is equivalent to Riemann's hypothesis and which involves the Stieltjes constants, appears in [70].

2.33. Hall-Montgomery Constant. Let ψ be the unique solution on $(0, \pi)$ of the equation $\sin(\psi) - \psi\cos(\psi) = \pi/2$ and define $K = -\cos(\psi) = 0.3286741629....$ Consider any real multiplicative function f whose values are constrained to $[-1, 1]$. Hall & Tenenbaum [71] proved that, for some constant $C > 0$,

$$\sum_{n=1}^{N} f(n) \le CN \exp\left\{-K \sum_{p \le N} \frac{1 - f(p)}{p}\right\} \qquad \text{for sufficiently large } N,$$

and that, moreover, the constant K is sharp. (The latter summation is over all prime numbers p.) This interesting result is a lemma used in [72]. A table of values of sharp constants K is also given in [71] for the generalized scenario where f is complex, $|f| \le 1$ and, for all primes p, $f(p)$ is constrained to certain elliptical regions in $\mathbb{C}$.

3.6. Sobolev Isoperimetric Constants. In section 3.6.1, $\sqrt{\lambda} = 1$ represents the principal frequency of the sound we hear when a string is plucked; in section 3.6.3, $\sqrt{\lambda} = \theta$ represents likewise when a kettledrum is struck. (The square root was missing in both.) The units of frequency, however, are not compatible between these two examples. More relevant material is found in [73, 74].

3.15. Van der Corput's Constant. We examined only the case in which f is a real twice-continuously differentiable function on the interval $[a, b]$; a generalization to the case where f is n times differentiable, $n \ge 2$, is discussed in [75, 76] with some experimental numerical results for $n = 3$.

4.3. Achieser-Krein-Favard Constants. While on the subject of trigonometric polynomials, we mention Littlewood's conjecture [77]. Let $n_1 < n_2 < \ldots < n_k$ be integers and let c_j, $1 \le j \le k$, be complex numbers with $|c_j| \ge 1$. Konyagin [78] and

McGehee, Pigno & Smith [79] proved that there exists $C > 0$ so that the inequality

$$\int_0^1 \left| \sum_{j=1}^{k} c_j e^{2\pi i n_j \xi} \right| d\xi \geq C \ln(k)$$

always holds. It is known that the smallest such constant C satisfies $C \leq 4/\pi^2$; Stegeman [80] demonstrated that $C \geq 0.1293$ and Yabuta [81] improved this slightly to $C \geq 0.129590$. What is the true value of C?

4.7. Berry-Esseen Constant. Significant progress on the asymptotic case (as $\lambda \to 0$) is described in [82, 83, 84]. A different form of the inequality is found in [85].

5.4. Golomb-Dickman Constant. Let $P^+(n)$ denote the largest prime factor of n and $P^-(n)$ denote the smallest prime factor of n. We mentioned that

$$\sum_{n=2}^{N} \ln(P^+(n)) \sim \lambda N \ln(N) - \lambda(1-\gamma)N, \qquad \sum_{n=2}^{N} \ln(P^-(n)) \sim e^{-\gamma} N \ln(\ln(N)) + cN$$

as $N \to \infty$, but did not give an expression for the constant c. Tenenbaum [86] found that

$$c = e^{-\gamma}(1+\gamma) + \int_1^{\infty} \frac{\omega(t) - e^{-\gamma}}{t} dt + \sum_p \left\{ e^{-\gamma} \ln\left(1 - \frac{1}{p}\right) + \frac{\ln(p)}{p-1} \prod_{q \leq p} \left(1 - \frac{1}{q}\right) \right\},$$

where the sum over p and product over q are restricted to primes. A numerical evaluation is still open.

The longest tail $L(\varphi)$, given a random mapping $\varphi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, is called the **height** of φ in [87, 88, 89] and satisfies

$$\lim_{n\to\infty} \mathrm{P}\left(\frac{L(\varphi)}{\sqrt{n}} \leq x\right) = \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(-\frac{k^2 x^2}{2}\right)$$

for fixed $x > 0$. For example,

$$\lim_{n\to\infty} \mathrm{Var}\left(\frac{L(\varphi)}{\sqrt{n}}\right) = \frac{\pi^2}{3} - 2\pi \ln(2)^2.$$

The longest rho-path $R(\varphi)$ is called the **diameter** of φ in [90] and has moments

$$\lim_{n\to\infty} \mathrm{E}\left[\left(\frac{R(\varphi)}{\sqrt{n}}\right)^p\right] = \frac{\sqrt{\pi} p}{2^{p/2}\Gamma((p+1)/2)} \int_0^{\infty} x^{p-1}\left(1 - e^{\mathrm{Ei}(-x) - I(x)}\right) dx$$

for fixed $p > 0$. Complicated formulas for the distribution of the largest tree $P(\varphi)$ also exist [88, 89, 91].

5.6. Otter's Tree Enumeration Constants. Higher-order asymptotic series for T_n, t_n and B_n are given in [26]. Also, the asymptotic analysis of series-parallel posets [92] is similar to that of trees. See [93, 94] for more about k-gonal 2-trees, as well as a new formula for α in terms of rational expressions involving e.

5.7. Lengyel's Constant. Constants of the form $\sum_{k=-\infty}^{\infty} 2^{-k^2}$ and $\sum_{k=-\infty}^{\infty} 2^{-(k-1/2)^2}$ appear in [95, 96].

5.10. Self-Avoiding Walk Constants. Hueter [97, 98] rigorously proved that $\nu_2 = 3/4$ and that $7/12 \leq \nu_3 \leq 2/3$, $1/2 \leq v_4 \leq 5/8$ (if the mean square end-to-end distance exponents ν_3, v_4 exist; otherwise the bounds apply for

$$\underline{\nu}_d = \liminf_{n\to\infty} \frac{\ln(r_n)}{2\ln(n)}, \qquad \overline{\nu}_d = \limsup_{n\to\infty} \frac{\ln(r_n)}{2\ln(n)}$$

when $d = 3, 4$). She confirmed that the same exponents apply for the mean square radius of gyration s_n for $d = 2, 3, 4$; the results carry over to self-avoiding trails as well [99].

5.12. Hard Square Entropy Constant. McKay [100] observed the following asymptotic behavior:

$$F(n) \sim (1.06608266...)(1.0693545387...)^{2n}(1.5030480824...)^{n^2}$$

based on an analysis of the terms $F(n)$ up to $n = 19$. He emphasized that the form of right hand side is conjectural, even though the data showed quite strong convergence to this form.

5.14. Digital Search Tree Constants. The constant Q is transcendental via a general theorem on values of modular forms due to Nesterenko [35, 36]. A correct formula for θ is

$$\theta = \sum_{k=1}^{\infty} \frac{k 2^{k(k-1)/2}}{1 \cdot 3 \cdot 7 \cdots (2^k - 1)} \sum_{j=1}^{k} \frac{1}{2^j - 1} = 7.7431319855...$$

(the exponent $k(k-1)/2$ was mistakenly given as $k + 1$ in [101], but the numerical value is correct). Also, the constant α appears in [102] and the constant Q^{-1} appears in [96].

5.15. Optimal Stopping Constants. When discussing the expected rank R_n, we assumed that no applicant would ever refuse a job offer! If each applicant only accepts an offer with known probability p, then [103]

$$\lim_{n\to\infty} R_n = \prod_{i=1}^{\infty} \left(1 + \frac{2}{i} \frac{1 + pi}{2 - p + pi}\right)^{\frac{1}{1+pi}}$$

which is 6.2101994550... in the case when $p = 1/2$.

Suppose that you view successively terms of a sequence X_1, X_2, X_3, ... of independent random variables with a common distribution function F. You know the function F, and as X_k is being viewed, you must either stop the process or continue. If you stop at time k, you receive a payoff $(1/k)\sum_{j=1}^{k} X_j$. Your objective is to maximize the expected payoff. An optimal strategy is to stop at the first k for which $\sum_{j=1}^{k} X_j \geq \alpha_k$, where α_1, α_2, α_3, ... are certain values depending on F. Shepp [104, 105] proved that $\lim_{k\to\infty} \alpha_k/\sqrt{k}$ exists and is independent of F as long as F has zero mean and unit variance; further,

$$\lim_{k\to\infty} \frac{\alpha_k}{\sqrt{k}} = x = 0.8399236756...$$

is the unique zero of $2x - \sqrt{2\pi}\,(1-x^2)\exp\left(x^2/2\right)\left(1+\operatorname{erf}(x/\sqrt{2})\right)$.

Also, consider a random binary string $Y_1 Y_2 Y_3 \ldots Y_n$ with $\Pr(Y_k = 1) = 1 - \Pr(Y_k = 0)$ independent of k and Y_k independent of the other Ys. Let H denote the pattern consisting of the digits

$$\underbrace{1000...0}_{l} \qquad \text{or} \qquad \underbrace{0111...1}_{l}$$

and assume that its probability of occurrence for each k is

$$\Pr\left(Y_{k+1}Y_{k+2}Y_{k+3}\ldots Y_{k+l} = H\right) = \frac{1}{l}\left(1-\frac{1}{l}\right)^{l-1} \sim \frac{1}{el} = \frac{0.3678794411...}{l}.$$

You observe sequentially the digits Y_1, Y_2, Y_3, ... one at a time. You know the values n and p, and as Y_k is being observed, you must either stop the process or continue. Your objective is to stop at the final appearance of H up to Y_n. Bruss & Louchard [106] determined a strategy that maximizes the probability of meeting this goal. For $n \geq \beta l$, this success probability is

$$\frac{2}{135}e^{-\beta}\left(4 - 45\beta^2 + 45\beta^3\right) = 0.6192522709...$$

as $l \to \infty$, where $\beta = 3.4049534663...$ is the largest zero of the cubic $45\beta^3 - 180\beta^2 + 90\beta + 4$. Further, the interval $[0.367..., 0.619...]$ constitutes "typical" asymptotic bounds on success probabilities associated with a wide variety of optimal stopping problems in strings.

5.18. Percolation Cluster Density Constants. An integral similar to that for $\kappa_B(p_c)$ on the triangular lattice appears in [107].

5.21. k-Satisfiability Constants. On the one hand, the lower bound for $r_c(3)$ was improved to 3.42 in [108] and further improved to 3.52 in [109]. On the other hand, the upper bound 4.506 for $r_c(3)$ in [110] has not been confirmed; the preceding two best upper bounds were 4.596 [111] and 4.571 [112].

5.23. Monomer-Dimer Constants. Friedland & Peled [113] revisited Baxter's computation of A and confirmed that $\ln(A) = 0.66279897....$ They examined the three-dimensional analog, A', of A and found that $0.7652 < \ln(A') < 0.7863$.

6.1. Gauss' Lemniscate Constant. Consider the following game [114]. Players A and B simultaneously choose numbers x and y in the unit interval; B then pays A the amount $|x - y|^{1/2}$. The value of the game (that is, the expected payoff, assuming both players adopt optimal strategies) is $M/2 = 0.59907....$

6.5. Plouffe's Constant. This constant is included in a fascinating mix of ideas by Smith [115], who claims that "angle-doubling" one bit at a time was known centuries ago to Archimedes and was implemented decades ago in binary cordic algorithms (also mentioned in section 5.14). Another constant of interest is $\arctan(\sqrt{2}) = 0.9553166181...$, which is the base angle of a certain isosceles spherical triangle (in fact, the unique non-Euclidean triangle with rational sides and a single right angle).

6.6. Lehmer's Constant. Rivoal [116] has studied the link between the rational approximations of a positive real number x coming from the continued cotangent representation of x, and the usual convergents that proceed from the regular continued fraction expansion of x.

6.9. Minkowski-Bower Constant. See [117, 118] for a generalization of the Minkowski question mark function.

7.1. Bloch-Landau Constants. In the definitions of the sets F and G, the functions f need only be analytic on the open unit disk D (in addition to satisfying $f(0) = 0$, $f'(0) = 1$). On the one hand, the weakened hypothesis doesn't affect the values of B, L or A; on the other hand, the weakening is essential for the existence of $f \in G$ such that $m(f) = M$.

The bounds $0.62\pi < A < 0.7728\pi$ were improved by several authors, although they studied the quantity $\tilde{A} = \pi - A$ instead (the omitted area constant). Barnard & Lewis [119] demonstrated that $\tilde{A} \leq 0.31\pi$. Barnard & Pearce [120] established that $\tilde{A} \geq 0.240005\pi$, but Banjai & Trefethen [121] subsequently computed that $\tilde{A} = (0.2385813248...)\pi$. It is believed that the earlier estimate was slightly in error. See [122, 123, 124] for related problems.

The spherical analog of Bloch's constant B, corresponding to meromorphic functions f mapping D to the Riemann sphere, was recently determined by Bonk & Eremenko [125]. This constant turns out to be $\arccos(1/3) = 1.2309594173....$ A proof as such gives us hope that someday the planar Bloch-Landau constants will also be exactly known.

More relevant material is found in [126].

7.5. Hayman Constants. An update on the Hayman-Wu constant appears in [127].

7.6. Littlewood-Clunie-Pommerenke Constants. The lower limit of summation in the definition of S_2 should be $n = 0$ rather than $n = 1$; that is, the coefficient b_0 need not be zero. Numerical evidence for both the Carleson-Jones conjecture and Brennan's conjecture was found by Kraetzer [128]. Theoretical evidence supporting the latter appears in [129], but a complete proof remains undiscovered.

8.1. Geometric Probability Constants. The convex hull of random point sets in the unit disk (rather than the unit square) is mentioned in [130].

8.19. Circumradius-Inradius Constants. The phrase "Z-admissible" in the caption of Figure 8.22 should be replaced by "Z-allowable".

Table of Constants. The formula corresponding to 0.8427659133... is $(12\ln(2))/\pi^2$.

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