Jumping succession rules and their generating functions

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Abstract

We study a generalization of the concept of succession rule, called *jumping succession rule*, where each label is allowed to produce its sons at different levels, according to the production of a fixed succession rule. By means of suitable linear algebraic methods, we obtain simple closed forms for the numerical sequences determined by such rules and give applications concerning classical combinatorial structures. Some open problems are proposed at the end of the paper.

1 Doubled succession rules

Consider a $2 \times n$ rectangle and suppose to tile it using 1×2 domino pieces. Clearly, if one uses vertical pieces only in the tiling, there is exactly one solution to the problem, whereas allowing vertical and horizontal pieces gives F_n possible solutions, where F_n is the n-th Fibonacci number, as it is well-known. These two, very simple enumerative results are clearly related, and it seems obvious that the latter can be derived from the former one, which is completely trivial. Our aim is to develop a general setting to deal with this kind of problems by slightly extending the concept of succession rule and the ECO method.

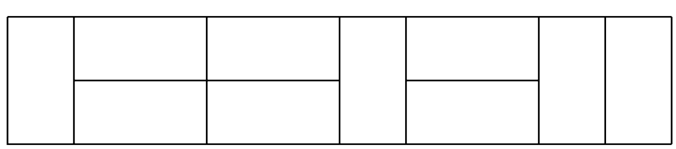

Figure 1: The tiling of a $2 \times n$ rectangle using Fibonacci pieces.

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A *succession rule* Ω is a system constituted by an *axiom* (a), $a \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$, and a set of *productions* of the form:

$$(k) \rightsquigarrow (e_1(k))(e_2(k)) \dots (e_k(k)), \qquad k \in M \subset \mathbb{N}^+,$$

where $e_i : \mathbb{N}^+ \to \mathbb{N}^+$, explaining how to derive, for any given label (k), $k \in \mathbb{N}^+$, its *successors* $(e_1(h)), (e_2(h)), \dots, (e_k(h))$. In most of the cases for a succession rule Ω, we use the more compact notation:

$$\left\{ \begin{array}{l} (a) \\ (k) \rightsquigarrow (e_1(k))(e_2(k)) \dots (e_k(k)), \end{array} \right. \tag{1}$$

to mean that there can be infinitely many productions in the system, but at most one for each integer $k \in \mathbb{N}^+$.

The rule Ω can be represented by means of a *generating tree*, that is a rooted tree whose vertices are the labels of Ω; (a) is the label of the root and each node labelled (k) produces k sons labelled $(e_1(k)), \dots, (e_k(k))$, respectively. We refer to [2] for further details and examples. A succession rule Ω defines a sequence of positive integers $\{f_n\}_{n \geq 0}$, being f_n the number of the nodes at level n in the generating tree defined by Ω. By convention the root is at level 0, so $f_0 = 1$. The function $f_\Omega(x) = \sum_{n \geq 0} f_n x^n$ is the *generating function* derived from Ω.

The concept of succession rules was first introduced in [3] by Chung et al. to study reduced Baxter permutations; later, West applied succession rules to the enumeration of permutations with forbidden subsequences [16]. Moreover, they are a fundamental tool used by the ECO method [2], which is a general method for the enumeration of combinatorial objects essentially based on the definition of a recursive construction for a class of objects by means of an operator which performs a "local expansion" on the objects themselves. Let p be a *discriminating parameter* on a class of objects $\mathcal{O}$, that is $p : \mathcal{O} \to \mathbb{N}^+$, such that $|\mathcal{O}_n| = |\{O \in \mathcal{O} : p(O) = n\}|$ is finite. An operator ϑ on the class $\mathcal{O}$ is a function from $\mathcal{O}_n$ to $2^{\mathcal{O}_{n+1}}$, where $2^{\mathcal{O}_{n+1}}$ is the power set of $\mathcal{O}_{n+1}$.

Proposition 1.1 *[2] Let ϑ be an operator on $\mathcal{O}$. If ϑ satisfies the following conditions:*

1. *for each $O' \in \mathcal{O}_{n+1}$, there exists $O \in \mathcal{O}_n$ such that $O' \in \vartheta(O)$,*
2. *for each $O, O' \in \mathcal{O}_n$ with $O \neq O'$, $\vartheta(O) \cap \vartheta(O') = \emptyset$,*

then the family of sets $\mathcal{F}_{n+1} = \{\vartheta(O) : O \in \mathcal{O}_n\}$ is a partition of $\mathcal{O}_{n+1}$.

Once the parameter p is fixed, if we are able to define an operator ϑ which satisfies conditions 1. and 2., then Proposition 1.1 allows us to construct each object $O' \in \mathcal{O}_{n+1}$ from an object $O \in \mathcal{O}_n$, and each object $O' \in \mathcal{O}_{n+1}$ is obtained from exactly one $O \in \mathcal{O}_n$.

The generating tree associated to the couple $(\mathcal{O}, \vartheta)$, is a rooted tree whose vertices are the objects of $\mathcal{O}$. The objects having the same value of the parameter p lie at the same level, and the sons of an object are the objects it produces through ϑ.

A slight generalization of the notion of succession rule is provided by the concept of *coloured succession rules*. Roughly speaking, a rule is said to be coloured when more than one production is allowed for at least one label. The usual notation to indicate a two-coloured rule is the following:

$$\begin{cases} (a) \\ (k) \rightsquigarrow (e_1(k)) \dots (e_t(k))(\overline{e_{t+1}(k)}) \dots (\overline{e_k(k)}), \\ (\overline{k}) \rightsquigarrow (c_1(k)) \dots (c_s(k))(\overline{c_{s+1}(k)}) \dots (\overline{c_k(k)}), \end{cases} \tag{2}$$

For more details about these topics, see [7].

Given a succession rule of the form (1), we define the *rule operator* L_Ω (briefly, L) associated with Ω [7, 8] as:

$$L_\Omega : \mathbb{R}[x] \to \mathbb{R}[x]$$

$$L_\Omega(\mathbf{1}) = x^a;$$

$$L_\Omega(x^k) = x^{e_1(k)} + \dots + x^{e_k(k)};$$

$$L_\Omega(x^k) = kx^k, \qquad \text{if the label } (k) \text{ is not in the generating tree of } \Omega \ ,$$

and then extending by linearity on $\mathbb{R}[x]$ (considered as a $\mathbb{R}$-vector space). In general, we use the power notation to express the iterated application of L: $L^{n+1}(\mathbf{1}) = L(L^n(\mathbf{1}))$. For any $n \in \mathbb{N}$ we have:

$$f_n = [L^{n+1}(\mathbf{1})]_{x=1} = [DL^n(\mathbf{1})]_{x=1},$$

where D is the derivative operator with respect to the variable x. In [7, 8] many properties of the rule operators are given.

The next definition is the key step in our extension of ECO method.

Given a succession rule Ω as in (1), we call *doubled succession rule* associated with Ω the following expression:

$$\Omega' : \begin{cases} (2a) \\ (2k) \overset{1}{\rightsquigarrow} (2e_1(k)) \dots (2e_k(k)) \\ (2k) \overset{2}{\rightsquigarrow} (2e_1(k)) \dots (2e_k(k)). \end{cases} \quad (3)$$

In order to understand the meaning of this definition we introduce the concept of generating tree associated with Ω', or *doubled generating tree*: it is precisely a rooted labelled tree whose edges can have "length" 1 or 2. The *lengthened level* (briefly, level) of a node N in a doubled generating tree is then defined as follows:

i) if N is the root, then its level is equal to 0;

ii) otherwise, let F be the father of N; in this case, the level of N is equal to the level of F plus the length of the edge from F to N.

In a word, the level of a node N is the sum of the lengths of the edges connecting the root to N. The root of the doubled generating tree is labelled $(2a)$ and every node at level l (labelled $(2k)$) has exactly k sons at level $l+1$ (labelled $(2e_1(k), \dots, (2e_k(k))$, resp.) and k sons at level $l+2$ (with the same labels). We remark that a similar notion has been used in [9, 10]. Anyway, these works deal with specific examples only, without providing a general theory for doubled rules.

At this stage, it is not difficult to see that our starting problem fits into this framework. Indeed, given the (unique) "vertical" tiling of the $2 \times n$ rectangle, we obtain the (unique) "vertical" tiling of the $2 \times (n+1)$ rectangle simply by adding a vertical domino piece on the right; this can be trivially described by the succession rule:

$$\Omega : \begin{cases} (1) \\ (1) \rightsquigarrow (1). \end{cases} \quad (4)$$

On the other hand, if we consider a generic tiling of the $2 \times n$ rectangle by vertical and horizontal dominoes, we can add on the right one vertical domino (so obtaining a tiling for the $2 \times (n+1)$ rectangle) or two horizontal dominoes (in this way obtaining a tiling for the $2 \times (n+2)$ rectangle). Because of the simplicity of this example, it is very easy to show that every tiling of the $2 \times (n+1)$ rectangle derives from exactly one tiling (either of the $2 \times n$ rectangle or of the $2 \times (n-1)$ rectangle). This construction can be described by doubling the succession rule Ω, so obtaining the rule:

$$
\Omega' : \begin{cases} (2) \\ (2) \overset{1}{\rightsquigarrow} (2) \\ (2) \overset{2}{\rightsquigarrow} (2). \end{cases} \tag{5}
$$

The first levels of its generating tree are represented in Figure 2:

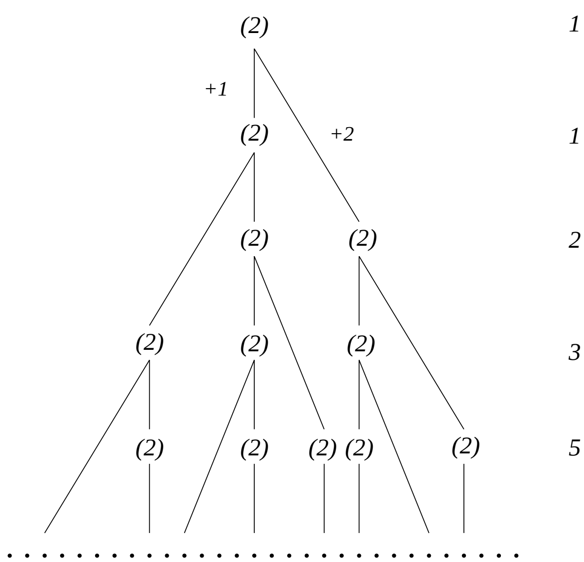

Figure 2: The first levels of the generating tree of the doubled rule (5).

It is immediate to see that the sequence enumerated by the above doubled generating tree is that of Fibonacci numbers: indeed, the number of nodes at each level is the sum of the cardinalities of the two preceding levels.

2 Fibonacci transform

Consider a succession rule Ω of the form (1), and suppose that $(s_n)_{n\geq 0}$ is the numerical sequence determined by Ω. If Ω' is the doubled succession rule associated with Ω, can we determine the sequence $(s'_n)_{n\geq 0}$ related to Ω'? The central result of this section is precisely the solution of this problem.

Before proving our main theorem, we need to state a few definitions.

Let L be the rule operator associated with Ω; the series:

$$
\sum_{n\geq 0} L^{n+1}(\mathbf{1})t^n \tag{6}
$$

is a formal power series in the variables x and t and it is called the *bivariate generating function* of the generating tree determined by Ω. In particular, the sequence of the numbers $[L^{n+1}(\mathbf{1})]_{x=1}$ is precisely the one defined by Ω, and the coefficient of x^k in the polynomial $L^{n+1}(\mathbf{1})$ represents the number of nodes labelled (k) at level n.

If Ω' is the doubled rule associated with Ω, the *normalization* of Ω' is the rule:

$$\widetilde{\Omega}' : \begin{cases} (a) \\ (k) \overset{1}{\rightsquigarrow} (e_1(k)) \dots (e_k(k)) \\ (k) \overset{2}{\rightsquigarrow} (e_1(k)) \dots (e_k(k)) \end{cases} \tag{7}$$

which is obtained by Ω' simply by dividing each label by 2. It is clear that the generating tree defined by $\widetilde{\Omega}'$ loses the "ECO-property", i.e. every node labelled (k) possesses $2k$ sons instead of k; however, Ω' and $\widetilde{\Omega}'$ count the same sequence, and $\widetilde{\Omega}'$ can be better treated in the formalism of rule operators. We remark that systems like $\widetilde{\Omega}'$ are also called *pseudo ECO-systems* [6].

Proposition 2.1 *The bivariate generating function of the generating tree defined by $\widetilde{\Omega}'$ has the form:*

$$\left(\frac{1}{1 - tL - t^2L}\right)(L(\mathbf{1})) = \sum_{n\geq 0}(tL + t^2L)^n(L(\mathbf{1})), \tag{8}$$

being $\frac{1}{M}$ the compositional inverse of the operator M.

Proof. Denote by $p_n(x)$ the polynomial such that the coefficient of x^k is the number of nodes labelled (k) at level n of the generating tree of $\widetilde{\Omega}'$. Clearly $p_0(x) = x^a$, $p_1(x) = x^{e_1(a)} + \ldots + x^{e_a(a)}$ and, in general, $p_n(x) = L^{n+1}(\mathbf{1})$. Now observe that a node at level n is the son of a node at level $n-1$ or of a node at level $n-2$. Then the following polynomial recurrence holds:

$$p_n(x) = L(p_{n-1}(x)) + L(p_{n-2}(x)). \tag{9}$$

which is valid for every $n \geq 1$ (by defining $p_{-1}(x) = 0$).

According to (9), the generating function $f(x,t) = \sum_{n\geq 0} p_n(x)t^n$ satisfies:

$$f(x,t) = \sum_{n\geq 1} L(p_{n-1}(x))t^n + \sum_{n\geq 2} L(p_{n-2})(x)t^n + L(\mathbf{1})$$

which simplifies into:

$$f(x,t) = (tL + t^2L)(f(x,t)) + L(\mathbf{1})$$

that is

$$(1 - tL - t^2L)(f(x,t)) = L(\mathbf{1}).$$

Therefore $f(x,t)$ is obtained by simply inverting the operator $1 - tL - t^2L$, which is precisely our thesis. $\square$

Theorem 2.1 *The number sequence enumerated by $\widetilde{\Omega}'$ (or by Ω') is the sequence:*

$$s'_n = \sum_{k=0}^{n} \binom{n-k}{k} s_{n-k} = \sum_{k=0}^{n} \binom{k}{n-k} s_k \tag{10}$$

being $(s_n)_{n\geq 0}$ the sequence determined by Ω.

Proof. From Proposition 2.1 we have:

$$s'_n = [[t^n]f(x,t)]_{x=1} = \left[[t^n]\sum_{m\geq 0}(tL + t^2L)^m(L(\mathbf{1}))\right]_{x=1}.$$

Since

$$(tL + t^2L)^m = t^m(1+t)^mL^m = \sum_{k=0}^{m}\binom{m}{k}t^{m+k}L^m,$$

we obtain:

$$[t^n](tL + t^2L)^m = \sum_{k=0}^{n}\binom{n-k}{k}L^{n-k},$$

whence:

$$s'_n = \left[\sum_{k=0}^{n} \binom{n-k}{k} L^{n-k+1}(\mathbf{1}) \right]_{x=1} = \sum_{k=0}^{n} \binom{n-k}{k} s_{n-k}. \quad \square$$

The numbers s'_n of (10) count the nodes at level n of the generating tree of Ω'. From a combinatorial view point, each term $\binom{n-k}{k} s_{n-k}$ of the sum in (10) counts the number of the nodes N at level n such that the path from the root to N contains exactly $n-k$ edges of length 2.

We call *Fibonacci transform* of a numerical sequence $(s_n)_{n \geq 0}$ the sequence:

$$s'_n = \sum_{k=0}^{n} \binom{n-k}{k} s_{n-k}. \tag{11}$$

The reason for choosing this name lies in the following

Corollary 2.1 *(Lucas' identity) The Fibonacci transform of the sequence* $s_n = 1$, $\forall n \in \mathbb{N}$, *is the sequence of Fibonacci numbers.*

Observe that this corollary is also the solution of our starting problem.

We now consider an extension of the ECO method which represents the combinatorial interpretation of doubled succession rules. Let $\mathcal{O}$ be a class of combinatorial objects. A *doubled operator* ϑ is an operator on the class $\mathcal{O}$:

$$\vartheta : \mathcal{O}_n \to 2^{\mathcal{O}_{n+1} \cup \mathcal{O}_{n+2}}.$$

Proposition 2.2 *Let ϑ be a doubled operator on $\mathcal{O}$. If ϑ satisfies the following conditions:*

1. *for each $O' \in \mathcal{O}_n$, there exists $O \in \mathcal{O}_{n-2} \cup \mathcal{O}_{n-1}$ such that $O' \in \vartheta(O)$,*
2. *for each $O, O' \in \mathcal{O}_n \cup \mathcal{O}_{n+1}$ with $O \neq O'$, $\vartheta(O) \cap \vartheta(O') = \emptyset$,*

then the family of sets $\mathcal{F}_{n+2} = \{\vartheta(O) : O \in \mathcal{O}_n \cup \mathcal{O}_{n+1}\} \cap 2^{\mathcal{O}_{n+2}}$ is a partition of $\mathcal{O}_{n+2}$.

Clearly, the generating tree associated to the operator ϑ is a doubled generating tree.

Example 2.1 *Doubled Dyck paths and a combinatorial interpretation of a doubled succession rule.*

On the lattice plane $\mathbb{N} \times \mathbb{N}$, the class $\mathcal{C}$ of *Dyck paths* contains the paths made up of *rise* steps $(1,1)$ and *fall* steps $(1,-1)$, running from $(0,0)$ to $(2n,0)$ (see Fig. 3 (a)). The length of a Dyck path is the number of its steps. It is common knowledge that the number of $2n-$length Dyck paths is the nth *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$ (for an interesting survey, see [5]).

The last sequence of fall steps in a Dyck path is called its last descent. Let $\mathcal{C}_n$ be the set of Dyck paths having length $2n$, and ϑ the operator defined in [2] such that

$$\vartheta : \mathcal{C}_n \to 2^{\mathcal{C}_{n+1}},$$

which inserts a peak into any point belonging to the last descent of each path.

The succession rule Ω describing this operator on $\mathcal{C}$ is:

$$\Omega : \begin{cases} (1) \\ (h) \rightsquigarrow (2)(3)\dots(h)(h+1). \end{cases} \tag{12}$$

Let us consider the class $\mathcal{CC}$ of lattice paths made up by rise $(1,1)$, fall $(1,-1)$, *double-rise* $(2,2)$ and *double-fall* $(2,-2)$ steps, defined recursively as follows:

i) the empty path belongs to $\mathcal{CC}$;

ii) if C, D are paths in $\mathcal{CC}$, then the path obtained by adding a rise step (resp. a double-rise step) before C and a fall step (resp. a double-fall step) after C and then concatenating with D belongs to $\mathcal{CC}$.

We call these paths *doubled Dyck paths* (see Fig. 3, (b)). In a doubled Dyck path the *last descent* is the last sequence of fall/double-fall steps, and a *peak* (resp. *double peak*) is a rise (resp. double-rise) step followed by a fall (resp. double-fall) step.

The class of doubled Dyck paths is suitably introduced, starting from the class of Dyck paths, with the aim of handling a combinatorial structure whose recursive construction can be defined by means of a doubled operator. Indeed, let us consider the doubled operator ϑ' on $\mathcal{CC}$; if $\mathcal{CC}_n$ denotes the set of paths having length $2n$, then:

$$\vartheta' : \mathcal{CC}_n \to 2^{\mathcal{CC}_{n+1}} \cup 2^{\mathcal{CC}_{n+2}}.$$

The operator ϑ' inserts a peak, or a doubled peak, in each lattice point of the last descent of a doubled Dyck path, clearly excluding those points internal to double-fall steps (see Fig. 4).

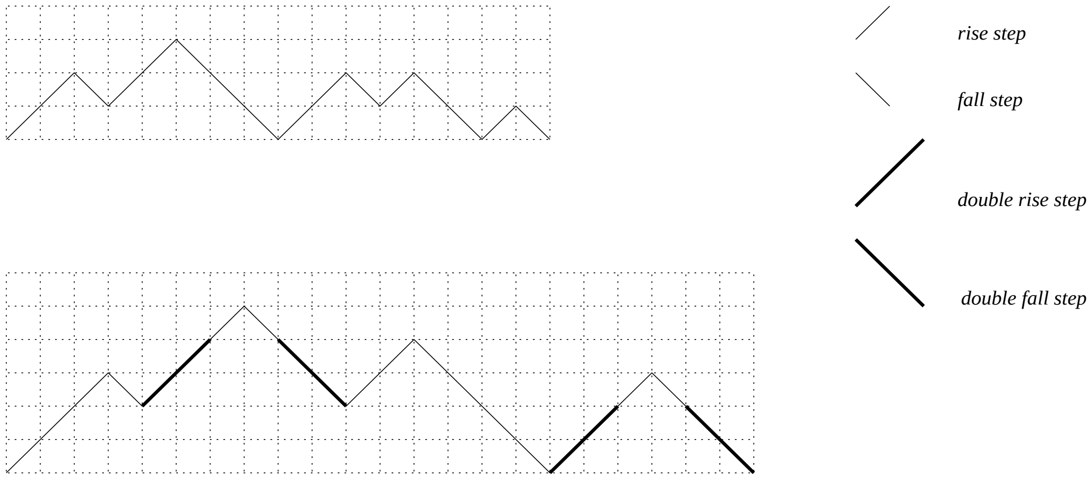

Figure 3: A Dyck path and a doubled Dyck path.

The operator ϑ' satisfies Proposition 2.2, and the doubled generating tree associated with ϑ' (see Fig. 5) determines a doubled succession rule Ω', which is the Fibonacci transform of Ω:

$$\Omega' : \begin{cases} (2) \\ (2h) \stackrel{1}{\rightsquigarrow} (4)(6)\dots(2h)(2h+2) \\ (2h) \stackrel{2}{\rightsquigarrow} (4)(6)\dots(2h)(2h+2). \end{cases} \tag{13}$$

Let us have a look at the enumeration of the class $\mathcal{CC}$ according to the path length. Theorem 2.1 ensures us that the number of doubled Dyck paths having length $2n$ is equal to

$$C'_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} C_{n-k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{k}{n-k} C_k. \tag{14}$$

Equality (14) has a very simple combinatorial interpretation: for any fixed length $2n$, for any $k = 0, \dots, \lfloor \frac{n}{2} \rfloor$, there are exactly $\binom{n-k}{k} C_{n-k}$ paths of $\mathcal{CC}_n$ having k doubled rise step.

Doubled Dyck paths can be represented as *doubled Dyck words*, defined by the unambiguous grammar:

$$S \rightarrow xS\overline{x}S|yyS\overline{yy}S|\epsilon,$$

being ϵ the empty word. The generating function is

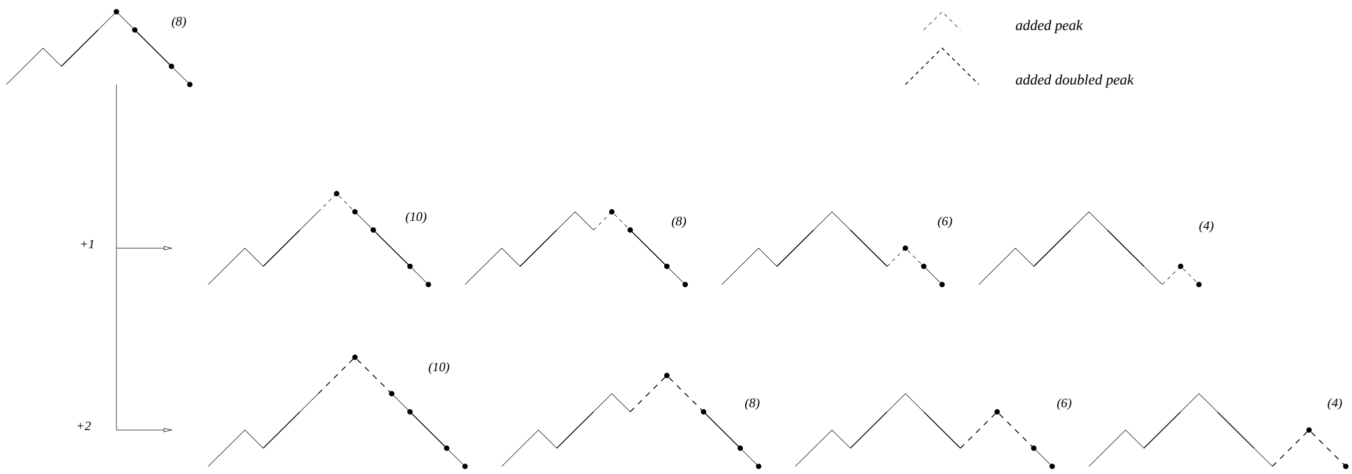

Figure 4: The doubled operator ϑ' on a doubled Dyck path. The marked points denote the sites where the operator performs the transformation.

$$\frac{1-\sqrt{1-4(x^2+x^4)}}{2(x^2+x^4)}$$

defining the numerical sequence $1, 1, 3, 9, 31, 113, 431, 1697, \ldots$, omitting the zeroes (sequence A052709 in [14]).

At the end of this section, we give a result which characterizes the set of generating functions of doubled succession rules. Recall that two rules are said to be *equivalent* when they define the same sequence.

Theorem 2.2 *Let Ω be a succession rule, and Ω' the doubled rule associated with Ω. Then a succession rule Ω'' exists such that Ω'' and Ω' are equivalent.*

Proof. We prove that, given a succession rule (1), the doubled succession rule Ω' associated with Ω, having the form (3), is equivalent to the following coloured rule:

$$\Omega'' : \begin{cases} (\overline{a}) \\ (k+1) \rightsquigarrow (e_1(k)+1)\ldots(e_k(k)+1)(\overline{k}) \\ (\overline{k}) \rightsquigarrow (e_1(k)+1)\ldots(e_k(k)+1). \end{cases} \tag{15}$$

Let L be the rule operator associated with Ω, and M the rule operator associated with Ω'':

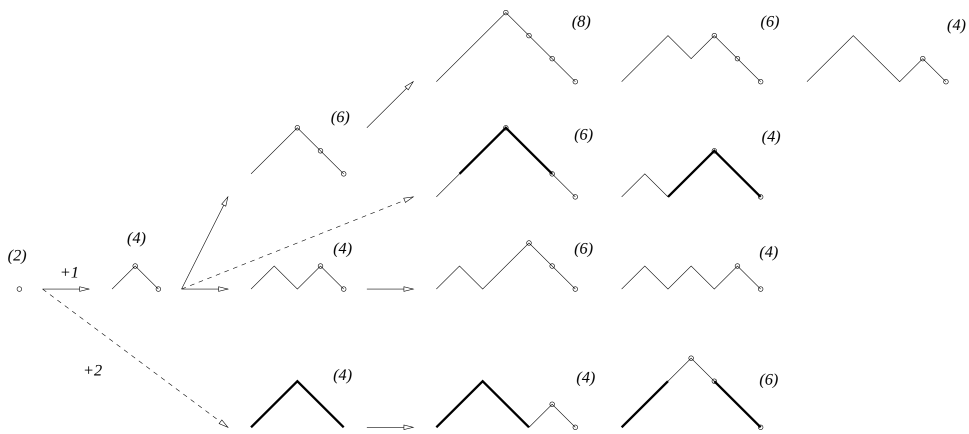

Figure 5: The first levels of the generating tree related to the doubled Catalan operator ϑ'.

$$M : x\mathbb{R}[x] \oplus \mathbb{R}[y] \longrightarrow x\mathbb{R}[x] \oplus \mathbb{R}[y]$$

$$M(\mathbf{1}) = y^a;$$

$$M(y^k) = xL(x^k);$$

$$M(x^{k+1}) = xL(x^k) + y^k.$$

The definition of the rule operator associated with a coloured succession rule can be found in [7]. It is easy to prove the following statements:

1) $M(xp(x)) = xL(p(x)) + p(x)$;

2) $M(p(y)) = xL(p(x))$;

3) $M^n(\mathbf{1}) = x\sum_{k=0}^{n-1}\binom{k-1}{n-k-1}L^k(x^a) + \sum_{k=0}^{n-2}\binom{k-1}{n-k-2}L^k(y^a)$.

As a consequence of these facts we have the desired result:

$$[M^n(\mathbf{1})]_{x=y=1} = \sum_{k=0}^{n-1}\binom{k}{n-k-1}s_k = s'_{n-1}. \quad \square$$

Simple as it is, Theorem 2.2 has a deep meaning from a theoretical viewpoint: in a word, it states that the set of generating functions of doubled succession rules is included into the set of generating functions of succession rules.

For example, we trivially obtain that the doubled rule (5) associated with the rule (4) defines Fibonacci numbers, like the rule:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (1)(2). \end{cases}$$

Moreover, the doubled rule (13), associated with Catalan numbers, is equivalent to the following rule:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (3) \\ (k+1) \rightsquigarrow (3)\dots(k+1)(\overline{k-1}) \\ (\overline{k}) \rightsquigarrow (3)\dots(k+2). \end{cases} \tag{16}$$

3 Jumping succession rules

The idea of doubling a succession rule can be slightly generalized in the following way.

Given the succession rule Ω of the form (1), and $i_1, \dots, i_m \in \mathbb{N}^+$ such that $0 < i_1 < \dots < i_m$, we call *jumping succession rule* of type $(i_1, \dots, i_m)$ associated with Ω the rule:

$$\Omega^{(i_1,\dots,i_m)} : \begin{cases} (ma) \\ (mk) \overset{i_1}{\rightsquigarrow} (me_1(k))\dots(me_k(k)) \\ \dots \\ (mk) \overset{i_m}{\rightsquigarrow} (me_1(k))\dots(me_k(k)). \end{cases} \tag{17}$$

Clearly, a doubled succession rule is a jumping rule of type $(1,2)$. Following the same philosophy of Section 2, we define the *normalization* of $\Omega^{(i_1,\dots,i_m)}$ as:

$$\widetilde{\Omega}^{(i_1,\dots,i_m)} : \begin{cases} (a) \\ (k) \overset{i_1}{\rightsquigarrow} (e_1(k))\dots(e_k(k)) \\ \dots \\ (k) \overset{i_m}{\rightsquigarrow} (e_1(k))\dots(e_k(k)). \end{cases} \tag{18}$$

The main enumerative results concerning jumping rules can be easily proved following the ideas developed in Section 2.

Proposition 3.1 *The bivariate generating function of the generating tree defined by* $\widetilde{\Omega}^{(i_1,\dots,i_m)}$ *has the form:*

$$\left(\frac{1}{1-t^{i_1}L-\ldots-t^{i_m}L}\right)\left(\sum_{i=1}^{i_1-1}L^{i+1}(\mathbf{1})t^i\right)$$
$$=\sum_{n\geq 0}(t^{i_1}L+\ldots+t^{i_m}L)^n\left(\sum_{i=1}^{i_1-1}L^{i+1}(\mathbf{1})t^i\right), \tag{19}$$

being L the rule operator associated with Ω.

Theorem 3.1 *If Ω counts the sequence $(s_n)_{n\geq 0}$, then the sequence enumerated by $\Omega^{(i_1,\ldots,i_m)}$ is:*

$$s'_n=\sum_{\alpha=0}^{i_1-1}\sum_{\substack{\mu_1,\ldots,\mu_m\\ \mu_1 i_1+\ldots+\mu_m i_m=n-\alpha}}\binom{\sum_{i=1}^m\mu_i}{\mu_1,\ldots,\mu_m}s_{(\sum_{i=1}^m\mu_i+\alpha)}, \tag{20}$$

where the expression $\binom{\theta}{\theta_1,\ldots,\theta_t}$, $\theta_1+\ldots+\theta_t=\theta$, denotes the usual multinomial coefficient. We call $(s'_n)_{n\geq 0}$ the Fibonacci transform of type $i_1,\ldots,i_m$ *of $(s_n)_{n\geq 0}$.*

Remark 3.1

1. s'_n is the sum of the number of the nodes at levels $n-i_1,\ldots,n-i_m$ in the "jumping generating tree".
2. If $i_1=1$, the expression for the numbers s'_n counted by $\Omega^{(1,i_2,\ldots,i_m)}$ is a bit more readable:

$$s'_n=\sum_{\substack{\mu_1,\ldots,\mu_m\\ \mu_1+\ldots+\mu_m i_m=n}}\binom{\sum_{i=1}^m\mu_i}{\mu_1,\ldots,\mu_m}s_{(\sum_{i=1}^m\mu_i)}. \tag{21}$$

3. It is clear that this result applied to $\Omega^{(1,2)}$ coincides with the result obtained for doubled rules, since in this case:

$$s'_n=\sum_{\substack{\mu_1,\mu_2\\ \mu_1+2\mu_2=n}}\binom{\mu_1+\mu_2}{\mu_1,\mu_2}s_{\mu_1+\mu_2}$$
$$=\sum_{\mu_2=0}^{n}\binom{n-\mu_2}{\mu_2}s_{n-\mu_2}. \tag{22}$$

Example 3.1 *Tribonacci numbers.*
Let Ω be

$$\Omega : \begin{cases} (1) \\ (1) \rightsquigarrow (1), \end{cases}$$

the jumping rule $\Omega^{(1,2,3)}$ defines the well-known Tribonacci numbers having $T_0 = 1, T_1 = 1, T_2 = 2$ as initial values. By applying equality (21), we obtain the following remarkable formula:

$$\begin{aligned} T_n &= \sum_{\substack{\mu_1, \mu_2, \mu_3 \\ \mu_1 + 2\mu_2 + 3\mu_3 = n}} \binom{\mu_1 + \mu_2 + \mu_3}{\mu_1, \mu_2, \mu_3} \\ &= \binom{n}{n,0,0} + \binom{n-1}{n-2,1,0} + \binom{n-2}{n-3,0,1} \\ &+ \binom{n-2}{n-4,2,0} + \binom{n-3}{n-5,1,1} + \binom{n-3}{n-6,3,0} \\ &+ \binom{n-4}{n-6,0,2} + \binom{n-4}{n-7,2,1} + \binom{n-5,}{n-8,1,2} + \dots \end{aligned}$$

which is the obviuos generalization to Tribonacci numbers of Lucas' identity. This equality was obtained by Shannon in [13] by a direct computation; the interest of our proof lies in the fact that it can be easily generalized to n-bonacci numbers, for every $n \in \mathbb{N}$.

3.1 Scattered succession rules and linear recurrences

We have just studied the generating tree obtained by "repeating" a succession rule Ω at various levels. A step forward could be done by allowing the repetition of Ω "more than one time" at each level.

We say that Ω' is a *scattered succession rule* associated with Ω whenever there exist positive integers $m_1, \dots, m_r, i_1, \dots, i_r$ such that $m = m_1 + \dots + m_r$ and:

$$\Omega' : \begin{cases} (ma) \\ (mk) \overset{i_1}{\rightsquigarrow} (me_1(k))^{m_1} \dots (me_k(k))^{m_1} \\ \dots \\ (mk) \overset{i_r}{\rightsquigarrow} (me_1(k))^{m_r} \dots (me_k(k))^{m_r}. \end{cases}$$

The *normalization* $\widetilde{\Omega'}$ of Ω' is defined in the usual way.

An interesting application of this definition can be obtained by considering the simple rule (4). In fact, the following proposition holds.

Proposition 3.2 *Suppose that the sequence* $(a_n)_{n\geq 0}$ *is defined by the linear recurrence* $a_n = m_1 a_{n-1} + \ldots + m_r a_{n-r}$, $m_1, \ldots, m_r \in \mathbb{N}$ *and having the initial values* $a_0 = 1$, $a_1 = m_1 a_0$, $a_2 = m_1 a_1 + m_2 a_0$,..., $a_{r-1} = m_1 a_{r-2} + \ldots + m_{r-1} a_0$. *Then* $(a_n)_{n\geq 0}$ *is the sequence determined by the scattered rule* Ω' *defined by:*

$$\Omega' : \begin{cases} (m) \\ (m) \overset{1}{\rightsquigarrow} (m)^{m_1} \\ \ldots \\ (m) \overset{r}{\rightsquigarrow} (m)^{m_r} \end{cases},$$

with $m = m_1 + \ldots + m_r$.

4 Exploded succession rules

Let Ω be a succession rule of the form (1) and h a positive integer. Consider the following jumping rule:

$$\Omega^{(1,2,\ldots,h)} : \begin{cases} (ha) \\ (hk) \overset{1}{\rightsquigarrow} (he_1(k)) \ldots (he_k(k)) \\ \ldots \\ (hk) \overset{h}{\rightsquigarrow} (he_1(k)) \ldots (he_k(k)). \end{cases} \tag{23}$$

Now let h tend to infinity: clearly the jumping rule $\Omega^{(1,2,\ldots,h)}$ cannot be expressed formally, whereas its normalization $\widetilde{\Omega}^{(1,2,\ldots,h)}$ can. More precisely, we can informally state that

$$\lim_{h\to\infty} \widetilde{\Omega}^{(1,2,\ldots,h)} = \widetilde{\Omega}^{\infty},$$

where

$$\widetilde{\Omega}^{\infty} : \begin{cases} (a) \\ (k) \overset{1}{\rightsquigarrow} (e_1(k)) \ldots (e_k(k)) \\ \ldots \\ (k) \overset{h}{\rightsquigarrow} (e_1(k)) \ldots (e_k(k)) \\ \ldots . \end{cases} \tag{24}$$

Every node possesses an infinite number of sons in the generating tree determined by $\widetilde{\Omega}^{\infty}$. The rule $\widetilde{\Omega}^{\infty}$ is called the *exploded succession rule* associated with Ω.

Next we study the bivariate generating functions and the number sequences given by (24). Quite surprisingly, we get rather simple expressions and closed forms in contrast with the (formal) difficulties when passing from doubled rules to arbitrary jumping rules.

Proposition 4.1 *The bivariate generating function related to $\widetilde{\Omega}^{\infty}$ has the form:*

$$\frac{t-1}{1-t-tL}(L(\mathbf{1})) = (t-1)\cdot\sum_{n\geq 0}(1+L)^n t^n (L(\mathbf{1})). \tag{25}$$

Proof. Consider the bivariate generating function of the jumping rule $\widetilde{\Omega}^{(1,2,\ldots,h)}$:

$$\frac{1}{1-tL-t^2L-\ldots-t^hL}(L(\mathbf{1})) = \frac{1}{1-tL(1+t+\ldots+t^{h-1})}(L(\mathbf{1})).$$

By letting h tend to infinity we get:

$$\begin{aligned}\frac{1}{1-tL\cdot\sum_{h\geq 0}t^h}(L(\mathbf{1})) &= \frac{1}{1-tL\frac{1}{1-t}}(L(\mathbf{1})) \\ &= \frac{1-t}{1-t-tL}(L(\mathbf{1}))\end{aligned}$$

which is the desired generating function. □

Theorem 4.1 *The sequence $(s_n^{\infty})_{n\geq 0}$ determined by $\widetilde{\Omega}^{\infty}$ is:*

$$\begin{aligned} s_0^{\infty} &= 1; \\ s_n^{\infty} &= \sum_{k=0}^{n-1}\binom{n-1}{k}s_{k+1}, \qquad n\geq 1. \end{aligned} \tag{26}$$

We will say that $(s_n^{\infty})_{n\geq 0}$ is the exploded Fibonacci transform *of the sequence $(s_n)_{n\geq 0}$.*

Proof. We manipulate the generating function obtained in Proposition 4.1 in the usual way:

$$\begin{aligned} f(x,t) &= (1-t)\sum_{n\geq 0}(1+L)^n t^n (L(\mathbf{1})) \\ &= \sum_{n\geq 0}(1+L)^n(L(\mathbf{1}))t^n - \sum_{n\geq 1}(1+L)^{n-1}(L(\mathbf{1}))t^n \\ &= L(\mathbf{1}) + \sum_{n\geq 1}(1+L)^{n-1}L^2(\mathbf{1})t^n. \end{aligned}$$

Thus, for $n \geq 1$, we have:

$$\begin{aligned} s_n^\infty &= [(1+L)^{n-1}L^2(\mathbf{1})]_{x=1} \\ &= \left[\sum_{k=0}^{n-1}\binom{n-1}{k}L^{k+2}(\mathbf{1})\right]_{x=1} \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}s_{k+1}, \end{aligned}$$

as desired. □

Example 4.1 1. Let

$$\Omega : \begin{cases} (1) \\ (1) \rightsquigarrow (1). \end{cases}$$

We already know that $\Omega^{(1,2)}$ counts the Fibonacci numbers, $\Omega^{(1,2,3)}$ counts the Tribonacci numbers, and so on. Which is the sequence counted by the exploded rule $\widetilde{\Omega}^\infty$? By applying theorem 4.1 we get:

$$s_n^\infty = \sum_{k=0}^{n-1}\binom{n-1}{k} = 2^{n-1}. \tag{27}$$

The table below shows the first terms of the sequences defined by $\Omega^{(1,2,3,\dots,h)}$:

$k\backslash n$	0	1	2	3	4	5	6	…
1	1	1	1	1	1	1	1	…
2	1	1	2	3	5	8	13	…
3	1	1	2	4	7	13	24	…
4	1	1	2	4	8	15	29	…
5	1	1	2	4	8	16	31	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
∞	1	1	2	4	8	16	32	…

Thus *the total number of nodes at level n in the generating tree determined by* $\widetilde{\Omega}^\infty$ *is equal to* 2^{n-1}. Now we give a nice combinatorial interpretation of this result. For any fixed n, the set of nodes at level n in the generating tree characterized by $\widetilde{\Omega}^\infty$ can be described using the words of length n of the language $\mathcal{L}$ on the alphabet $\Sigma = \{x_1, x_2, x_3, \ldots x_n\}$ generated by the regular grammar:

$$S \rightarrow x_1 S | x_2^2 S | x_3^3 S | \ldots | x_n^n S | \epsilon.$$

Indeed, for any node N at level n, let us consider the path from the root to N. Following such a path, each edge of length i ($i \leq n$) is coded by x_i^i. Thus we obtain a word of $\mathcal{L}$ having length n. For instance, the nodes at level $n = 4$ are coded by the words $x_1x_1x_1x_1$, $x_1x_1x_2x_2$, $x_1x_2x_2x_1$, $x_2x_2x_1x_1$, $x_2x_2x_2x_2$, $x_1x_3x_3x_3$, $x_3x_3x_3x_1$, $x_4x_4x_4x_4$.

Therefore we give another proof of (27) by providing a bijection between n-length words of $\mathcal{L}$, and $(n-1)$-length paths in the discrete plane, running from $(0,0)$ and using *rise* steps $(1,1)$ or *fall* steps $(1,-1)$. Each word $w \in \mathcal{L}$ can be univocally decomposed into blocks:

$$w = B_1 B_2 \ldots B_h, \qquad B_i \in \Sigma^+,$$

such that $B_i = x_l^l$, $i = 1, \ldots, h$. For example the word $x_2x_2x_2x_2x_1x_3x_3x_3x_1$ is constituted by the blocks x_2x_2, x_2x_2, x_1, $x_3x_3x_3$, x_1. Now we recursively define the function ψ on the words of $\mathcal{L}$ as follows:

$$\psi(\epsilon) = \psi(x_i) = \text{the empty path}, \qquad x_i \in \Sigma;$$

$$\psi(x_i x_j) = \begin{cases} \text{rise step} & \text{if } x_i \text{ and } x_j \text{ belong to the same block,} \\ \text{fall step} & \text{otherwise;} \end{cases}$$

$$\psi(w) = \psi(x_1x_2)\psi(x_2x_3)\ldots\psi(x_{n-1}x_n) \qquad \text{being } w = x_1x_2\ldots x_n,\ x_i \in \Sigma.$$

It is easy to prove that, for each $n \geq 1$, ψ is a bijection between n-length words in $\mathcal{L}$ and $(n-1)$-length paths. Figure 6 shows the bijection for $n = 4$.

2. By generalizing the above example we can consider:

$$\Omega_a : \begin{cases} (a) \\ (a) \rightsquigarrow (a)^a, \end{cases}$$

defining the sequence $(a^n)_{n\geq 0}$. A simple computation shows that the sequence counted by the exploded succession rule $\widetilde{\Omega}_a^\infty$ is precisely $(a(a+1)^{n-1})_{n\geq 0}$.

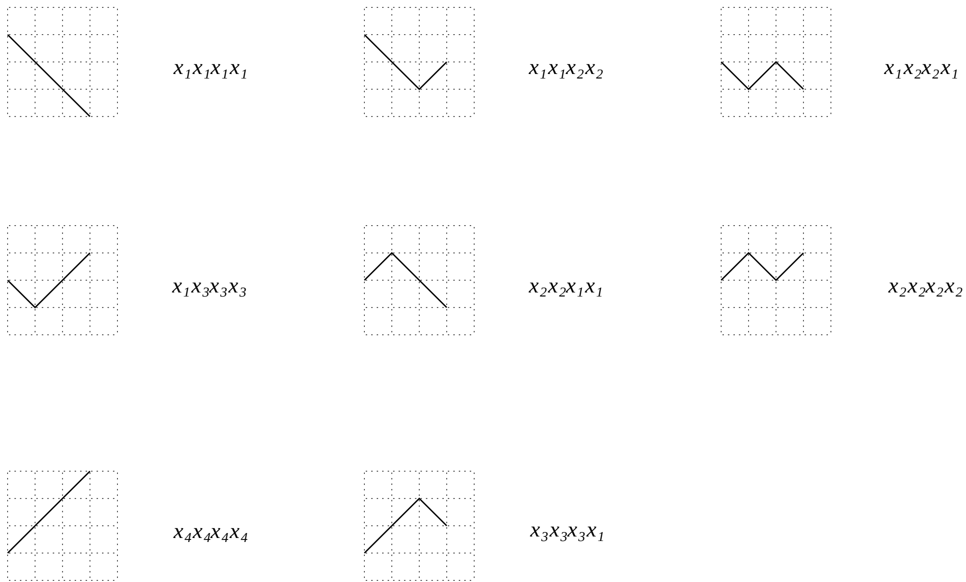

Figure 6: A bijective proof for the infinite Fibonacci transform of the sequence $1, 1, 1, 1, \ldots$.

Example 4.2 Let Ω be the rule (12) defining Catalan numbers. Let us now consider the rules $\Omega^{(1,2,3,\ldots,k)}$, $k \geq 1$; for any fixed k, the rule $\Omega^{(1,2,3,\ldots,k)}$ enumerates the language defined by the unambiguous context-free grammar:

$$S \to x_1 S | x_2^2 S | \ldots | x_k^k S^2 | \epsilon.$$

Then the generating function $f_k(x)$ of the rule $\Omega^{(1,2,3,\ldots,k)}$ is easily determined:

$$f_k(x) = \frac{1 - \sqrt{1 - 4(x + x^2 + \ldots + x^k)}}{2(x + x^2 + \ldots + x^k)}.$$

Letting k tend to infinity we have the generating function $f_\infty(x)$ for the exploded rule $\widetilde{\Omega}^\infty$:

$$f_\infty(x) = \frac{1 - x - \sqrt{1 - 6x + 5x^2}}{2x}.$$

This generating function defines a sequence f_n^∞ which is strictly related to Catalan numbers: the numbers are $1, 1, 3, 10, 36, 137, 543, 2219, 9285, \ldots,$ (A002212 in [14]), and count two different structures:

1. f_{n+1}^∞ is the number of 3-coloured Motzkin paths having length n ([15]);
2. f_n^∞ is the number of edge-rooted polyhexes having n hexagons ([11]).

These facts still ask for a combinatorial explanation.

Example 4.3 Let $(B_n)_{n\geq 0}$ be the sequence of Bell numbers; by definition, B_n counts the way to partition an n-set into nonempty subsets. We define the sequence $(\overline{B}_n)_{n\geq 0}$ of *shifted Bell numbers* by setting $\overline{B}_0 = 1$ and $\overline{B}_{n+1} = B_n$ for all $n \in \mathbb{N}$. A succession rule Ω counting these numbers is the following:

$$\Omega : \begin{cases} (\overline{1}) \\ (\overline{1}) \rightsquigarrow (1) \\ (k) \rightsquigarrow (k)^{k-1}(k+1). \end{cases}$$

This is a typical example of a *coloured succession rules.* It is not difficult to extend all the notions defined in this paper to coloured rules. In particular, we can consider the exploded succession rule $\widetilde{\Omega}^{\infty}$; by the usual properties of Bell numbers, we observe that the shifted Bell numbers constitute a "quasi-fixed point" for the infinite Fibonacci transform, since:

$$\overline{B}_n^{\infty} = \sum_{k=0}^{n-1} \binom{n-1}{k} \overline{B}_{k+1} = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k = B_n = \overline{B}_{n+1}. \tag{28}$$

A result analogous to Theorem 2.2 holds for exploded succession rules.

Theorem 4.2 *Let Ω be a succession rule, and Ω^{∞} the exploded succession rule associated with Ω. Then the succession rule*

$$\Omega' : \begin{cases} (a) \\ (a) \rightsquigarrow (e_1(a)+1)\dots(e_a(a)+1) \\ (k+1) \rightsquigarrow (e_1(k)+1)(e_2(k)+1)\dots(e_k(k)+1)(k+1), \end{cases} \tag{29}$$

is equivalent to Ω^{∞}.

Example 4.4 Let Ω be the rule defining Fibonacci numbers, having (2) as axiom:

$$\begin{cases} (2) \\ (1) \rightsquigarrow (2), \\ (2) \rightsquigarrow (1)(2) \end{cases}$$

According to Theorem 4.2 the exploded succession rule Ω^∞ associated with Ω is equivalent to the following:

$$\begin{cases} (2) \\ (2) \rightsquigarrow (2)(3), \\ (3) \rightsquigarrow (2)(3)(3), \end{cases}$$

which defines the odd Fibonacci numbers!

Example 4.5 The exploded rule of Catalan numbers, already examined in Example 4.2, is equivalent to the rule:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (3), \\ (k) \rightsquigarrow (3)(4)\dots(k)(k)(k+1), \end{cases}$$

One can go further and iterate the application of the transform defined in Theorem 4.2 to a given succession rule. Let $\mathcal{S}$ be the set of succession rules, and let $T : \mathcal{S} \to \mathcal{S}$ be the operator such that, for any rule Ω, $T(\Omega)$ is the rule defined by (29), equivalent to the exploded succession rule $\widetilde{\Omega}^\infty$ associated with Ω. Now let us define:

$$T^0(\Omega) = \Omega$$

$$T^n(\Omega) = T(T^{n-1}(\Omega)) \qquad n \geq 1$$

Now let Ω be the rule (12) defining Catalan numbers. We easily obtain the following facts, which extend our previous results:

i) for any $n \geq 0$, $T^n(\Omega)$ has the form:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (n+2), \\ (k) \rightsquigarrow (n+2)(n+3)\dots(k-1)(k)^{n+1}(k+1); \end{cases}$$

ii) for any $n \geq 0$, $T^n(\Omega)$ enumerates $(n+2)$-coloured Motzkin paths according to the length of the path.

In a word, the combinatorial meaning of Theorem 4.2 is that exploded succession rules do not enlarge the set of generating functions of succession rules.

5 Further work

1. Given a sequence $(D_n)_{n\geq 0}$, is it possible to find a sequence $(C_n)_{n\geq 0}$ such that $(D_n)_{n\geq 0}$ is its Fibonacci transform? This is simply the problem of inverting a combinatorial sum, and it has been solved, for example, in the classical text [12], where it is classified as a Chebyshev inverse relation. The solution is:

$$C_n = \sum_{k=0}^{n}(-1)^k \left(\binom{n+k-1}{k} - \binom{n+k-1}{k-1}\right) D_{n-k}.$$

   Instead, it would be interesting to know when the sequence $(C_n)_{n\geq 0}$ can be represented by means of a suitable succession rule, since in this case we are able to describe D_n using a doubled succession rule. Of course, these problems can be stated for the Fibonacci transform of any type, but their solution seems much more complicated.

2. If a sequence $(C_n)_{n\geq 0}$ can be described by means of a succession rule, does the same happen for its Fibonacci transform? We have seen that the answer is positive if we allow coloured rules, but the problem remains open if we restrict to non coloured ones. A solution to this question would allow to iterate the Fibonacci transform, as we did in Example 4.5 for the exploded Fibonacci transform.

3. Shifted Bell numbers are a "quasi-fixed point" for the exploded Fibonacci transform. What about the Fibonacci transforms of any other type?

4. Given a double-indexed sequence $\sigma_{n,k}$, we can define, for any sequence $(C_n)_{n\geq 0}$:

$$C_n^\sigma = \sum_{k=0}^{n} \sigma_{n-k,k} C_{n-k}.$$

   This is clearly done in analogy with Fibonacci transform. Can we say anything about the sequence $(C_n^\sigma)_{n\geq 0}$? Is it possible to give a description of this transform in terms of something similar to succession rules, at least when $\sigma_{n,k}$ is a sequence of combinatorial interest (Stirling numbers, etc.)?

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