Moments, Narayana Numbers, and the Cut and Paste for Lattice Paths

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January 28, 2002.

Abstract. Let $\mathcal{U}(n)$ denote the set of lattice paths that run from $(0,0)$ to $(n,0)$ with permitted steps $(1,1)$, $(1,-1)$, and perhaps a horizontal step. Let $\mathcal{E}(n+2)$ denote the set of paths in $\mathcal{U}(n+2)$ that run strictly above the horizontal axis except initially and finally. First we review *the cut and paste bijection* which relates points under paths of $\mathcal{E}(n+2)$ to points on paths of $\mathcal{U}(n)$. We apply it to obtain enumerations, some involving the Narayana distribution. We extend the bijection to a formula relating arbitrary factorial moments for the paths of $\mathcal{E}(n+2)$ to moments for the paths of $\mathcal{U}(n)$. This formula produces some additional results for moments and for the total area of the paths of $\mathcal{E}(n+2)$.

Key phases: lattice path moments, Catalan numbers, Narayana distribution, Schröder numbers, square-triangular numbers.

1 Introduction

Consider lattice paths in the integer plane represented as concatenations of the directed steps types: $U := (1,1)$, $D := (1,-1)$, and, perhaps, $H := (h,0)$ where h is a positive integer. When the steps are weighted, the weight of a path P, denoted by $|P|$, is the product of the weights of its steps. The weight of a path set $\mathcal{X}$, denoted $|\mathcal{X}|$, is the sum of the weights of its paths. For a given step set $\mathcal{S}$, let $\mathcal{U}(n)$ denote the set of all *unrestricted* paths running from $(0,0)$ to $(n,0)$. Let $\mathcal{C}(n)$ denote the set of paths in $\mathcal{U}(n)$ *constrained* never to pass beneath the horizontal axis. Let $\mathcal{E}(n)$ denote the set of paths in $\mathcal{C}(n)$ that are *elevated* strictly above the horizontal axis except at their initial and final points. E.g., for the unit-weighted steps of $\mathcal{S} = \{U, D\}$ and for $n \geq 0$, we have that $|\mathcal{U}(2n)|$ is the central binomial coefficient $\binom{2n}{n}$, $\mathcal{C}(2n)$ are the Dyck paths of length $2n$, and $|\mathcal{C}(2n)| = |\mathcal{E}(2n+2)|$ is a Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

For any step set $\mathcal{S}$ and any path P running from $(0,0)$ to $(n,0)$, let

$$(0,p_0),(1,p_1),(2,p_2),\ldots,(x,p_x),\ldots,(n,p_n) \tag{1}$$

denote the lattice points traced by the path. (When P uses an $(h,0)$ step for $h \geq 2$, the trace points need not be at the ends of steps.) For any real valued function f defined on the integers and for any path set $\mathcal{X}(n)$, $\mathcal{X}(n) \subseteq \mathcal{U}(n)$, the *weighted moment with respect to the formula* $f(y)$ is designated as

$$\mu(\mathcal{X}(n), f(y)) = \sum_{P \in \mathcal{X}(n)} |P| \sum_{x=0}^{n} f(p_x).$$

For $f(y) = 1/(n+1)$ and $\mathcal{X}(n) \subseteq \mathcal{U}(n)$, $\mu(\mathcal{X}(n), 1/(n+1)) = |X(n)|$. By the trapezoid rule, if $f(y) = y$, $\mu(\mathcal{E}(n), y)$ is the sum of the weighted areas of the regions bounded by the paths of $\mathcal{E}(n)$ and the horizontal axis. When we permit only the unit-weighted steps of $\mathcal{S} = \{U, D\}$, we will see later that $\mu(\mathcal{E}(2n+2), y) = 4^n$ for $n \geq 0$.

This paper is a direct continuation of the paper [6] which introduces *the cut and paste* bijective method relating lattice points under elevated paths of $\mathcal{E}(n+2)$ to points on the unrestricted paths of $\mathcal{U}(n)$. In [6] the method yielded results about zeroth, first, and second moments for $\mathcal{E}(n+2)$, in particular

$$\mu(\mathcal{E}(n+2), y^2) = \mu(\mathcal{U}(n), 1) = (n+1)|\mathcal{U}(n)|. \tag{2}$$

Section 2 will review the cut and paste method. Section 3 will give some illustrations of the method obtained by restricting its domain and codomain. *In particular, the cut and paste delivers the Narayana numbers.* Section 4 will extend the method to a result that relates factorial moments for $\mathcal{E}(n+2)$ to factorial moments for $\mathcal{U}(n)$. The paper concludes with additional means for handling moments and with consequences of Section 4 including results related to the Schröder, central Delannoy, and square-triangular numbers.

Some notation and background. In this paper n will denote an arbitrary nonnegative integer. Usually, U, D, and H $(= (h, 0))$ will denote unit-weighted steps, while U_t and H_s will denote steps weighted by the indeterminates t and s. For notational brevity, we will allow h to be either a positive integer or '∞'. Effectively, $\{U, D, (\infty, 0)\} = \{U, D\}$, and the power z^∞ will make no contribution to any power series.

For any $\mathcal{S} = \{U_t, D, H_s\}$, consider the following generating functions: $c(z) := \sum_{n\geq 0} |\mathcal{C}(n)|z^n$, $e(z) := tz^2c(z) = \sum_{n\geq 0} |\mathcal{E}(n+2)|z^{n+2}$, and $u(z) := \sum_{n\geq 0} |\mathcal{U}(n)|z^n$.

From the known decompositions of paths sets we have,

$$c(z) = 1 + sz^h c(z) + tz^2 c(z)^2 \tag{3}$$

$$u(z) = 1 + sz^h u(z) + 2tz^2 c(z)u(z) \tag{4}$$

To see (4) note that every path in $\mathcal{U}(n)$ either has zero length, begins with H, or begins with U or D followed by a constrained path or its reflection and later returns to the horizontal axis for the first time. Identity (3) follows in a similar manner. Solving these yields

$$\begin{aligned} e(z) &= tz^2c(z) = (1 - sz^h - \sqrt{(1 - sz^h)^2 - 4tz^2})/2, \\ u(z) &= 1/\sqrt{(1 - sz^h)^2 - 4tz^2}. \end{aligned} \tag{5}$$

We remark that Example 6.3.8 of [9] extends easily to an alternative derivation of (5).

Recall the rising factorial, $z^{\overline{k}}$, defined so $z^{\overline{k}} = z(z+1)\cdots(z+k-1)$ for positive integer k, $z^{\overline{0}} = 1$, and $z^{\overline{k}} = 0$ for negative integer k. Then, for $k \geq 0$, $\binom{r+k}{k} = (r+1)^{\overline{k}}/k!$. For any statement A, we define its truth value by $\chi(A)$ so that $\chi(A) = 1$ if A is true, and $= 0$ otherwise.

2 The cut and paste method

Here we define *the cut and paste method*, which was presented with more detail, including its invertibility, in [6]. Let $\mathcal{S} = \{U, D, H\}$. First we need the notion of a *dot*. Given a path $P \in \mathcal{E}(n+2)$, given a lattice point (x, y) lying strictly under P but weakly above the horizontal axis, and given an integer k, a *dot* is a triple, $[P, (x, y), k]$. The index k permits the existence of more than one distinguishable dot at some points. With the notation of (1), the domain for our proposed bijection is

$$\mathrm{DOTS}(n+2) := \{[P, (x, y), k] : P \in \mathcal{E}(n+2), 0 < x < n+2, 0 \leq y < p_x, -p_x+y < k < p_x-y\}.$$

This domain can be partitioned into triangular arrays of dots, with one array corresponding to each lattice point on the trace of each elevated path in $\mathcal{E}(n+2)$. Thus there will be $(n+1)|\mathcal{E}(n+2)|$ triangular arrays. E.g., if $P = UUDUUDDD$ and if $x = 5$, then $p_5 = 3$ and the corresponding array appears as

$$\begin{array}{ccccc} & & [P,(5,2),0] & & \\ & [P,(5,1),-1] & [P,(5,1),0] & [P,(5,1),1] & \\ [P,(5,0),-2] & [P,(5,0),-1] & [P,(5,0),0] & [P,(5,0),1] & [P,(5,0),2] \end{array}$$

The codomain for the proposed bijection is a set of pointed paths, each path being pointed, i.e. marked, by a distinguished lattice point on its trace. Hence the codomain is

$$\mathrm{POINTS}(n) := \{(P, (x, p_x)) : P \in \mathcal{U}(n), 0 \leq x \leq n\}.$$

We now define

$$\phi : \mathrm{DOTS}(n+2) \to \mathrm{POINTS}(n) \tag{6}$$

First assume $k \geq 0$. Each $[P, (x, y), k] \in \mathrm{DOTS}(n+2)$ determines four points on P (See Fig. 1):

- Let θ be the point on P directly above (x, y); i.e., $\theta := (x, p_x)$.
- Let $\epsilon = (\epsilon_1, \epsilon_2)$ be the nearest point on P to the left of (x, y) such that $\epsilon_2 = p_x - k - 1$. (This indicates the role k plays in defining the bijection.)
- Let $\lambda = (\lambda_1, \lambda_2)$ be the nearest point on P to the left of (x, y) such that $\lambda_2 = y$.
- Let $\rho = (\rho_1, \rho_2)$ be the nearest point on P to right of (x, y) such that $\rho_2 = y$.

Let L_1 be that subpath of P running from $(0, 0)$ to λ; let L_2 be that subpath of P running from λ to ϵ; let R_1 be that subpath of P running from ϵ to ρ; let R_2 be that subpath of P running from ρ to $(n+2, 0)$. Some of these subpaths may be empty. Here $P = L_1 L_2 R_1 R_2$.

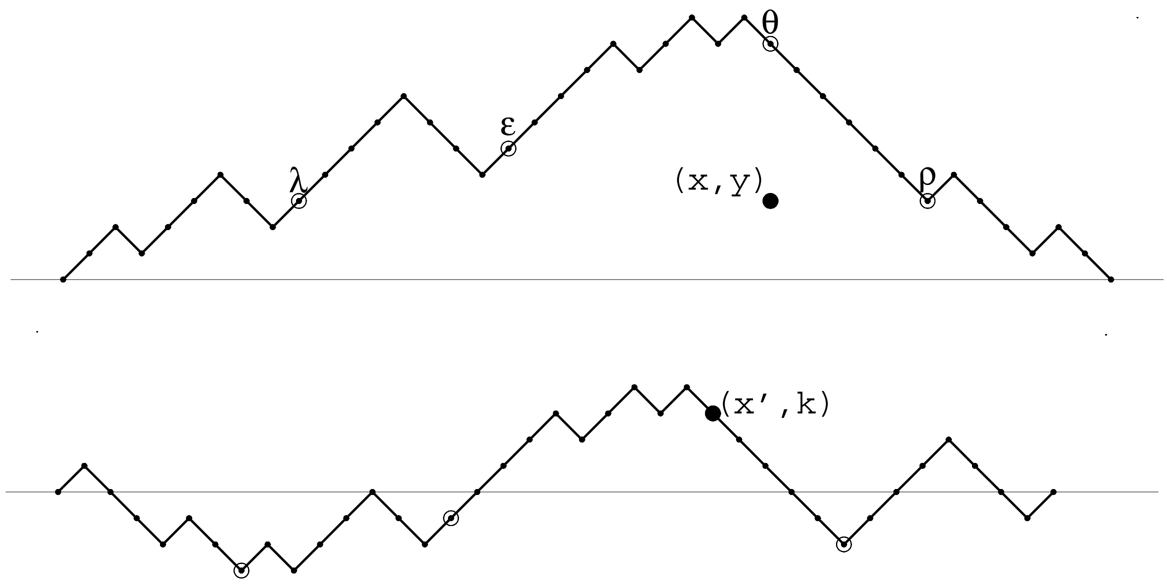

Figure 1: For $k = 3$, the dot $[P, (27,3), 3]$ in DOTS(40) and its image $[P', (25,3)]$ in POINTS(38).

Let $\overline{L_1 R_1}$ be the path obtained from the concatenation $L_1 R_1$ by deleting its first and last steps. When $y = 0$, L_1 has zero length, and we will use $\overline{\overline{R_1}}$ to denote $\overline{L_1 R_1}$. Define

$$\phi([L_1 L_2 R_1 R_2, (x, y), k]) = (R_2 \overline{L_1 R_1} L_2, (x', k)) \tag{7}$$

where the point θ is moved along with $\overline{R_1}$ so that $x' = x + n + \lambda_1 - \epsilon_1 - \rho_1 - 1$.

If $k < 0$, let REFL(P') denote the reflection of the path P' about the x-axis and define

$$\phi([P, (x, y), k]) = (\text{REFL}(P'), (x', k))$$

where $\phi([P, (x, y), |k|]) = (P', (x', |k|))$.

3 Examples of the cut and paste

Here we illustrate the cut and paste method by restricting its domain and codomain to prove some known results, mainly concerning the Narayana distribution and the large Schröder numbers. Other such examples appear in [6].

3.1 Cardinality results analogous to the cycle lemma

For $\mathcal{S} = \{U_t, D, H_s\}$ let $\mathcal{E}(n, m)$ and $\mathcal{U}(n, m)$ denote those subsets of $\mathcal{E}(n)$ and $\mathcal{U}(n)$, respectively, where each path has m U-steps.

Proposition 1 *For $m \geq 0$,*

$$(m + 1)|\mathcal{E}(n + 2, m + 1)| = t|\mathcal{U}(n, m)|.$$

When $\mathcal{S} = \{U, D\}$, this formula reduces to $(n+1)|\mathcal{E}(2n+2)| = |\mathcal{U}(2n)|$, and thus the cut and paste explains the factor $(n+1)$. The paper [6] compares this explanation with that given by the classical cycle lemma of [4].

To obtain the proposition, place $m+1$ dots on the x-axis under each path P in $\mathcal{E}(n+2, m+1)$ so that exactly one dot is located directly below the final point of each U step. More specifically, apply the restricted bijection

$$\phi : \{[P, (x,0), 0] : P \in \mathcal{E}(n+2, m+1) \text{ and } x \text{ below a final point of a } U \text{ step}\}$$

$$\to \{[P', (0,0)] : P' \in \mathcal{E}(n, m)\}.$$

We see that the weight of the domain is $(m+1)|\mathcal{E}(n+2, m+1)|$ while the weight of the codomain is $t|\mathcal{U}(n, m)|$, where the factor t corresponds to the deletion of a U in the cut and paste. $\square$

We define the elevated large Schröder paths to be the paths in $\overline{\mathcal{E}}(n+2)$ having $\overline{\mathcal{S}} = \{U, D, (2,0)\}$. (Here and in the following, when we embellish '$\mathcal{S}$', we embellish '$\mathcal{E}$' and '$\mathcal{U}$' correspondingly.) We thus take the large Schröder numbers to be defined in terms of the cardinality of these path sets. Specifically, $(|\overline{\mathcal{E}}(2n+2)|)_{n\geq 0} = (1, 2, 6, 22, 90, 394, ...)$. (Sequence A006318 of [8].)

The proposition shows that large Schröder numbers can be formulated as

$$|\overline{\mathcal{E}}(2n+2)| = \sum_{m\geq 0} |\overline{\mathcal{E}}(2n+2, m+1)| =$$

$$\sum_{m\geq 0} \frac{1}{(m+1)} |\overline{\mathcal{U}}(2n, m)| = \sum_{m\geq 0} \frac{(m+n)!}{(m+1)!\ m!\ (n-m)!}. \tag{8}$$

3.2 The Narayana distribution in terms of oddly positioned up steps

Consider the step set $\widetilde{\mathcal{S}} = \{U_t, U, D\}$ where U_t is a step of weight t that must be oddly positioned on any path of $\widetilde{\mathcal{E}}(2n+2)$, U is a unit-weighted step that must be evenly positioned on any path of $\widetilde{\mathcal{E}}(2n+2)$, and D is unit weighted.

Proposition 2 *For $n \geq 1$,*

$$|\widetilde{\mathcal{E}}(2n+2)| = \sum_{i=1}^{n} \frac{1}{i} \binom{n-1}{i-1} \binom{n}{i-1} t^i = \sum_{i=1}^{n} \frac{1}{n} \binom{n}{i} \binom{n}{i-1} t^i, \tag{9}$$

where $\frac{1}{n}\binom{n}{i}\binom{n}{i-1}$ is a Narayana number. (Sequence A001263 in [8].)

For $t = 1$, $(|\widetilde{\mathcal{E}}(2n+2)|)_{n\geq 1} = (1, 2, 5, 14, 42, ...)$, the Catalan numbers less the first term. For $t = 2$, $(|\widetilde{\mathcal{E}}(2n+2)|)_{n\geq 1} = (2, 6, 22, 90, 394, ...)$, the large Schröder numbers less the first term.

For this proof, use the unit weighted steps, U and H. For any path Q, let $W_o(Q)$ ($W_e(Q)$, respectively) denote the number of oddly (evenly, respectively) positioned U steps on Q, translated if necessary to begin at $(0,0)$. Let $\mathcal{E}(2n+2,i)$ denote the subset of $\mathcal{E}(2n+2)$ whose paths have i oddly positioned steps. Let

$$
\begin{aligned}
A &:= \{[P,(x,0),0] : P \in \mathcal{E}(2n+2,i) \text{ and } x \text{ is a final abscissa of an oddly positioned } U\},\\
B &:= \{(P,(0,0)) : P \in \mathcal{U}(2n) \text{ and the lowest point of } P \text{ has an even ordinate}\},\\
C &:= \{P'' : P'' \text{ runs from } (0,0) \text{ to } (2n-1,1) \text{ and } W_e(P'') = i-1\}.
\end{aligned}
$$

Restricting the cut and paste establishes the bijection $\phi : A \to B$. However, since this map does not preserve the number of oddly positioned U steps, we need a bijection $\nu : B \to C$ correcting this situation for which the composition $\nu \circ \phi$ is weight preserving.

For $y = 0$, P factors as $L_2 R_1$ and its image under ϕ factors as $\overline{\overline{R_1}}\ L_2$. We now define $\nu(\overline{\overline{R_1}}\ L_2)$. Immediately, $W_e(\overline{\overline{R_1}}) = W_o(R_1) - 1$. If L_2 has zero length then $W_o(L_2) = W_e(L_2) = 0$. If L_2 has positive length, write $\overline{\overline{R_1}}\ L_2$ as a sequence of steps: $\overline{\overline{R_1}}\ L_2 = r_1 r_2 \cdots r_{2n-x+1} \ell_1 \ell_2 \cdots \ell_{x-1}$. The step ℓ_1 begins with a negative even ordinate, namely, $1-p_x$. Let ℓ_j denote the last step on $\overline{\overline{R_1}}\ L_2$ that begins with ordinate equal -1. Put

$$\overline{\overline{R_1}}\ L_2' := r_1 r_2 \cdots r_{2n-x+1} \ell_j \ell_{j+1} \cdots \ell_{x-1} \ell_1 \ell_2 \cdots \ell_{j-1}.$$

Since on $\overline{\overline{R_1}}\ L_2'$ the relocated step ℓ_1 is the last step to begin with ordinate $2 - p_x$, L_2 is recoverable from L_2'. Moreover, $W_e(L_2') = W_o(L_2)$.

If L_2 has zero length, let P' denote $\overline{\overline{R_1}}$; otherwise, let P' denote $\overline{\overline{R_1}}\ L_2'$ and notice that we have $W_e(P') = W_o(P) - 1$. If P' terminates with a D step, let P'' denote the path obtained from P' when this final D is deleted. For the sake of recovering P' notice that the lowest point on P'' has even ordinate. On the other hand, if P' terminates with a U step, let P'' denote the path obtained when this (evenly positioned) last step is removed and the leftmost lowest D step is changed into an (evenly positioned) U step. Here, for the sake of recovering P', notice that the lowest point on P'' is unique and has an odd ordinate. In either case, P'' terminates at $(2n-1,1)$ and $W_e(P'') = W_e(P') = W_o(P) - 1 = i - 1$. Thus we define $\nu(\overline{\overline{R_1}}\ L_2) = P''$.

To determine $|C|$, observe that each path must have $i-1$ of its $n-1$ even step positions filled with a U step and must have $n-(i-1)$ of its n odd step positions filled with a U step. Hence $|C| = \binom{n-1}{i-1}\binom{n}{n-i+1} = \frac{i}{n}\binom{n}{i}\binom{n}{i-1}$, while $|A| = i|\mathcal{E}(2n+2,i)|$. $\square$

3.3 The Narayana distribution in terms of peaks

For this application we count elevated paths with respect to the number of bicolored peaks. Again we derive the Narayana distribution and the large Schröder numbers. On any path, a 'right-hand turn' or a 'peak' is the intermediate vertex of a consecutive UD pair. For $\widehat{\mathcal{S}} = \{U, D\}$, let $\widehat{\mathcal{E}}(n,b,r)$ denote the set of elevated paths using the steps U and D and having b blue peaks and r red peaks.

Proposition 3 *For $1 \le b+r \le n$,*

$$|\widehat{\mathcal{E}}(2n+2,b,r)| = \frac{1}{b+r}\binom{n-1}{b+r-1}\binom{n}{b+r-1}\binom{b+r}{b}.$$

For each path P in $\widehat{\mathcal{E}}(2n+2,0,i)$, place i dots on the x-axis below the peaks (all being red) of the path. With $k=0$ each dot is mapped by ϕ to a point $\phi([P,(x,0),0]) = [\overline{\overline{R_1}}L_2,(0,0)]$ where the image path begins with a D step and has $i-1$ right-hand turns. If we tilt each image path counterclockwise by 45 degrees, one can check that in the tilted path there would be $\binom{n-1}{i-1}$ ways to choose the abscissae and $\binom{n}{i-1}$ ways to choose the ordinates for the intermediate vertices of the right-hand turns, where these turns uniquely determine the path. Thus, $i|\widehat{\mathcal{E}}(2n+2,0,i)| = \binom{n-1}{i-1}\binom{n}{i-1}$. Now, allowing b of the peaks to be independently colored blue, while the remainder are red, yields the factor $\binom{b+r}{b}$. $\Box$

When we disallow blue peaks, the proposition shows that $\widehat{\mathcal{E}}(2n+2,0,r)$ is a Narayana number. When we do not limit the coloring or the number of peaks, we see that the number of paths in the $\widehat{\mathcal{E}}(2n+2)$ with independently bicolored peaks is the large Schröder number:

$$\begin{aligned}\sum_b\sum_r|\widehat{\mathcal{E}}(2n+2,b,r)| &= \sum_b\sum_i|\widehat{\mathcal{E}}(2n+2,b,i-b)| = \\ \sum_i\sum_b|\widehat{\mathcal{E}}(2n+2,b,i-b)| &= \sum_{i=1}^{n}\frac{1}{i}\binom{n-1}{i-1}\binom{n}{i-1}2^i \qquad (10)\end{aligned}$$

Now return to the large Schröder paths, considered in (8), which used $\overline{\mathcal{S}} = \{U,D,(2,0)\}$. In that notation, $\overline{\mathcal{E}}(2n,n+1-j)$ will be the set of elevated paths having j of the $(2,0)$ steps. There is a simple matching between $\cup_r\widehat{\mathcal{E}}(2n+2,b,r)$ and $\overline{\mathcal{E}}(2n+2,n+1-b)$ that is obtained by transforming each UD pair with a blue intermediate vertex into a $(2,0)$ step and by removing the color red. Hence the number of paths in $\overline{\mathcal{E}}(2n+2)$ is also counted by large Schröder numbers of (10).

3.4 The 'area' under peaks

We will consider the total area under the paths of $\mathcal{E}(n+2)$ more extensively in Section 6. Here, for $\mathcal{S} = \{U_t,D,H_s\}$, we will sum the heights of peaks over all of the constrained paths in $\mathcal{C}(n)$. Equivalently, by the manner in which dots are arrayed under each trace point of the paths of $\mathcal{E}(n+2)$, we will find the weighted cardinality of the dots with $k=1$ under the peaks of the paths of $\mathcal{E}(n+2)$. By the cut and paste one can check that the restricted bijection is

$$\begin{aligned}&\phi : \{[P,(x,y),1] \in \text{DOTS}(n+2) : (x,p_x) \text{ is a peak }\}\\ &\quad\to \{(P,(x,1)) \in \text{POINTS}(n) : p_{x-1} = p_{x+1} = 0\}.\end{aligned}$$

Here, each dot in the restricted domain is mapped to an unrestricted path with a marked UD with intermediate vertex having ordinate 1. Each marked path results from the concatenation of three paths, namely, an unrestricted path from $(0,0)$ to the marked UD followed by an unrestricted path to $(n,0)$. Hence, with tz^2 corresponding to the marked UD, we have

Proposition 4

$$\sum_{n\geq 0}\sum_{P\in\mathcal{C}(n)}\sum_{0<x<n}\chi((x,p_x)\ is\ a\ peak)p_x z^n = tz^2u(z)^2.$$

Consequently, for $\mathcal{S}=\{U,D\}$, the power series for the sum of heights of the peaks on constrained paths is $tz^2u(z)^2=z^2(1-4z^2)^{-1}$, whose coefficients are powers of 4.

4 Relating moments for $\mathcal{E}(n+2)$ to those for $\mathcal{U}(n)$

To obtain our principal consequence of the the cut and paste bijection, we assign a value to each dot and its image. Consider any real valued function, ρ, defined on $\mathbb{Z}\times\mathbb{Z}$. With r viewed as an index, the cut and paste yields trivially

$$\sum_{[P,(x,y),k]\in\mathrm{DOTS}(n+2)}\rho(k,r)=\sum_{(P,(x,k))\in\mathrm{POINTS}(n)}\rho(k,r)=\sum_{(P,(x,y))\in\mathrm{POINTS}(n)}\rho(y,r). \quad (11)$$

We call a formula $\rho(y,r)$ a 'b-moment' if, for arbitrary $(y,r)\in\mathbb{Z}\times\mathbb{Z}$,

$$\rho(y,r+1)=\chi(y\geq 0)\sum_{0\leq j\leq y}\rho(j,r). \quad (12)$$

We use the name, 'b-moment', since a b-moment is easily seen to be a linear combination of binomial coefficients. Thus, if ρ is a b-moment, the left side of (11) becomes

$$\begin{aligned}\sum_P\sum_x\sum_{y=0}^{p_x-1}\sum_{k=0}^{p_x-y-1}\rho(k,r) &= \sum_P\sum_x\sum_{y=0}^{p_x-1}\rho(p_x-y-1,r+1)\\ &= \sum_P\sum_x\rho(p_x-1,r+2).\end{aligned}$$

This identity and (11), which is a consequence of the cut and paste, yields

Proposition 5 *For $\mathcal{S}=\{U_t,D,H_s\}$, for integer r, and for any b-moment $\rho(y,r)$,*

$$\mu(\mathcal{E}(n+2),\rho(y-1,r+2))=t\mu(\mathcal{U}(n),\rho(y,r)). \quad (13)$$

Specifically,

$$\mu(\mathcal{E}(n+2),\tbinom{y-1+r+2}{y-1})=t\mu(\mathcal{U}(n),\tbinom{y+r}{y}) \quad (14)$$

and for $r\geq 0$,

$$\frac{\mu(\mathcal{E}(n+2),y^{\overline{r+2}})}{(r+2)!}=\frac{t\mu(\mathcal{U}(n),y^{\overline{r}})}{r!}. \quad (15)$$

As the classic example, we use (14) to prove (2):

$$\mu(\mathcal{E}(n+2), y^2) = \mu(\mathcal{E}(n+2), 2\tbinom{y+1}{y-1} - \tbinom{y}{y-1})$$

$$= \mu(\mathcal{U}(n), 2\tbinom{y}{y} - \tbinom{y-1}{y}) = \mu(\mathcal{U}(n), 2\chi(y \geq 0) - \chi(y = 0)) = \mu(\mathcal{U}(n), 1),$$

where the last identity holds since over the paths of $\mathcal{U}(n)$ there are the same number of points with positive ordinate as with negative ordinate.

5 Recurrences of moment generating functions

We will recast a recurrence for factorial moments for elevated paths like one given by Chapman [3]. (See also [7] and [12]). We will then use Proposition 5 to convert that recurrence into one for moments for unrestricted paths. For a given b-moment $\rho(y, r)$ and for $\mathcal{S} = \{U_t, D, H_s\}$, let

$$e_r(z) := \sum_{n \geq 0} \mu(\mathcal{E}(n+2), \rho(y-1, r)) z^{n+2} \quad \text{and} \quad u_r(z) := \sum_{n \geq 0} \mu(\mathcal{U}(n), \rho(y, r)) z^n.$$

Proposition 6 *For integer r,*

$$(1 - tz^2 c(z)^2) e_r(z) = e_{r-1}(z) \tag{16}$$

$$(1 - tz^2 c(z)^2) u_r(z) = u_{r-1}(z) \tag{17}$$

$$e_r(z) = \frac{1}{2}(1 + (1 - sz^h) u(z)) e_{r-1}(z) \tag{18}$$

We prove (16) in the form

$$e_r(z) = e_{r-1}(z) + tz^2 c(z)^2 e_r(z). \tag{19}$$

Since ρ is a b-moment we have $\rho(i, r) = \rho(i, r-1) + \rho(i-1, r)$ and hence

$$\sum_{P \in \mathcal{E}(n+2)} \sum_{x=1}^{n+1} \rho(p_x - 1, r) = \sum_{P \in \mathcal{E}(n+2)} \sum_{x=1}^{n+1} \rho(p_x - 1, r-1) + \sum_{P \in \mathcal{E}(n+2)} \sum_{x=1}^{n+1} \rho(p_x - 2, r). \tag{20}$$

Consider the rightmost double sum. Each $P \in \mathcal{E}(n+2)$, satisfies $P = UQD$ where $Q \in \mathcal{C}(n)$ and Q can be factored uniquely so that each factor, Q', is a translation of some elevated path. Suppose Q' can be translated to belong to $\mathcal{E}(n')$, for $2 \leq n' \leq n$. Each time Q' appears as a factor in some of the concatenations forming the paths of $\mathcal{C}(n)$, it is preceded by a (perhaps void) constrained path and followed by a (perhaps void) constrained path; thus Q' makes $\sum_{i+i'=n-n'} |\mathcal{C}(i)||\mathcal{C}(i')|$ appearances in the paths of $\mathcal{C}(n)$. Since each factor Q' begins and ends with points of ordinate 1, i.e., since each Q' is 'doubly elevated', the moment contribution to $\sum_{P\in\mathcal{E}(n+2)}\sum_{x=1}^{n+1}\rho(p_x-2,r)$ of Q', translated to begin at $(0,0)$, is $\sum_{x=1}^{n'-1}\rho(q'_x-1,r)$ times its frequency of appearances. Hence,

$$\sum_{P\in\mathcal{E}(n+2)}\sum_{x=1}^{n+1}\rho(p_x-2,r)=\sum_{n'=2}^{n}\sum_{Q\in\mathcal{E}(n')}\sum_{i+i'=n-n'}|\mathcal{C}(i)||\mathcal{C}(i')|\sum_{x=1}^{n'-1}\rho(q'_x-1,r) \tag{21}$$

and the corresponding generating function is $tz^2c(z)^2e_r(z)$. The identities (20) and (21) yield (19). An application of Proposition 5 to (16) yields (17). A straightforward computation yields (18). □

6 Further examples

6.1 Areas and intercepts

Take $\mathcal{S}=\{U,D,H\}$ and let $\rho(y,r)=\binom{r+y}{y}$. Since $\rho(y-1,1)=y$, the left side of (13) becomes the first moment $\sum_{P\in\mathcal{E}(n+2)}\sum_x p_x$, which by the trapezoid rule is the total area bounded between the horizontal axis and the paths of $\mathcal{E}(n+2)$. Since $\rho(y,-1)=\chi(y=0)$, the right side of (13) becomes $\sum_{P\in\mathcal{U}(n)}\sum_x\chi(p_x=0)$ which is the total number of intercepts of the horizontal axis by the paths of $\mathcal{U}(n)$. More generally, if we consider $\rho(y,r)=\binom{r+y-y_0}{y-y_0}$ for nonnegative y_0, then we can use (13) to show

Proposition 7 *For $\mathcal{S}=\{U,D,H\}$ and for $y_0\geq 0$, the total area of the regions under the paths of $\mathcal{E}(n+2)$ and above the horizontal line $y=y_0$ is equal to the number of intercepts of that line by the paths of $\mathcal{U}(n)$.*

Next we give further results concerning intercepts, area, and the generating function $u(z)^2$, whose formula is obtained from (5).

Proposition 8 *For $\mathcal{S}=\{U,D,H\}$, the generating function for the number of intercepts of the horizontal axis by the step end points on the traces of the paths of $\mathcal{U}(n)$ satisfies*

$$\sum_{n\geq 0}\sum_{P\in\mathcal{U}(n)}\sum_{0\leq x\leq n}\chi(\ (x,0)\ \textit{is a step end point on}\ P\ )z^n=u(z)^2.$$

This proposition follows by observing that each intercept contributing to the inner summations results from the concatenation of two paths, an unrestricted path from $(0,0)$ to the intercept followed by an unrestricted path from the intercept to $(n,0)$. □

Proposition 9 *For $\mathcal{S}=\{U,D,H\}$, the generating function for the number of intercepts of the horizontal axis by lattice points on the traces of paths of $\mathcal{U}(n)$ satisfies*

$$\sum_{n\geq 0}\mu(\mathcal{U}(n),\chi(y=0))z^n=(1+(h-1)z^h)u(z)^2. \tag{22}$$

Equivalently, the generating function for the total area under the paths of $\mathcal{E}(n+2)$ satisfies

$$\sum_{n\geq 0}\mu(\mathcal{E}(n+2),y)z^{n+2}=z^2(1+(h-1)z^h)u(z)^2.$$

Intercepts which are end points of steps make a contribution to the generating function as in Proposition 8. When a step lies on the horizontal axis, the $h-1$ intercepts which are interior to a step can be collapsed along with the step to become the intercept of a step end point on a path of belonging to $\mathcal{U}(n-h)$ and thus make an adjusted contribution to the right side of (22). The second identity follows by the initial remarks of this section. □

6.2 Two comparable models Here we examine some comparable, yet different, models whose area results are derivable from the the cut and paste, through Proposition 9. In the two cases below we will consider instances of a recurrence: If $a(n)$ is defined so that $\sum_n a(n)z^n = u(z)^2$, then

$$a(n) = 4ta(n-2) + 2sa(n-h) - s^2a(n-2h) \tag{23}$$

which follows the comparison of coefficients and (5).

Case for $s = t+1$ **and** $h = 1$**:** Here the step set is $\mathcal{S}^* = \{U_t, D, (0,1)_{t+1}\}$. For $t = 1$, $e^*(z) = 1z^2 + 2z^3 + 5z^4 + 14z^5 + 42z^6...$, a generating function for the Catalan numbers less the first term. For $t = 2$, $e^*(z) = 2z^2 + 6z^3 + 22^4 + 90z^5 + 394z^6...$, a generating function for the large Schröder numbers less its first term; it is not a curiosity that the cardinalities of $\mathcal{E}^*(n+2)$ and $\widetilde{\mathcal{E}}(2n+2)$ (see (9)) agree as one can establish a straightforward isomorphism $\alpha : \mathcal{U}^*(n) \to \widetilde{\mathcal{U}}(2n)$ using the step replacement rules: $\alpha(U_t) = U_tU$, $\alpha(D) = DD$, and $\alpha(H_{t+1}) = \{U_tD, DU\}$.

For $t = 1$ and $t = 2$, $u^*(z)$ is the generating function for the central binomial coefficients and the central Delannoy numbers, respectively. (See Sequence A001850 of [8] and Section 6.5 of [9].)

The weighted area under the elevated paths of $\mathcal{E}^*(n+2)$, satisfies

$$\sum_{n\geq 0} \mu(\mathcal{E}^*(n+2), y)z^{n+2} = tz^2\mathcal{U}^*(z)^2 = \frac{tz^2}{((t+1)z-1)^2 - 4tz^2}$$

For $t = 1$, $z^2u^*(z) = \frac{z^2}{1-4z}$, a generating function whose coefficients are powers of 4. We remark that there is a simple bijection from $\mathcal{E}^*(n+2)$ to a set of parallogram polyominoes of perimeter $2n+4$ which is area preserving as discussed in [11]. Further attention to this result appears in [5].

For $t = 2$, $2z^2u^*(z)^2 = \frac{2z^2}{1-6z+z^2} = 2z^2 + 12z^3 + 70z^4 + 408z^5...$, where the coefficients are double the square-triangular numbers, or, equivalently, every other Pell number. (See sequences A000129, A001109, and A001542 in [8].) By (23), for $t = 2$, $u^*(z)^2 = \sum_n a^*(n)z^n$ satisfies

$$a^*(n) = 6a^*(n-1) - a^*(n-2) \tag{24}$$

subject to $a^*(2) = 2$ and $a^*(3) = 12$. This recurrence in terms of the total the areas of zebras (i.e., column-bicolored parallogram polyominoes) was a principal topic of [11].

Case for $t = 1$**:** Here $\overline{\mathcal{S}} = \{U, D, (0,h)_s\}$. For $h = \infty$, $\overline{e}(z)$ is a generating function for the Catalan numbers. For $s = 1$ and $h = 1$, $\overline{e}(z)$ is the generating function for the Motzkin numbers. For $s=1$ and $h=2$, $\overline{\mathcal{C}}(n)$ are the usual large Schröder paths with $\overline{e}(z)$ being the generating function for the large Schröder numbers, as noted after formula (10).

For $h=\infty$, $\overline{u}(z)$ gives the central binomial coefficients. For $s=h=1$, $\overline{u}(z)$ gives the central trinomial numbers. For $s=1$ and $h=2$, $\overline{u}(z)$ gives the central Delannoy numbers.

By Proposition 9 the area under the elevated paths of $\overline{\mathcal{E}}(n+2)$, satisfies

$$\sum_{n\geq 0}\mu(\overline{\mathcal{E}}(n+2),y)z^{n+2}=\frac{1+(h-1)z^h}{(sz^h-1)^2-4z^2}.$$

For $h=\infty$, the coefficients of this power series are powers of 4. (See [13].) For $s=1$ and $h=2$, this power series becomes $\frac{z^2(1+z^2)}{1-6z^2+z^4}=1z^2+7z^4+41z^6+239z^8+1393z^{10}...$, where the coefficients correspond to pairwise sums of consecutive square-triangular numbers, or equivalently to every other pairwise sum of consecutive Pell numbers, as noted in [2]. (See sequence A002315 in [8].) For $\overline{\mathcal{E}}(2n+2)$, we remark that the square-triangular numbers give both the sums of the ordinates of the trace points restricted to be end points of steps and the sums of the ordinates of the trace points which are the mid points of steps. By (23), for $s=1$ and $h=2$, $\overline{u}(z)^2=\sum_n \overline{a}(n)z^n$ satisfies

$$\overline{a}(n)=6\overline{a}(n-2)-\overline{a}(n-4) \tag{25}$$

subject to $\overline{a}(2)=1$ and $\overline{a}(4)=7$, as noted in [2]. Compare this recurrence to (24). For recent considerations of (25) and other references, see [1]. For a bijective approach to the recurrences for the cardinality, area, and the second moments for large Schröder paths see [10].

6.3 Moments for unrestricted paths about the horizontal axis

Consider the rising factorial moments of the distances from the horizontal axis for trace points of the paths of $\mathcal{U}(n)$. While our computations are not consequences of the cut and paste, we include them since they are analogous to the proofs of Propositions 6 and 8.

Proposition 10 *For $m>0$,*

$$\sum_{n\geq 0}\sum_{P\in\mathcal{U}(n)}\sum_{0\leq x\leq n}|p_x|^{\overline{m}}z^n=2u(z)^2\sum_{n\geq 0}\mu(\mathcal{E}(n+2),y^{\overline{m}})z^n.$$

To prove this, note that since $|p_x|^{\overline{m}}>0$ there is no contribution to the left side from the intercepts. Also note that each path in $\mathcal{U}(n)$ can be factored into subpaths which begin and end on the horizontal axis, each factor being either an elevated path P' whose translation belongs to $\mathcal{E}(n'+2)$ or its reflection. Since the factor P', or its reflection (requiring the multiple 2), is preceded and followed by a unrestricted path and since $\sum_{1\leq x\leq n'+1}|p'_x|^{\overline{m}}$ (when P' is translated to begin at the origin) is a summand of $\mu(\mathcal{E}(n'+2),y^{\overline{m}})$, the proposition now follows in a manner of the proof of (16). $\square$.

6.4 Two Catalan configurations

Here we give two configurations enumerated by the Catalan numbers but not appearing in the catalog of Exercise 6.19 of [9].

Proposition 11 *For $\mathcal{S} = \{U, D\}$ and for $n \geq 0$, the total number of intercepts of the horizontal axis by the Dyck paths running from $(0,0)$ to $(2n,0)$ is $\frac{1}{n+2}\binom{2n+2}{n+1}$. Moreover, if the values 1, -2, and 1, are assigned respectively to the points of ordinate 0, 1, and 2 on the unrestricted paths running from $(0,0)$ to $(2n,0)$, then the sum of these values over all these paths is the same Catalan number.*

More generally, for $\mathcal{S} = \{U_t, D, H_s\}$, we claim that

$$\sum_{n\geq 0} \mu(\mathcal{U}(n), (-1)^y \tbinom{2}{y}) z^n = \sum_{n\geq 0} \mu(\mathcal{U}(n), \tbinom{y-3}{y}) z^n =$$

$$\sum_{n\geq 0} t^{-1} \mu(\mathcal{E}(n+2), \tbinom{y-2}{y-1}) z^n = \sum_{n\geq 0} \mu(\mathcal{C}(n), \chi(y=0)) z^n = c(z)^2.$$

To see the last identity, notice that $\mu(\mathcal{C}(n), \chi(y=0))$ is the weight of the set of intercept-marked paths form from constrained paths of $C(n)$. Since each intercept is realized as the concatenation of a constrained path from $(0,0)$ to the intercept with a constrained path from the intercept to $(n,0)$, $\mu(\mathcal{C}(n), \chi(y=0)) = \sum_i |C(i)||C(n-i|$. To finish the proof of the proposition, notice that $z^2 c(z)^2 = c(z) - 1$ for $\mathcal{S} = \{U, D\}$. $\square$

Acknowledgement. The author thanks Elisa Pergola for her contribution to the proof of Proposition 2.

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