Bipartite, k-Colorable and k-Colored Graphs

Steven Finch

June 5, 2003

A labeled graph G is **bipartite** if its vertex set V can be partitioned into two disjoint subsets A and B, $V = A \cup B$, such that every edge of G is of the form (a, b), where $a \in A$ and $b \in B$.

Let k be a positive integer and $K = \{1, 2, \ldots, k\}$. A labeled graph G is **k-colorable** if there exists a function $V \to K$ with the property that adjacent vertices must be colored differently. Clearly G is bipartite if and only if G is 2-colorable.

Define $c_{n,k}$ to be the number of k-colorable graphs with n vertices. We have $c_{n,1} = 1$ for $n \geq 1$ since a 1-colorable graph G cannot possess any edges. We also have $c_{1,k} = 1$ for $k \geq 1$, $c_{2,k} = 2$ for $k \geq 2$, $c_{3,2} = 7$ by Figure 1, $c_{3,3} = 8$, $c_{4,2} = 41$ by Figure 2, and $c_{4,3} = 63$. More generally, $c_{n,n-1} = 2^{n(n-1)/2} - 1$ since the total number of labeled graphs with n vertices is $2^{n(n-1)/2}$ and, of these, only the complete graph cannot be $(n-1)$-colored.

Does there exist a formula for $c_{n,k}$? The answer is yes if $k = 2$, but evidently no for $k \geq 3$. We'll examine this issue momentarily, but first define a related notion.

A **k-colored graph** is a labeled k-colorable graph together with its coloring function. Let $\gamma_{n,k}$ be the number of k-colored graphs with n vertices. The point is that a k-colorable graph counts several times as a k-colored graph. Clearly $\gamma_{n,1} = 1$, $\gamma_{1,k} = k$, $\gamma_{2,2} = 6$ by Figure 3, $\gamma_{2,3} = 15$ by Figure 4, and $\gamma_{3,2} = 26$ by Figure 5.

When $k = 2$, the following formulas can be proved [1, 2, 3]:

$$\gamma_{n,2} = \sum_{j=0}^{n} \binom{n}{j} 2^{j(n-j)}$$

$$c_{n,2} = n! \cdot \left(\text{the } n^{\text{th}} \text{ degree Maclaurin series coefficient of } \sqrt{\Gamma(x)} \right)$$

where

$$\Gamma(x) = \sum_{i=0}^{\infty} \gamma_{i,2} \frac{x^i}{i!}$$

For arbitrary k, we have the following recursion [4, 5]:

$$\gamma_{n,k} = \sum_{j=0}^{n} \binom{n}{j} 2^{j(n-j)} \gamma_{j,k-1}$$

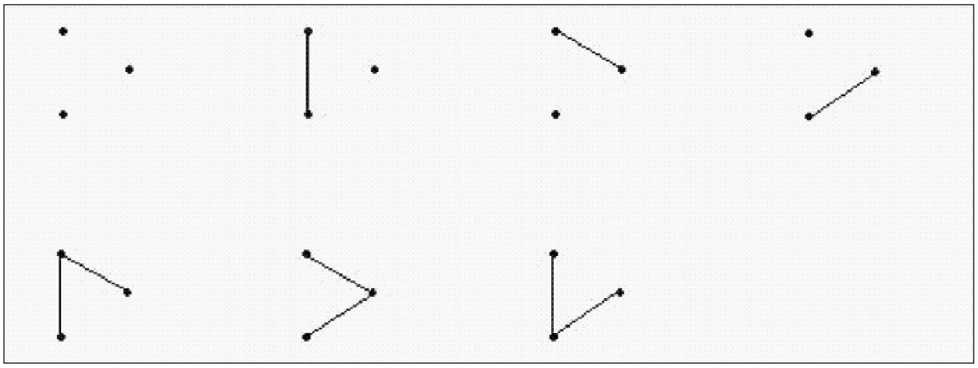

Figure 1: There are 7 labeled bipartite graphs with 3 vertices.

with initial conditions $\gamma_{0,k} = 1$ and $\gamma_{n,0} = 0$ for $n \geq 1$. Alternatively, we have a closed-form expression involving multinomial coefficients:

$$\gamma_{n,k} = \sum_{N} \binom{n}{n_1, n_2, \ldots, n_k} 2^{\frac{1}{2}\left(n^2 - n_1^2 - n_2^2 - \cdots - n_k^2\right)}$$

where the summation is over all nonnegative integer k-vectors $N = (n_1, n_2, \ldots, n_k)$ satisfying $n_1 + n_2 + \cdots + n_k = n$. There is, however, no known analogous formula for $c_{n,k}$ when $k \geq 3$.

Computations show that [4, 6]

$$\{\gamma_{n,2}\}_{n=1}^{\infty} = \{2, 6, 26, 162, 1442, 18306, 330626, 8488962 \ldots\}$$

$$\{c_{n,2}\}_{n=1}^{\infty} = \{1, 2, 7, 41, 376, 5177, 103237, 2922446 \ldots\}$$

and suggest that $\gamma_{n,2}/c_{n,2} \to 2$ as $n \to \infty$. We also have

$$\{\gamma_{n,3}\}_{n=1}^{\infty} = \{3, 15, 123, 1635, 35043, 1206915, 66622083, 5884188675, \ldots\}$$

$$\{c_{n,3}\}_{n=1}^{\infty} = \{1, 2, 8, 63, 958, 27554, \ldots\}$$

but there is insufficient data on $c_{n,3}$ to clearly suggest the asymptotic behavior of $\gamma_{n,3}/c_{n,3}$. Prömel & Steger [7], however, proved that

$$\lim_{n\to\infty} \frac{\gamma_{n,k}}{c_{n,k}} = k!$$

for each $k \geq 2$. In words, a random k-colorable graph is almost surely uniquely k-colorable (up to a permutation of colors). This is an important result since it allows us to utilize at least one term of the $\gamma_{n,k}$ asymptotics to estimate the growth of $c_{n,k}$.

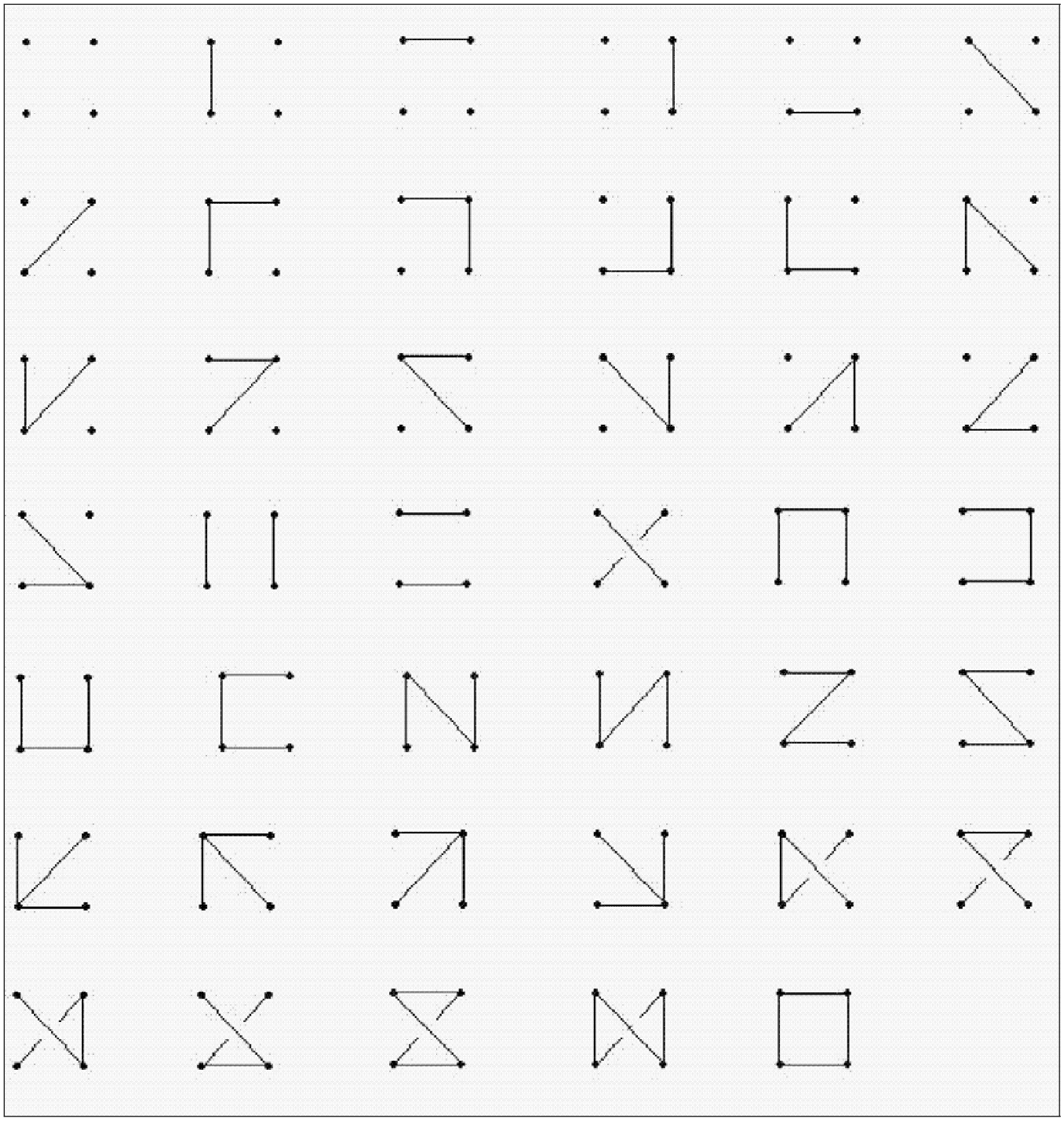

Figure 2: There are 41 labeled bipartite graphs with 4 vertices.

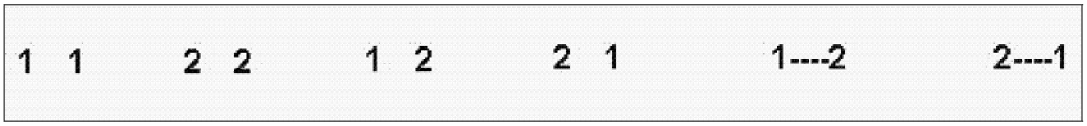

Figure 3: There are 6 labeled 2-colored graphs with 2 vertices.

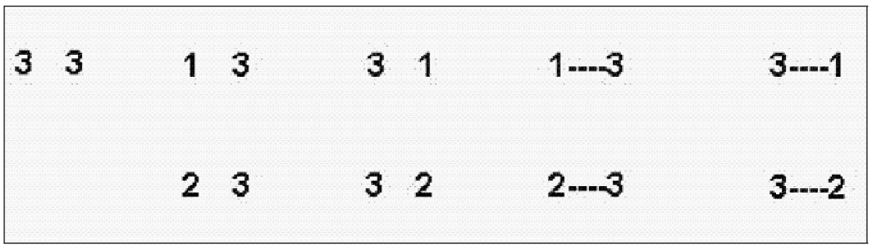

Figure 4: There are 15 labeled 3-colored graphs with 2 vertices (these 9 plus the preceding 6).

We turn now to a result due to Wright [8, 9, 10, 11, 12]: if $n \equiv a \bmod k$, where $0 \leq a < k$, then

$$\gamma_{n,k} \sim C(k,a) \cdot 2^{\frac{1}{2}(1-\frac{1}{k})n^2} \cdot k^n \cdot \left(\frac{k}{\ln(2) \cdot n}\right)^{\frac{k-1}{2}}$$

as $n \to \infty$, where $C(k,a)$ is a constant that depends on n only via its residue modulo k. In fact,

$$C(k,a) = k^{\frac{1}{2}} \cdot (\ln(2))^{\frac{k-1}{2}} \cdot (2\pi)^{-\frac{k-1}{2}} \cdot L_k(a)$$

and the infinite series $L_k(a)$ will be defined for $k = 2$, 3 and 4 shortly.

0.1. 2-Colored Graph Asymptotics. To characterize the growth of $\gamma_{n,k}$, by the above, it is sufficient to determine $C(k,a)$ for each $0 \leq a < k$. We have here

$$\begin{aligned} L_2(a) &= \sum_{r=-\infty}^{\infty} 2^{-\frac{1}{2}r^2 - \frac{1}{2}(a-r)^2 + \frac{1}{4}a^2} \\ &= \sum_{r=-\infty}^{\infty} 2^{-\frac{1}{4}(a-2r)^2} = \begin{cases} 2.1289368272\ldots & \text{if } a = 0 \\ 2.1289312505\ldots & \text{if } a = 1 \end{cases} \end{aligned}$$

These two constants also appear with regard to the asymptotic enumeration of partially ordered sets [13] and of linear subspaces of $\mathbb{F}_2^n$ [14], where $\mathbb{F}_2$ is the binary field

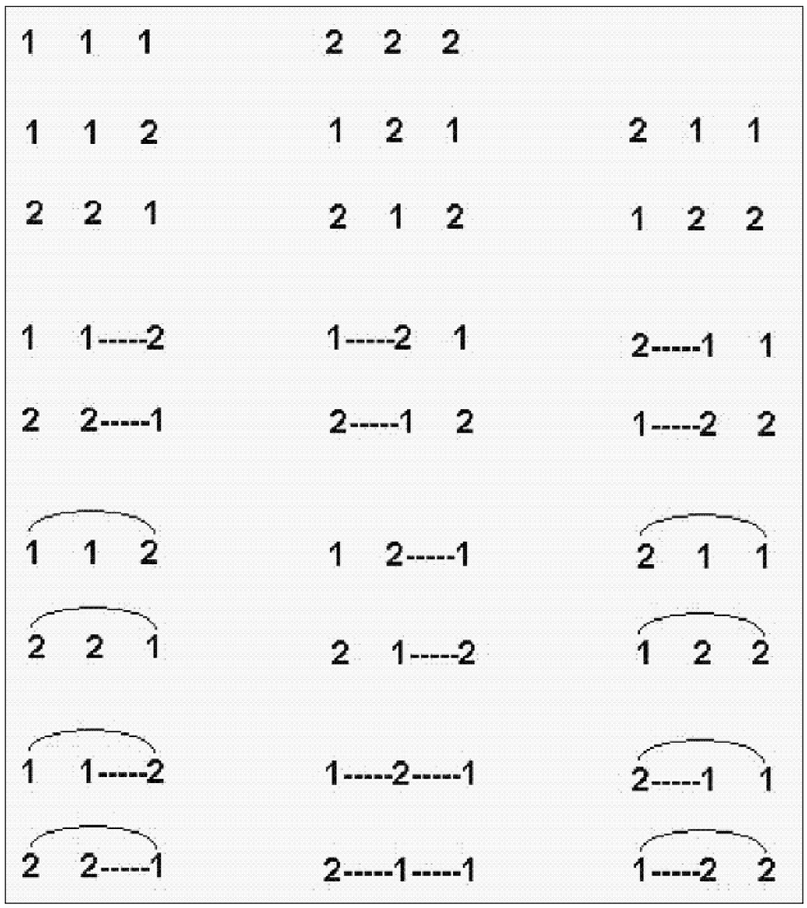

Figure 5: There are 26 labeled 2-colored graphs with 3 vertices.

with arithmetic modulo 2. Therefore

$$C(2,a)=\begin{cases} 1.0000013097\ldots=1+\varepsilon & \text{if } a=0 \\ 0.9999986902\ldots=1-\varepsilon & \text{if } a=1 \end{cases}$$

where $\varepsilon=1.3097396978\ldots\times10^{-6}$. In fact, all of the constants $C(k,a)$ we examine are close to 1; thus we shall focus on difference with 1 henceforth.

0.2. 3-Colored Graph Asymptotics. We have here

$$\begin{aligned} L_3(a) &= \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} 2^{-\frac{1}{2}r^2-\frac{1}{2}s^2-\frac{1}{2}(a-r-s)^2+\frac{1}{6}a^2} \\ &= \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} 2^{-\frac{1}{3}\left(a^2-3ar+3r^2-3as+3rs+3s^2\right)} \end{aligned}$$

and therefore

$$C(3,a)=\begin{cases} 1+2\varepsilon & \text{if } a=0 \\ 1-\varepsilon & \text{if } a=1 \text{ or } 2 \end{cases}$$

where $\varepsilon=1.7060611047...\times10^{-8}$.

0.3. 4-Colored Graph Asymptotics. All planar graphs are 4-colorable by the famous Four Color Theorem. We have here [4, 6]

$$\{\gamma_{n,4}\}_{n=1}^{\infty}=\{4,28,340,7108,254404,15531268,1613235460,284556079108,\ldots\}$$

$$\{c_{n,4}\}_{n=1}^{\infty}=\{1,2,8,64,1023,32596,\ldots\}$$

$$\begin{aligned} L_4(a) &= \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} 2^{-\frac{1}{2}r^2-\frac{1}{2}s^2-\frac{1}{2}t^2-\frac{1}{2}(a-r-s-t)^2+\frac{1}{8}a^2} \\ &= \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} 2^{-\frac{1}{8}\left(3a^2-8ar+8r^2-8as+8rs+8s^2-8at+8rt+8st+8t^2\right)} \end{aligned}$$

and therefore

$$C(4,a)=\begin{cases} 1+\delta & \text{if } a=0 \\ 1-\varepsilon & \text{if } a=1 \text{ or } 3 \\ 1-\delta+2\varepsilon & \text{if } a=2 \end{cases}$$

where $\delta=4.2421496651\ldots\times10^{-9}$ and $\varepsilon=2.5731271141...\times10^{-12}$. A simple relationship between δ and ε is not apparent.

Higher-order asymptotics for $\gamma_{n,k}$ are possible, due to Wright [8]; we hope to examine the corresponding constants later. Observe that terms beyond the first need not necessarily apply for $c_{n,k}$.

A random k-colorable graph is almost surely connected [10, 12, 15] and is almost surely k-chromatic (meaning that $k-1$ colors won't suffice to color all n vertices). The asymptotics discussed above therefore apply to these important subclasses as well.

Enumerating unlabeled k-colorable graphs (that is, non-isomorphic types of labeled k-colorable graphs) is also a difficult computational problem [16]. A general result due to Prömel [17] provides that $c_{n,k}/n!$ is the associated asymptotic formula.

REFERENCES

[1] H. S. Wilf, *generatingfunctionology*, 2nd ed., Academic Press, 1994, pp. 86-89; MR1277813 (95a:05002).

[2] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, unpublished manuscript, 2001.

[3] D. A. Klarner, The number of graded partially ordered sets, *J. Combin. Theory* 6 (1969) 12–19; MR0236035 (38 #4333).

[4] R. C. Read, The number of k-colored graphs on labelled nodes, *Canad. J. Math.* 12 (1960) 410–414; MR0114775 (22 #5594).

[5] F. Harary and E. M. Palmer, *Graphical Enumeration*, Academic Press, 1973, pp. 16-18; MR0357214 (50 #9682).

[6] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A000684, A000685, A000686, A047863, A047864, A084279, and A084280.

[7] H. J. Prömel and A. Steger, Random l-colorable graphs, *Random Structures Algorithms* 6 (1995) 21–37; MR1368832 (96j:05095).

[8] E. M. Wright, Counting coloured graphs, *Canad. J. Math.* 13 (1961) 683–693; MR0133252 (24 #A3086).

[9] E. M. Wright, Counting coloured graphs. II, *Canad. J. Math.* 16 (1964) 128–135; MR0177913 (31 #2171).

[10] R. C. Read and E. M. Wright, Coloured graphs: A correction and extension, *Canad. J. Math.* 22 (1970) 594-596; MR0263695 (41 #8296).

[11] E. M. Wright, Counting coloured graphs. III, *Canad. J. Math.* 24 (1972) 82–89; MR0289368 (44 #6559).

[12] F. Harary and R. W. Robinson, Labeled bipartite blocks, *Canad. J. Math.* 31 (1979) 60–68; MR0518706 (80b:05036).

[13] S. R. Finch, Lengyel's constant: Chains in the subset lattice of S, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 317–318.

[14] S. R. Finch, Transitive relations, topologies and partial orders, unpublished note (2003).

[15] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A001832, A084283, and A084284.

[16] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A033995, A076315, and A076316.

[17] H. J. Prömel, Counting unlabeled structures, *J. Combin. Theory Ser. A* 44 (1987) 83–93; MR0871390 (87m:05014).