Fractal Sequences, Part 1: Overview

Infinity is where things happen that don't.
— S. Knight

Ever since Benoit Mandelbrot published his book on fractals [1], we've become accustomed to seeing fantastic and beautiful fractal images such as the ones at the bottom of the pages of this article.

A fractal is a mathematical object that exhibits *self similarity* — it looks the same at any scale. If you zoom in on an image of a fractal, you see the same structure no matter how far you go, at least to the resolution of the image. In an actual fractal, there is no limit.

Fractal Sequences

In the case of sequences, a fractal sequence contains an infinite number of copies of itself, embedded within itself, as strange as this may seem.

The idea can be shown by the sequence

1, 2, 1, 3, 2, 1, 4, 3, 2, 1, 5, 4, 3, 2, 1, 6, 5, 4, …

Striking out the first instance of every value,

~~1~~, ~~2~~, 1, ~~3~~, 2, 1, ~~4~~, 3, 2, 1, ~~5~~, 4, 3, 2, 1, ~~6~~, 5, 4, …

↓

1, 2, 1, 3, 2, 1, 4, 3, 2, 1, 5, 4, …

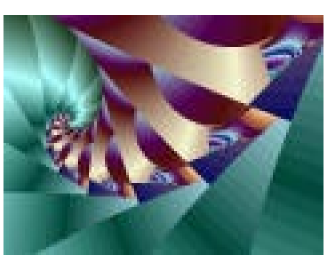
↓

1, 2, 1, 3, 2, 1, 4, 3, 2, 1, 5, 4, …

which is the same as the original sequence, as far as it goes. Of course, we can't show the complete sequence — it is infinite — for this you have to have faith.

Rules that remove the values in fractal sequences to show their self similarity are called *fractal decimation* rules.

Bounded Fractal Sequences



The Morse-Thue sequence [2] is a binary fractal sequence, consisting only of 0s and 1s:

0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, …

For this sequence, striking out even-numbered values leaves the original sequence:

0, ~~1~~, 1, ~~0~~, 1, ~~0~~, 0, ~~1~~, 1, ~~0~~, 0, ~~1~~, 0, ~~1~~, 1, ~~0~~, …

↓

0, 1, 1, 0, 1, 0, 0, 1, …

↓

0, 1, 1, 0, 1, 0, 0, 1, …

The Morse-Thue sequence can be generalized to include more different values. The three-valued Morse-Thue sequence is

0, 1, 2, 1, 2, 0, 2, 0, 1, 1, 2, 0, 2, 0, 1, 0, 1, …

and the four-valued Morse-Thue sequence is

0, 1, 2, 3, 1 ,2, 3, 0, 2, 3, 0, 1, 3, 0, 1, 2, 1, …

Generalized Morse-Thue sequences also are fractal sequences. Can you find decimation rules that show their self similarity?

Another binary fractal sequence is the rabbit sequence [3]:

1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, …

To show that this sequence is self-similar, underline 1, 0 pairs, replace them by 1s, and replace non-underlined 1s by 0s:

1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, …

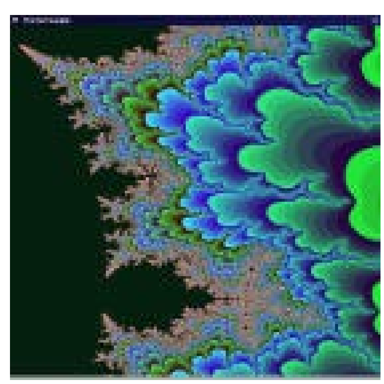
↓

1, 0, 1, 1, 0, 1, 0, 1, 1, 0, …

↓

1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, …

Again, this is the original sequence.



Unbounded Fractal Sequences

Many fractal sequences increase without bounds. Examples are sig-

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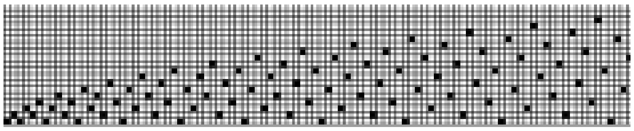
nature sequences [2], which provide characterizations of irrational numbers (numbers like $\sqrt{7}$ that cannot be represented by fractions). Here is a grid plot of the signature sequence for ϕ, the golden ratio:



Signature sequences are one category of Kimberling fractal sequences [3].

Two operations that, when applied to Kimberling fractal sequences, produce fractal sequences are *upper trimming* and *lower trimming*. Upper trimming strikes out the first instance of every value, as illustrated by the first example in this article. Lower trimming subtracts 1 from each value and discards 0s. For the first example in this article, it goes like this:

1, 2, 1, 3, 2, 1, 4, 3, 2, 1, 5, 4, 3, 2, 1, 6, 5, 4, …

↓

0, 1, 0, 2, 1, 0, 3, 2, 1, 0, 4, 3, 2, 1, 0, 5, 4, 3, …

↓

1, 2, 1, 3, 2, 1, 4, 3, 2, 1, 5, 4, 3, …

↓

1, 2, 1, 3, 2, 1, 4, 3, 2, 1, 5, 4, 3, …

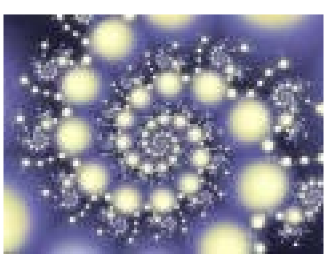
This is the same as the original sequence. This is not always true of lower trimming, but the result always is some fractal sequence.

Periodic Sequences

A periodic sequence is a sequence in which a subsequence repeats [4]. An example is the sequence of Fibonacci numbers, mod 6. The repeat length is 24:

1, 1, 2, 3, 5, 2, 1, 3, 4, 1, 5, 0, 5, 5, 4, 3,
1, 4, 5, 3, 2, 5, 1, 0

Infinite periodic sequences are, technically speaking, self similar. Decimation rules for periodic sequences are simple: Remove the initial repeat. In fact removing any repeat or any combination of repeats does the same thing: There is an infinite number of decimation rules for an infinite periodic sequence.



Periodic sequences have an important role in weave design, but they lack the intriguing aspects of other kinds of fractal sequences. Periodic sequences are best left to other contexts.

Adapting Fractal Sequences to Weave Design

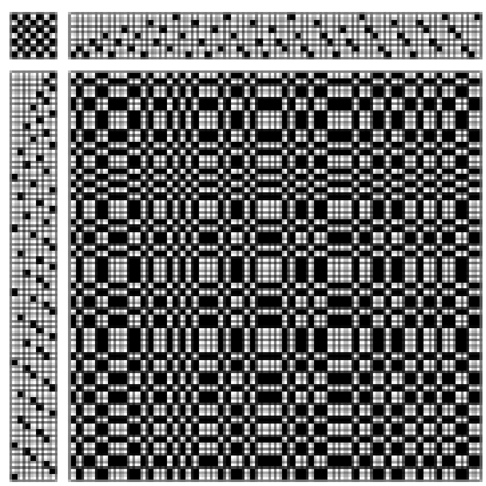
Threading and Treadling Sequences

The obvious use of fractal sequences in weave design is as threading and treadling sequences. For this purpose, fractal sequences can be divided into two classes: Those whose values fit within the constraints of a loom and those that do not.

The former can be used directly — at least portions of them. For fractal sequences, such as the Morse-Thue sequences, that have 0s, simply adding 1 to each value produces a sequence that works for the 1-based numbering of shafts and treadles.

Fractal sequences that have values exceeding the constraints of a loom can be converted to the desired range using modular reduction [5].

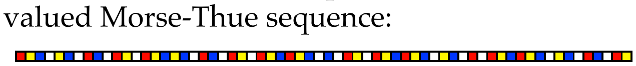
For example, the signature sequence for the golden ratio, reduced (shaft) modulo 7, treadles as drawn in with a tabby tie-up produces this draft:



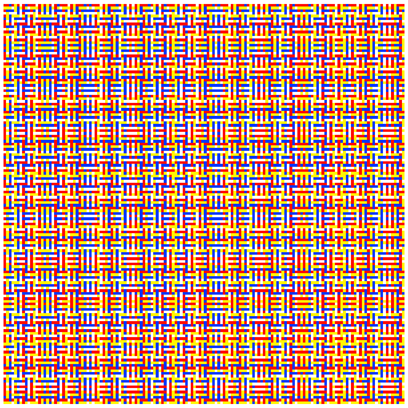
Color Sequences

Fractal sequences also can be used to derive warp and weft color sequences by assigning a color to each different value in the sequence.

Here is a color sequence based on the 4-valued Morse-Thue sequence:



A plain weave with this color sequence used for the warp and weft is

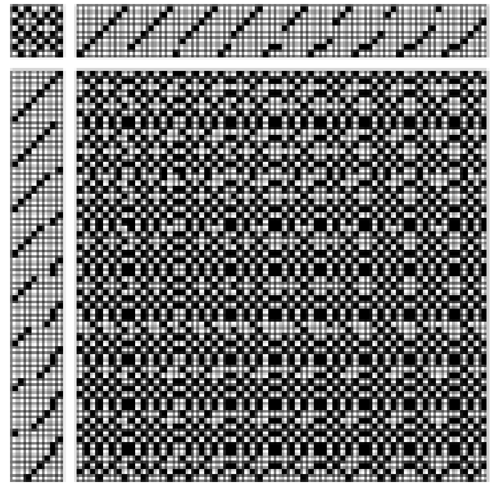


Selecting Fractal Sequences for Weave Design

You can't expect to get weave designs from fractal sequences that rival the fractal images at the bottoms of the pages at this article: These images are too complex to be woven in a loom-controlled fashion. Tapestry weaving is an intriguing possibility, however.

The fractal sequences that are the most interesting from a design viewpoint are those that develop successive variations and extensions of an initial "theme". The Morse-Thue and rabbit sequences are excellent examples of this type of fractal sequence.

Such sequences often give designs that appear at first glance to be periodic but on closer examination show continual variations. An example is this draft based on the 8-valued Morse-Thue sequence:



Prospecting for Fractal Sequences

The few examples of fractal sequences given here can get you started on designing. But if you find the results interesting, you may want to try other fractcal sequences.

The best source for integer sequences of all kinds is the *On-Line Encyclopedia of Integer Sequences* [2]. There you can search for sequences by entering a few initial terms or using keywords. The keywords fractal and self-similar produce long lists of relevant sequences. You can also look for Morse-Thue and the synonym Thue-Morse.

Sequences in the *Encyclopedia* also can be looked up by their identifying numbers, which consist of an A followed by 6 digits. Some numbers for examples given here and related ones are:

A010060	Morse-Thue Sequence
A005614	Rabbit Sequence
A053838	2-Valued Morse-Thue Sequence
A053839	4-Valued Morse-Thue Sequence

When you look up a sequence by number, the description may not be what you expect: A sequence may have many origins. Similarly, there are different forms of some sequences.

The site also has a page listing many self-similar sequences with simple decimation rules [6].

References

1. Mandelbrot, Benoit B. *The Fractal Geometry of Nature*. W. H. Freeman, 1977.

2. Schroeder, Manfred. *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise*. W. H. Freeman, 1991

3. On-Line Encyclopedia of Integer Sequences:
http://www.research.att.com/~njas/sequences/

4. Residue Sequences in Weave Design, Ralph E. Griswold, 2003:
http://www.cs.arizona.edu/patterns/weaving/webdocs/gre_res.pdf

5. Drafting with Sequences, Ralph E. Griswold, 2002:
http://www.cs.arizona.edu/patterns/weaving/webdocs/gre_seqd.pdf

6. Some Self-Similar Integer Sequences, Michael Gilleland, 2002:
http://www.research.att.com/~njas/sequences/selfsimilar.html

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