An infinite set of Heron triangles with two rational medians

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1 Introduction

If we denote the sides of a triangle by (a, b, c) then the area is given by

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} \tag{1}$$

where $s = (a+b+c)/2$ is the *semiperimeter*. This formula is usually attributed to Heron of Alexandria circa 100 BC - 100 AD. However, it was already known to Archimedes prior to 212 BC [5, p. 105].

Our investigation is limited to triangles with rational sides. Even with sides of rational length, "Heron's" formula shows that the area need not be rational; any triangle with three rational sides and rational area is called a *Heron triangle*. The smallest such triangle with integer sides is the familiar $(5, 4, 3)$ right triangle (with area 6) shown in Figure 1. If we let (k, l, m) denote the medians that are

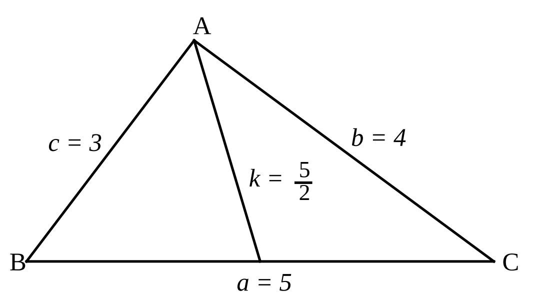

Figure 1: The (5,4,3) right triangle

incident with the respective sides (a, b, c), they can be expressed in terms of the sides:

$$k = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2},\ l = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2},\ m = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}\ . \tag{2}$$

The medians of the $(5, 4, 3)$ triangle are $(k, l, m) = \left(5/2, \sqrt{13}/2, \sqrt{73}/2\right)$. This triangle has rational area and one rational median—from the midpoint of the hypotenuse to the vertex at the right angle. It is an interesting exercise to prove that integer right triangles have precisely one rational median [1, p. 31]—the median to the hypotenuse.

But can any Heron triangle have *two* rational medians? In 1905, Schubert [3, p. 199] claimed that no such triangle could exist. As Dickson points out [3, p. 208], Schubert's proof was flawed but no such triangle was forthcoming. Despite this flaw, the parametrization used by Schubert turns out to be extremely useful in helping to uncover a key underlying pattern.

2 The Schubert Parameters

Consider the triangle in Figure 2, showing one of the medians with its adjacent angles. If we apply the trigonometric identity

$$\cot\left(\frac{\alpha}{2}\right) = \frac{\sin\alpha}{1-\cos\alpha}$$

to the angle α_a say, in Figure 2, then it is clear that the corresponding half-angle cotangent is rational only if $\sin\alpha_a$ and $\cos\alpha_a$ are rational. Since

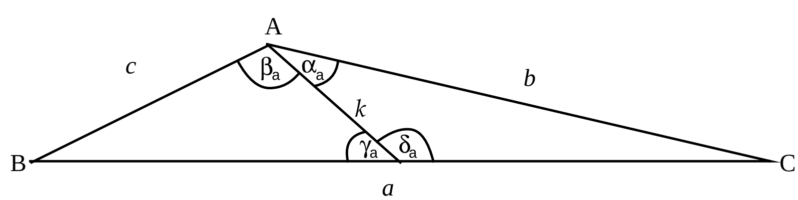

Figure 2: The angles related to Schubert's parameters

$$\sin\alpha_a = \frac{\Delta}{bk} \quad \text{and} \quad \cos\alpha_a = \frac{b^2+k^2-(a/2)^2}{2bk},$$

we see that $\sin\alpha_a$, $\cos\alpha_a$ and hence $\cot\left(\alpha_a/2\right)$ are rational for any Heron triangle with a rational median k. The same argument applies to all the angles $\alpha_a, \beta_a, \gamma_a, \delta_a$ adjacent to median k so all the half-angle cotangents are rational in this case. To ensure an unambiguous naming scheme for these parameters we impose a counter-clockwise orientation on the triangle around its centroid. Then the angles that the median to side a makes with the triangle, beginning with the two at the vertex, are labeled $\alpha_a, \beta_a, \gamma_a, \delta_a$ as in Figure 2. The respective half-angle cotangents are denoted by M_a, P_a, X_a, Y_a. We call the set of rational numbers (M, P, X, Y) 'Schubert parameters'; it is understood that if no subscript is present then the parameters are all obtained from the same median.

$M_a = \frac{4\Delta}{4bk+a^2-3b^2-c^2}$	$P_a = \frac{4\Delta}{4ck+a^2-b^2-3c^2}$
$X_a = \frac{4\Delta}{2ak-b^2+c^2}$	$Y_a = \frac{4\Delta}{2ak+b^2-c^2}$

Table 1: Schubert parameters for a triangle with sides (a, b, c)

For the $(5, 4, 3)$ Heron triangle, we obtain $(M_a, P_a, X_a, Y_a) = \left(\frac{3}{1}, \frac{2}{1}, \frac{4}{3}, \frac{3}{4}\right)$. The half-angle cotangents X and Y satisfy $XY = 1$, while the three half-angle cotangents M, P, and X satisfy an important relationship first proved by Schubert:

$$(M - 1/M) - (P - 1/P) = 2(X - 1/X) \ . \tag{3}$$

Although only two parameters suffice to describe any triangle, we usually consider three parameters (M, P, X). It is important to note that if (M, P, X) does satisfy equation (3), then so do 32 related 3-tuples. These occur because equation (3) is invariant under the following operations:

(i) replace any parameter by its negated inverse, or

(ii) interchange M and P while also inverting X, or

(iii) simultaneously invert all three of the parameters.

Since all such 3-tuples correspond to the same Heron triangle, we occasionally use an alternate representation.

Conversely, if we know any set of Schubert parameters, (M, P, X) say, then we can calculate the ratio of the sides (a, b, c) from

$$\frac{a}{c} = \frac{2\left(X + \frac{1}{X}\right)}{P + \frac{1}{P}} \qquad\qquad \frac{b}{c} = \frac{M + \frac{1}{M}}{P + \frac{1}{P}} \ . \tag{4}$$

This specifies the triangle up to homothety (a similarity transformation), which is sufficient for our purposes.

In the process of trying to describe all rational-sided triangles with three rational medians the first author discovered that any rational-sided triangle, (a, b, c), with two rational medians is given by the parametrization (see [1, p. 38])

$$\begin{aligned}
a &= \tau\{(-2\phi\theta^2 - \phi^2\theta) + (2\theta\phi - \phi^2) + \theta + 1\} \\
b &= \tau\{(\phi\theta^2 + 2\phi^2\theta) + (2\theta\phi - \theta^2) - \phi + 1\} \\
c &= \tau\{(\phi\theta^2 - \phi^2\theta) + (\theta^2 + 2\theta\phi + \phi^2) + \theta - \phi
\end{aligned} \tag{5}$$

for (τ, ϕ, θ) constrained such that $\tau > 0$, $0 < \theta, \phi < 1$, and $\phi + 2\theta > 1$. In this case, if the parameters (τ, θ, ϕ) are rational, then the corresponding triangle must have rational sides and two rational medians, namely k and l, but not

	Sides			Medians	Area
a	b	c	k	l	
73	51	26	$\frac{35}{2}$	$\frac{97}{2}$	420
626	875	291	572	$\frac{433}{2}$	55440
4368	1241	3673	1657	$\frac{7975}{2}$	2042040
14791	14384	11257	$\frac{21177}{2}$	11001	75698280
28779	13816	15155	$\frac{3589}{2}$	21937	23931600
1823675	185629	1930456	$\frac{2048523}{2}$	$\frac{3751059}{2}$	142334216640

Table 2: Sides, medians, area of discovered Heron triangles

necessarily rational area. The scaling factor τ is usually set to one. Solving for θ and ϕ gives

$$\theta = \frac{c - a \pm \sqrt{2c^2 + 2a^2 - b^2}}{a + b + c} \qquad \text{and} \qquad \phi = \frac{b - c \pm \sqrt{2b^2 + 2c^2 - a^2}}{a + b + c}. \qquad (6)$$

Any triangle obtained from a rational triple (M, P, X) has rational sides, rational area, and one rational median, while a triangle obtained from a rational pair (θ, ϕ) has rational sides and two rational medians. It is the unveiling of the interplay of these two parametrizations of a triangle that ultimately allows us to make progress on the question mentioned in the introduction.

3 Search results and hint of a connection

In 1986, both authors, unaware of each other's work, began searching for Heron triangles with two rational medians. One particularly efficient method is to enumerate over the rational parameters (θ, ϕ) in equations (5) and then check if the area of the corresponding triangle is rational. This technique allowed us to obtain the last two triangles in Table 2; meanwhile naive exhaustion struggled to reach the fourth triangle in the list. So Heron triangles with two rational medians do exist. Naturally we wondered how to find, or better yet generate, more such triangles. The first author noted that the first, second, fifth, and sixth triangles of Table 2 have related internal angles and asked how this could be exploited.

4 Discovery of the sequence of squares

In October 1989, the second author discovered a remarkable connection between the X_a and X_b parameters of related triangles. By selecting the "appropriate"

level i	triangle	$M_a(i)$	$P_a(i)$	$X_a(i)$	$M_b(i)$	$P_b(i)$	$X_b(i)$
0	(2,1,1)	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{2}{3}$
1	1st	$\frac{4}{1}$	$\frac{2}{3}$	$\frac{8}{3}$	$\frac{35}{6}$	$\frac{84}{5}$	$\frac{7}{40}$
2	2nd	$\frac{18}{1}$	$\frac{35}{6}$	$\frac{63}{10}$	$\frac{176}{105}$*	$\frac{77}{360}$*	$\frac{99}{32}$*
3	5th	$\frac{75}{98}$	$\frac{176}{105}$	$\frac{539}{800}$	$\frac{3080}{111}$	$\frac{14504}{275}$	$\frac{147}{1850}$
4	6th	$\frac{1344}{605}$	$\frac{3080}{111}$	$\frac{363}{4736}$	$\frac{3256}{165585}$*	$\frac{5312}{255189}$*	$\frac{36480}{70301}$*

Table 3: Triangles with a common $\{M_b(i), P_a(i+1)\}$ ratio.

Schubert parameters and inverting where necessary (denoted by an asterisk), it became possible to arrange the four triangles into a logical chain such that the M_b parameter from one triangle was equal to the P_a parameter of the next triangle. We label these first four triangles of the chain (see Table 3) by level 1, 2, 3 and 4 respectively, and insert the degenerate triangle $(2, 1, 1)$, with rational area and medians, at level 0 to start the chain logically. The crucial observation occurred by comparing the $X_b(i)$ and $X_a(i+1)$ ratios of consecutive triangles. From levels 1 and 2 we observed that $\frac{40\cdot 7}{63\cdot 10} = \left(\frac{2}{3}\right)^2$. Similarly, levels 2, 3 and 3, 4 imply that $\frac{99\cdot 32}{800\cdot 539} = \left(\frac{3}{35}\right)^2$ and $\frac{147\cdot 1850}{363\cdot 4736} = \left(\frac{35}{88}\right)^2$. In other words, there is a distinct pattern of rational squares in the first few products of the numerators and denominators of the $X_b(i)$ and $X_a(i+1)$ parameters. Furthermore, the denominator of one square becomes the numerator of the next square. Now all one needs to specify the next triangle in the chain is the denominator of the X product ratio since this would determine $P(i+1)$, $X(i+1)$ and hence $M(i+1)$ via Schubert's equation. For example, we set $P_a(5) = M_b(4)$. Then since

$$\frac{36480 \cdot 70301}{\text{numerator}(X_a(5)) \cdot \text{denominator}(X_a(5))} = \left(\frac{88}{k}\right)^2$$

and since $P_a(5)$ and $X_a(5)$ must lead to a rational value of $M_a(5)$ in Schubert's equation (3), one finds that $k = 37$ and hence $X_a(5) = \frac{780330}{581}$. Now calculate the Schubert parameters corresponding to the other rational median in this triangle and repeat the process. This leads to the sequence of ratios

$$\left(\frac{1}{2}\right)^2, \left(\frac{2}{3}\right)^2, \left(\frac{3}{35}\right)^2, \left(\frac{35}{88}\right)^2, \left(\frac{88}{37}\right)^2, \left(\frac{37}{4731}\right)^2, \left(\frac{4731}{107134}\right)^2, \ldots$$

This permitted us to generate the next few triangles. For example, the fifth Heron-2-median triangle has sides given by $(2442655864, 2396426547, 46263061)$.

5 Connection to Somos sequences

There the matter stood for 5 years, until the two authors were able to re-establish contact. The main question was: How was the rational square sequence

i	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	$X_b(i)$
	Numerator Factors				Denominator Factors				Parameter
0	$2\cdot 1$	-1	-1	1	$1\cdot 1$	1	3	1	$\frac{2}{3}$
1	$1\cdot 1$	1	1	-7	$2\cdot 2$	2	5	-1	$\frac{7}{40}$
2	$2\cdot 2$	1	1	8	$1\cdot 3$	3	11	1	$\frac{32}{99}$
3	$1\cdot 3$	-7	-7	-1	$2\cdot 5$	5	37	1	$-\frac{147}{1850}$
4	$2\cdot 5$	8	8	-57	$1\cdot 11$	11	83	-7	$\frac{36480}{70301}$
5	$1\cdot 11$	-1	-1	391	$2\cdot 37$	37	274	8	$\frac{4301}{6001696}$
6	$2\cdot 37$	-57	-57	-455	$1\cdot 83$	83	1217	-1	$\frac{109393830}{8383913}$

Table 4: Decomposition of the parameter $X_b(i)$

i	1	2	3	4	5	6	7	8	9	10
S_i	1	1	2	3	5	11	37	83	274	1217
T_i	1	-1	1	1	-7	8	-1	-57	391	-455

Table 5: The S and T series

determined, and could a formula be found for it? After intense correspondence from late 1994 to early 1995, we obtained some interesting results.

The problem with the method described in the previous section is that it requires the factorization of numbers that are growing very rapidly. Furthermore, there is still some ambiguity about inverting certain parameters and not others.

We found that all of the (M, P, X) parameters could be formed as a combination of two series. Notice that the numerator of the X_b parameter in Table 4 is the product $a_1 \cdot a_2 \cdot a_3 \cdot a_4$ and the denominator is likewise the product $b_1 \cdot b_2 \cdot b_3 \cdot b_4$, where each of the a_i and b_i are shifts of one or another of two special sequences. There are similar relationships for all the Schubert parameters for our set of triangles in terms of these two series, which we denote by S and T. We observed that each series seemed to satisfy an order eight recurrence, namely,

$$S_i = \frac{2^{\chi(i)} \cdot 3^{\chi(i+1)} \cdot S_{i-7} \cdot S_{i-1} + S_{i-4}^2}{S_{i-8}} \qquad \text{and}$$

$$T_i = \frac{6^{\chi(i+1)} \cdot T_{i-7} \cdot T_{i-1} + T_{i-4}^2}{T_{i-8}} \qquad ,$$

where

$$\chi(i) = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

Since these two series were so fundamental, one author sent a query to the *On-Line Encyclopædia of Integer Sequences* (`sequencesresearch.att.com`), authored by Neil J. A. Sloane. It quickly posted back that the first, S series, was indeed a Somos 5 sequence [4, p. 41], and gave the recursion formula

$$A_i = \frac{A_{i-1} \cdot A_{i-4} + A_{i-2} \cdot A_{i-3}}{A_{i-5}} \; . \tag{7}$$

We realised that the T series satisfied the same recurrence with different initial terms. In terms of the order 5 recurrence we have

$$S_i = \begin{cases} 1, 1, 2, 3, 5 & \text{for } i = 1, \dots, 5 \\ A_i & \text{for } i \geq 6. \end{cases} \qquad T_i = \begin{cases} 1, -1, 1, 1, -7 & \text{for } i = 1, \dots, 5 \\ A_i & \text{for } i \geq 6. \end{cases} \tag{8}$$

The half-angle cotangents of our chain of Heron triangles with two rational medians are given in terms of the series S and T by

$$\begin{aligned} M_a(i) &= -\frac{S_{i+1} \cdot S_{i+2}^2 \cdot T_i}{S_i \cdot T_{i+1} \cdot T_{i+2}^2} & M_b(i) &= \frac{S_{i+1} \cdot S_{i+4} \cdot T_{i+1} \cdot T_{i+4}}{S_{i+2} \cdot S_{i+3} \cdot T_{i+2} \cdot T_{i+3}} \\ P_a(i) &= -\frac{S_{i+1} \cdot S_{i+2} \cdot T_{i+1} \cdot T_{i+2}}{S_i \cdot S_{i+3} \cdot T_i \cdot T_{i+3}} & P_b(i) &= -\frac{S_{i+2}^2 \cdot S_{i+3} \cdot T_{i+4}}{S_{i+4} \cdot T_{i+2}^2 \cdot T_{i+3}} \\ X_a(i) &= 2^{(-1^{i+1})} \cdot \frac{S_i \cdot S_{i+2}^2 \cdot T_{i+3}}{S_{i+3} \cdot T_i \cdot T_{i+2}^2} & X_b(i) &= 2^{(-1^{i})} \cdot \frac{S_{i+1} \cdot T_{i+2}^2 \cdot T_{i+4}}{S_{i+2}^2 \cdot S_{i+4} \cdot T_{i+1}}. \end{aligned} \tag{9}$$

Equations (9) permitted us to rapidly compute many corresponding triangles using multiprecision packages (`MAPLE` and `PARI`) and each such triangle invariably had rational area and two rational medians.

6 Searching for a Closed form for S and T sequences

Having obtained recurrence relations for S_i and T_i, we hoped that a closed formula would allow us to prove some of the results that we had so far observed only numerically. A second posting to the `sci.math.research` newsgroup prompted a number of interesting responses but by far the most impressive came from Noam Elkies, who gave two closed formulae for the S_i sequence and indirectly provided a formula for the T_i sequence. What follows borrows heavily from his reply.

Numerical evidence suggests that the sequence S_i also satisfies recurrence relations of the form

$$\begin{aligned} S_{i-2}S_{i+2} &= 2S_{i-1}S_{i+1} - S_i^2 \qquad && \text{if } i \text{ is even,} \\ S_{i-2}S_{i+2} &= 3S_{i-1}S_{i+1} - S_i^2 \qquad && \text{if } i \text{ is odd.} \end{aligned}$$

It is possible to combine these into a single identity by defining

$$\sigma_i = \begin{cases} S_i, & \text{if } i \text{ is even,} \\ rS_i, & \text{if } i \text{ is odd.} \end{cases}$$

Replacing S_i with σ_i or σ_i/r as appropriate and then equating the preceding two recurrences, one finds that $r = \sqrt[4]{2/3}$. Hence, the σ_i satisfy the recurrence relation

$$\sigma_{i-2}\sigma_{i+2} = \sqrt{6}\sigma_{i-1}\sigma_{i+1} - \sigma_i^2 \ .$$

Because of the similarity of this to a Somos recurrence on sequences of elliptic theta functions, one attempts to fit a solution of the form

$$\sigma_i = bu^{i^2} \sum_{n=-\infty}^{+\infty} q^{n^2} z^{in}. \tag{10}$$

In fact, the parameters q, z, b, u can be obtained numerically from the condition that the formula for σ_i hold for the initial values. This leads to

$$\begin{aligned} q &= 0.02208942811097933557356088\ldots \\ z &= 0.1141942041600238048921321\ldots \\ b &= 0.9576898995913810138013844\ldots \\ u &= 0.7889128685374661530379575\ldots \end{aligned}$$

The theta function (10) is rapidly convergent and so we have a numerical, closed form expression to evaluate each σ_i and hence each S_i. Using the initial conditions for the T-sequence would lead to a similar theta function.

However, the numbers S_i can also be obtained "arithmetically" from the elliptic curve $\mathbb{C}^*/q^{2\mathbb{Z}}$ associated to our theta functions. By

(i) computing the j-invariant $j(E) = j(q^2)$ as a real number,

(ii) using its continued fraction to recognize $j(E)$ as the rational $11^6/612$,

(iii) computing the x-coordinate of the point z on the curve $\mathbb{C}^*/q^{2\mathbb{Z}}$, which determines the correct quadratic twist, and

(iv) reducing to standard minimal form,

Elkies finds the elliptic curve

$$E : y^2 + xy = x^3 + x^2 - 2x,$$

which is curve #102-A1 in Cremona's tables [2]. It has a point of order 2 at $(0, 0)$ and an infinite order point at $P = (x, y) = (2, 2)$. For $i = 1, 2, 3, 4, \ldots$ the x-coordinate of the i-th multiple of P on E in lowest terms is

$$\frac{2\cdot 1^2}{1^2}, \ \frac{1^2}{1^2}, \ \frac{2\cdot 2^2}{1^2}, \ \frac{3^2}{1^2}, \ \frac{2\cdot 5^2}{7^2}, \ \frac{11^2}{8^2}, \ \frac{2\cdot 37^2}{1^2}, \ \ldots \quad .$$

Indeed, the numerator of $i * P$ is always S_i^2 or $2S_i^2$ according as i is even or odd. Notice that the denominator is precisely T_i^2. The two sequences are very closely connected. Not only do they satisfy the same recurrence relation, but the initial conditions are no longer arbitrary; given one it is possible to construct the other.

Unfortunately, we were not able to use either of these closed forms to prove that the triangles generated from equations (9) and (4) always have rational area. However, the elliptic curve does turn up again and leads to such a proof from a different direction.

7 Triangles in the $\theta\phi$-plane lead to five elliptic curves

At this stage we used equations (9), (4), and (6) to generate the values of θ and ϕ corresponding to the first 100 terms of the two Somos sequences S_i and T_i. We plotted these parameters, considered as points corresponding to distinct Heron triangles with two rational medians, in the $\theta\phi$-plane (Figure 3) and the structure here was a surprise.

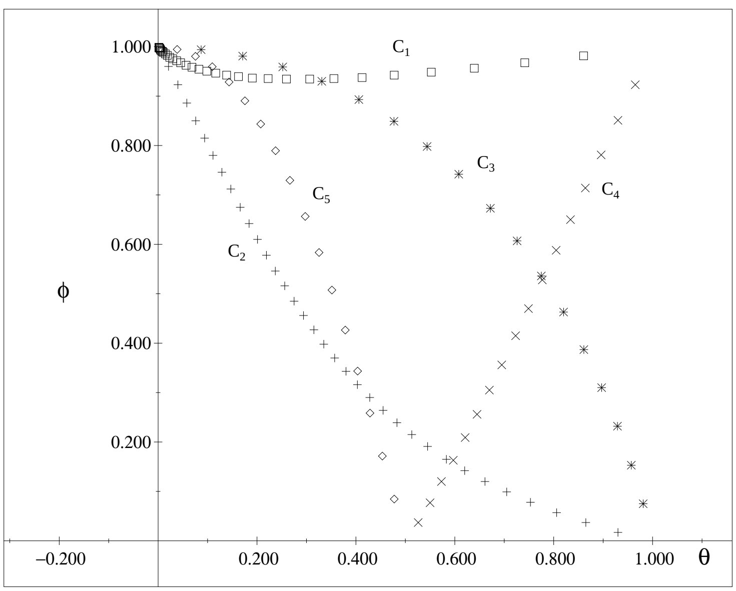

Figure 3: Heron triangles with 2 rational medians in the $\theta\phi$-plane

Rather than being randomly distributed in the region, the points seem to lie on five distinct curves. During this process we discovered that the points were being distributed to the five curves in a periodic way with a cycle length of

7. The points generated by the parameter set $(M_a(i), P_a(i), X_a(i))$ visited the curves in the order {1,2,3,4,1,2,5}. Similarly, the points generated by the set $(M_b(i), P_b(i), X_b(i))$ visited the curves in the order {2,1,4,3,2,1,5}. As a result, it was easy to isolate the rational coordinates of enough points on each curve to determine the corresponding equations:

$$\begin{aligned}
&C_1 : 27\theta^3\phi^3 - \theta\phi(\theta-\phi)(8\theta^2+11\theta\phi+8\phi^2) - 3\theta\phi(5\theta^2-\theta\phi+5\phi^2) \\
&\qquad - (\theta-\phi)(\theta^2+4\theta\phi+\phi^2) - (3\theta^2-7\theta\phi+3\phi^2) - 3(\theta-\phi) - 1 = 0, \\
&C_2 : 3\theta^2\phi^2 - 2\theta\phi(\theta-\phi) - (\theta^2+6\theta\phi+\phi^2) + 1 = 0, \\
&C_3 : \theta\phi(\theta-\phi)^3 - (\theta^4+11\theta^3\phi+3\theta^2\phi^2+11\theta\phi^3+\phi^4) \\
&\qquad - 2(\theta^3-\phi^3) + 10\theta\phi + 2(\theta-\phi) + 1 = 0, \\
&C_4 : \theta\phi(\theta-\phi) + \theta\phi + 2(\theta-\phi) - 1 = 0, \\
&C_5 : (\theta-1)^3\phi^2 + 2(\theta+1)(\theta^3+2\theta^2-2\theta+1)\phi + (2\theta-1)(\theta+1)^3 = 0.
\end{aligned}$$

We conjectured that all the rational points on these five curves produce triangles with rational area. Since the triangle has two rational medians, one can form (θ, ϕ) parameters for either median. We call these *dual* parameter sets for the triangle. The transformation that takes (θ, ϕ) to its dual point (θ', ϕ') is given by

$$\theta' = \frac{2\theta^2+\theta\phi+\theta+\phi-1}{3\theta\phi+\theta-\phi+1}, \qquad \phi' = \frac{-\theta\phi-2\phi^2+\theta+\phi+1}{3\theta\phi+\theta-\phi+1}.$$

Under this mapping the curves C_1 and C_2 are dual, as are C_3 and C_4, while C_5 is self-dual. Thus it is sufficient to prove that all rational points on the curves C_2, C_4, and C_5 say, correspond to Heron triangles with two rational medians.

Next, we find that C_2, C_4 and C_5 are all birationally equivalent to the same elliptic curve so we need to prove the conjecture only for C_4, say. These three curves are quadratic in ϕ and the respective discriminants are

$$\begin{aligned}
\mathrm{Disc}(C_2) &= 4(4\theta^4+8\theta^3+5\theta^2-2\theta+1), \\
\mathrm{Disc}(C_4) &= \theta^4+2\theta^3+5\theta^2-8\theta+4, \text{ and} \\
\mathrm{Disc}(C_5) &= 4\theta^2(\theta+1)^2(\theta^4+2\theta^3+5\theta^2-8\theta+4).
\end{aligned}$$

Since we are searching for rational points on each of the curves, we require the discriminant of each to be a rational square. All the rational points that force this correspond to rational points on the elliptic curve

$$Y^2 = X^4 + 2X^3 + 5X^2 - 8X + 4.$$

For C_2, we map X to $-1/\theta$ while for C_4 and C_5 we just map X to θ. Finally we were able to prove the following

Theorem 1 *Every rational point on the curve*

$$C_4 : \theta^2\phi - \theta\phi^2 + \theta\phi + 2\theta - 2\phi - 1 = 0$$

such that $0 < \theta, \phi < 1$ and $2\theta + \phi > 1$ corresponds to a triangle with rational sides, rational area, and two rational medians.

The proof requires several technical lemmas that will appear in a forthcoming paper. Here we just give an outline.

(i) The θ, ϕ inequalities are obtained from the triangle inequalities.

(ii) Reduce the squarefree part of the square of the area from degree 11 to degree 8 by applying the curve C_4 to Heron's formula (1).

(iii) Transform the curve C_4 to minimal Weierstraß form to obtain E, the elliptic curve found by Elkies in Section VI.

(iv) Finally, use induction in the group $E(\mathbb{Q})$ to show that any point that corresponds to a triangle with rational area leads, in all possible ways, to another point corresponding to a triangle with rational area.

8 Two Isolated Triangles

The story does not end here since two of the triangles found by computational search (the third and fourth entries of Table 2) do not lie on any of our five elliptic curves. Although these two triangles were found using equations (5), they are probably not parametrizable by equations (9) since the five curves were numerically obtained from the latter. Each of these isolated triangles has associated with it six triangles that have a rational median and rational area and share a common Schubert parameter ratio. What role these ratios play is as yet undetermined.

We are continuing further research into these two triangles, as we conjecture that all Heron triangles with two rational medians are produced by formulæ similar to those we have presented in this paper. However, finding more examples like these two appears difficult.

9 Acknowledgements

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