

Reflexive Numbers and Berger Graphs from Calabi–Yau Spaces

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Abstract

We review the Batyrev approach to Calabi-Yau spaces based on reflexive weight vectors. The Universal CY algebra gives a possibility to construct the corresponding reflexive numbers in a recursive way. A physical interpretation of the Batyrev expression for the Calabi-Yau manifolds is presented. Important classes of these manifolds are related to the simple-laced and quasi-simple-laced numbers. We discuss the classification and recurrence relations for them in the framework of quantum field theory methods. A relation between the reflexive numbers and the so-called Berger graphs is studied. In this correspondence the rôle played by the generalized Coxeter labels is highlighted. Sets of positive roots are investigated in order to connect them to possible new algebraic structures stemming from the Berger matrices.

1 Introduction

The Calabi-Yau manifolds belong to an interesting class of the Riemann spaces [1–5]. They are used in physics for the compactification of extra dimensions. For example, the proof of the duality between the string theories IIA, IIB and the heterotic $E_8 \times E_8$ model was based on such compactifications. The CY spaces are related to the Lie and Kac-Moody algebras. New symmetries based on ternary, quaternary, etc operations are investigated now in physics and mathematics [6–12] and appear also in the CY_n geometry [24–27]. They will be discussed also below in terms of the so-called Berger graphs. Our goal here is to study the reflexivity property of CY_n spaces through the theory of numbers, recurrence relations and quantum field theory methods.

A CY space can be realized as an algebraic variety \mathcal{M} in a weighted projective space $CP^{n-1}(\vec{k})$ [4, 5] where the weight vector reads $\vec{k} = (k_1, \dots, k_n)$. This variety is defined by

$$\mathcal{M} \equiv (\{x_1, \dots, x_n\} \in CP^{n-1}(\vec{k}) : \mathcal{P}(x_1, \dots, x_n) \equiv \sum_{\vec{m}} c_{\vec{m}} x^{\vec{m}} = 0), \quad (1)$$

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i.e., as the zero locus of a quasi-homogeneous polynomial of degree $d_k = \sum_{i=1}^n k_i$, with the monomials being $x^{\vec{m}} \equiv x_1^{m_1} \cdots x_n^{m_n}$. The points in $\mathbb{C}P^{n-1}$ satisfy the property of projective invariance $\{x_1, \dots, x_n\} \approx \{\lambda^{k_1} x_1, \dots, \lambda^{k_n} x_n\}$ leading to the constraint $\vec{m} \cdot \vec{k} = d_k$. The sum in the above expression is performed over all solutions \vec{m} of this equation and the coefficients $c_{\vec{m}}$ are arbitrary complex numbers.

The most important constraint for a CY candidate is the condition of reflexivity of the vector \vec{k} , which can be defined in terms of the Batyrev reflexive polyhedra [13]. Let us consider this condition in more detail. We can construct the vector $\vec{m}' = \vec{m} - \vec{1}$ where $\vec{m}' \cdot \vec{k} = 0$. Then it is convenient to define the lattice

$$\Lambda = \left\{ \vec{m}' \in \mathbb{Z}^n : \vec{m}' \cdot \vec{k} = 0 \right\} \quad (2)$$

with basis $\{e_i\}$. The dual of this lattice, Λ^* , has the basis $\{e_i^*\}$ with the orthonormality condition $e_i \cdot e_j^* = \delta_{ij}$. We define a polyhedron Δ as the convex hull of the lattice Λ and the polyhedron Δ^* as the convex hull of the dual lattice Λ^* . The reflexivity condition means that the polyhedron Δ is integer, its origin $\vec{0}$ is the only interior point, and its dual Δ^* is also integer and contains only one interior point. In this case the vector \vec{k} is considered to be reflexive. Using this criteria of reflexive vectors Batyrev proved the Mirror duality of $CY_{3,4}$ spaces [13]. Namely, for each CY, \mathcal{M} , there exists a Mirror CY partner, \mathcal{M}^* . This symmetry helped establish the duality between the type IIA and IIB string theories.

The correspondence between CY spaces and reflexive polyhedra led the way for their classification. In particular, for the case $CY_2 = K_3$ the 4319 three-dimensional polyhedra were found in Ref. [14,15]. Among them, 95 can be described with a single reflexive vector. The algorithm, constructed in Ref. [14,15], generated 473 800 776 reflexive four-dimensional polyhedra in the case of CY_3 spaces. A subclass of this large number can be described by one reflexive vector. Namely, 184 026 polyhedra belong to this subclass [14,15] (see also [16–19]).

Recently, an alternative to this classification was developed using some properties of the theory of reflexive vectors. This new approach was named “Universal Calabi–Yau Algebra” (UCYA) [16]. One of its main results is that all reflexive vectors of dimension n can be obtained from the reflexive vectors with lower dimension $1, \dots, n-1$. Consequently, every reflexive vector of dimension n can be constructed from the simplest reflexive vector $\vec{k} = (1)$. The key observation to realize this program was to use the concept of the r -arity composition law (with $r = 2, \dots, n$) for the subclass which can be described by a unique reflexive vector (in general CY spaces this subclass corresponds to the so-called level one). Using this composition law, it was shown how the level one CY_n space can be obtained from its slices of lower dimension $r = 1, \dots, n-1$, generating in this way the r -arity slice classification. For example, the 2-arity composition law in K_3 space gives us 90 out of the 95 reflexive vectors. These 90 vectors were unified in 22 chains having the same CY_1 -slice in K_3 . Four of the remaining reflexive weight vectors can be obtained with the 3-arity and the last one with the 4-arity. In CY_3 a similar 2-arity classification produces 4242 chains having the same K_3 slice of CY_3 [16,17].

To be more specific, the 22 chains in K_3 are generated by taking the reflexive vectors (1), (1,1), (1,1,1), (1,1,2) and (1,2,3) and extending them to dimension 4 by including additional zero components, i.e., (0,0,0,1), (0,0,1,1), (0,1,1,1), (0,1,1,2), (0,1,2,3) with all permutations of their components. In the 2-arity construction one should take all possible pairs of two extended vectors and select those “good pairs” which have a reflexive polyhedron in the intersection of the corresponding slices \vec{m} . The selected vectors can be added with integer coefficients. For example, from 50 possible extended vectors we can take the pair $\vec{k}_a^{\text{ext}} = (0, 1, 1, 1)$ and $\vec{k}_b^{\text{ext}} = (1, 0, 0, 0)$. Their intersection is defined as the solution of two constraints $\vec{m} \cdot \vec{k}_a^{\text{ext}} = d_{k_a} = 3$ and $\vec{m} \cdot \vec{k}_b^{\text{ext}} = d_{k_b} = 1$. These equations can be also written as

$$\Lambda = \left\{ \vec{m}' \in \mathbb{Z}^4 : \vec{m}' \cdot \vec{k}_a^{\text{ext}} = \vec{m}' \cdot \vec{k}_b^{\text{ext}} = 0 \right\}, \quad (3)$$

where $\vec{m}' = \vec{m} - \vec{1}$. The lattice of solutions for \vec{m}' corresponds to a two dimensional reflexive polyhedron or CY_1 , and, therefore, according to UCYA, the linear combination

$$m_a \vec{k}_a^{\text{ext}} + m_b \vec{k}_b^{\text{ext}} \quad (4)$$

with $m_a \leq d_{k_b}$ and $m_b \leq d_{k_a}$ of these two vectors forms the chain with the eldest vector having unit coefficients $m_a = m_b = 1$.

Another simple example is the combination of two vectors, \vec{k}_m with dimension m and \vec{k}_n with dimension n , in the form $(\vec{k}_m, \vec{0}_n) + (\vec{0}_m, \vec{k}_n)$ which is always an eldest reflexive vector of dimension $m + n$. In particular, by adding $\vec{k}_b = (0, 1, 1, 1)$ and $\vec{k}_a = (1, 0, 0, 0)$ with certain coefficients, we obtain the three vectors $(1, 1, 1, 1)$, $(2, 1, 1, 1)$ and $(3, 1, 1, 1)$. The intersection of the slices for the vectors \vec{k}_b and \vec{k}_a produces a two-dimensional reflexive slice. This slice divides the corresponding three-dimensional K_3 polyhedrons in two parts. On the left and right sides of the slice the set of points at the edges forms affine Dynkin diagrams. For example, in the reflexive polyhedra corresponding to the vector $(1, 1, 1, 1)$ from the $(1, 0, 0, 0)$ side we obtain the diagram for the algebra $A_{11}^{(1)}$ and from the $(0, 1, 1, 1)$ side we have the graph of $E_6^{(1)}$. The $(2, 1, 1, 1)$ and $(3, 1, 1, 1)$ members of this chain contain different Dynkin graphs. This property is universal and valid for all 22 chains. It is a generalization of the results of Candelas and Font [20–23] who found a dictionary for the Dynkin graphs of the Cartan–Lie algebra in the case of the Weierstrass slice using the type IIA and heterotic $E_8 \times E_8$ string duality. In the K_3 case a correspondence between extended reflexive vectors and Dynkin graphs was found [16, 17, 19], for example,

$$\begin{aligned}
(0, 0, 0, 1) &\rightarrow A_r^{(1)} \\
(0, 0, 1, 1) &\rightarrow D_r^{(1)} \\
(0, 1, 1, 1) &\rightarrow E_6^{(1)} \\
(0, 1, 1, 2) &\rightarrow E_7^{(1)} \\
(0, 1, 2, 3) &\rightarrow E_8^{(1)}.
\end{aligned} \tag{5}$$

Note that the maximal Coxeter label of the graphs at the right hand side of this correspondence coincides with the degree of the reflexive vectors at the left hand side. We shall discuss this point later. Our scheme offers the possibility of constructing new graphs for CY spaces in any dimension. For example, in K_3 all 4242 graphs for the reflexive numbers of the level one can be obtained following the above–mentioned approach [16]. Our analysis allows to classify the structure of these large CY spaces in terms of number theory and to construct the Berger graphs. These Berger graphs might correspond to unknown symmetries lying beyond Cartan–Lie algebras [24].

The number of algebraic CY_n varieties is very large and grows very rapidly with the dimension n of the space. A similar situation occurs with the number of reflexive weight vectors. For example, the number of eldest reflexive vectors of 2–arity is 1, 2, 22 and 4242 [16] for dimensions $n = 2, 3, 4$ and 5, respectively. To obtain the last number 4242 for $n=5$ using the arity construction we need 100 extended reflexive vectors (and all permutations of their components) (see Table 1). An important remark is that all reflexive weight vectors can be considered as new types of numbers because in the framework of UCYA the arithmetic of their adding and subtracting is well defined. In the “tree” classification of CY spaces the trunk line of the reflexive weight numbers corresponds to those with unit components, i.e., $(1), (1, 1), (1, 1, 1), \dots$. An interesting wider subclass is the so-called “simply–laced” numbers. A simply–laced number $\vec{k} = (k_1, \dots, k_n)$ with degree $d = \sum_{i=1}^n k_i$ is defined such that

$$\frac{d}{k_i} \in \mathbb{Z}^+ \text{ and } d > k_i. \tag{6}$$

For these numbers there is a simple way of constructing the corresponding affine Dynkin and Berger graphs together with their Coxeter labels. The Cartan and Berger matrices of these graphs are symmetric. In the well known Cartan case they correspond to the ADE series of simply–laced algebras. In dimensions $n = 1, 2, 3$ the numbers $(1), (1, 1), (1, 1, 1), (1, 1, 2), (1, 2, 3)$ are simply–laced. For $n = 4$ among all 95 reflexive numbers 14 are simply–laced, as it is shown in Table 1 [24–26]. The remaining 81 correspond to the so-called quasi–simply–laced case. Before constructing these graphs in the next section we proceed to review the concept of reflexivity and relate it to techniques used in the functional approach to Quantum Field Theory.

\aleph	$\vec{k}_{\delta\epsilon\epsilon x}^{(i)}$	$G(Gal)$	\aleph	$\vec{k}_{\delta\epsilon\epsilon x}^{(i)}$	$G(Gal)$
i	(0, 0, 0, 0, 1) [1]	5	46	(0, 2, 3, 4, 7)[16]	120
ii	(0, 0, 0, 1, 1) [2]	10	47	(0, 2, 3, 4, 9)[18]	120
iii	(0, 0, 1, 1, 1) [3]	10	48	(0, 2, 3, 5, 5)[15]	60
iv	(0, 0, 1, 1, 2) [4]	30	49	(0, 2, 3, 5, 7)[17]	120
v	(0, 0, 1, 2, 3) [6]	60	50	(0, 2, 3, 5, 8)[18]	120
1	(0, 1, 1, 1, 1) [4]	5	51	(0, 2, 3, 5, 10)[20]	120
2	(0, 1, 1, 1, 2)[5]	20	52	(0, 2, 3, 7, 9)[21]	120
3	(0, 1, 1, 1, 3) [6]	20	53	(0, 2, 3, 7, 12)[24]	120
4	(0, 1, 1, 2, 2) [6]	30	54	(0, 2, 3, 8, 11)[24]	120
5	(0, 1, 1, 2, 3)[7]	60	55	(0, 2, 3, 4, 7)[16]	120
6	(0, 1, 1, 2, 4) [8]	60	56	(0, 2, 3, 10, 15) [30]	120
7	(0, 1, 1, 3, 4)[9]	60	57	(0, 2, 4, 5, 9)[20]	120
8	(0, 1, 1, 3, 5)[10]	60	58	(0, 2, 4, 5, 11)[22]	120
9	(0, 1, 1, 4, 6) [12]	60	59	(0, 2, 5, 6, 7)[20]	120
10	(0, 1, 2, 2, 3)[8]	60	60	(0, 2, 5, 6, 13)[26]	120
11	(0, 1, 2, 2, 5) [10]	60	61	(0, 2, 5, 9, 11)[27]	120
12	(0, 1, 2, 3, 3)[9]	60	62	(0, 2, 5, 9, 16)[32]	120
13	(0, 1, 2, 3, 4)[10]	120	63	(0, 2, 5, 14, 21)[42]	120
14	(0, 1, 2, 3, 5)[11]	120	64	(0, 2, 6, 7, 15)[30]	120
15	(0, 1, 2, 3, 6) [12]	120	65	(0, 3, 3, 4, 5)[15]	60
16	(0, 1, 2, 4, 5)[12]	120	66	(0, 3, 4, 5, 6)[18]	120
17	(0, 1, 2, 4, 7)[14]	120	67	(0, 3, 4, 5, 7)[19]	120
18	(0, 1, 2, 5, 7)[15]	120	68	(0, 3, 4, 5, 8)[20]	120
19	(0, 1, 2, 5, 8)[16]	120	69	(0, 3, 4, 5, 12)[24]	120
20	(0, 1, 2, 6, 9) [18]	120	70	(0, 3, 4, 7, 10)[24]	120
21	(0, 1, 3, 4, 4) [12]	60	71	(0, 3, 4, 7, 14)[28]	120
22	(0, 1, 3, 4, 5)[13]	120	72	(0, 3, 4, 10, 13)[30]	120
23	(0, 1, 3, 4, 7)[15]	120	73	(0, 3, 4, 10, 17)[24]	120
24	(0, 1, 3, 4, 8)[16]	120	74	(0, 3, 4, 11, 18)[36]	120
25	(0, 1, 3, 5, 6)[15]	120	75	(0, 3, 4, 14, 21)[42]	120
26	(0, 1, 3, 5, 9)[18]	120	76	(0, 3, 5, 6, 7)[21]	120
27	(0, 1, 3, 7, 10)[21]	120	77	(0, 3, 5, 11, 14)[33]	120
28	(0, 1, 3, 7, 11)[22]	120	78	(0, 3, 5, 11, 19)[38]	120
29	(0, 1, 3, 8, 12) [24]	120	79	(0, 3, 5, 16, 24)[48]	120
30	(0, 1, 4, 5, 6)[16]	120	80	(0, 3, 6, 7, 8)[24]	120
31	(0, 1, 4, 5, 10) [20]	120	81	(0, 4, 5, 6, 9)[24]	120
32	(0, 1, 4, 6, 7)[18]	120	82	(0, 4, 5, 6, 15)[30]	120
33	(0, 1, 4, 6, 11)[22]	120	83	(0, 4, 5, 7, 9)[25]	120
34	(0, 1, 4, 9, 14)[28]	120	84	(0, 4, 5, 7, 16)[32]	120
35	(0, 1, 4, 10, 15)[30]	120	85	(0, 4, 5, 13, 22)[44]	120
36	(0, 1, 5, 7, 8)[21]	120	86	(0, 4, 5, 18, 27)[54]	120
37	(0, 1, 5, 7, 13)[26]	120	87	(0, 4, 6, 7, 11)[28]	120
38	(0, 1, 5, 12, 18)[36]	120	88	(0, 4, 6, 7, 17)[34]	120
39	(0, 1, 6, 8, 9)[24]	120	89	(0, 5, 6, 7, 9)[27]	120
40	(0, 1, 6, 8, 15)[30]	120	90	(0, 5, 6, 8, 11)[30]	120
41	(0, 1, 6, 14, 21) [42]	120	91	(0, 5, 6, 8, 19)[38]	120
42	(0, 2, 2, 3, 5)[12]	60	92	(0, 5, 6, 22, 33)[66]	120
43	(0, 2, 2, 3, 7)[14]	60	93	(0, 5, 7, 8, 20)[40]	120
44	(0, 2, 3, 3, 4) [12]	60	94	(0, 7, 8, 10, 25)[50]	120
45	(0, 2, 3, 4, 5)[14]	120	95	(0, 7, 8, 9, 12)[36]	120

Table 1: The 100 distinct types of five-dimensional extended projective vectors used to construct CY_3 spaces. The order of their permutation symmetry groups is also shown. Including these permutations, the total number of extended vectors is 10 270. The simply-laced vectors ($1+1+3+14=19$) are highlighted with bold face.

2 From Reflexive Numbers to Quantum Field Theory Methods

In this section we reconsider for CY spaces the condition of reflexivity proposed firstly by Batyrev [13]. We shall do it in a new approach where the reflexive numbers are studied starting from the simply-laced case with a subsequent generalization to quasi-simply-laced cases. The properties of these reflexive numbers turn out to be very interesting.

2.1 Geometrical Construction for Reflexivity

Let us start by recalling the definition of the degree d_k of a weight vector \vec{k}

$$d_k = \sum_{i=1}^n k_i, \quad (7)$$

where k_i are positive integer numbers. It is convenient to normalize this vector as follows

$$l_i \equiv k_i/d_k. \quad (8)$$

Then Eq. (7) looks simpler

$$\sum_{i=1}^n l_i = 1, \quad l_i < 1. \quad (9)$$

The numbers l_i are positive regular ratios. An additional constraint stems from the existence of solutions of the equation $\vec{m} \cdot \vec{k} = d_k$ which now reads

$$\vec{m} \cdot \vec{l} = 1. \quad (10)$$

This last equation has the solutions for the monoms $\{\vec{m}_i\}$ and an independent set of them with $i = 1, 2, \dots, n$ can be written as a matrix M . At this point let us note that $\vec{m} = \vec{1}$ ($1_i = 1$ for $i = 1, 2, \dots, n$) also satisfies Eq. (10) due to Eq. (9). Relation (10) for \vec{m}_i can be presented as the set of equations

$$(M l)_i = 1_i, \quad (11)$$

where M is a $n \times n$ matrix constructed from the non-negative integer numbers m_{ij}

$$(M)_{ij} = m_{ij} \quad (m_{ij} = 0, 1, 2, \dots). \quad (12)$$

The vectors \vec{m}_i for $i = 1, 2, \dots, n$ correspond to n points in the n -dimensional space with non-negative integer components m_{ij} . These vectors are considered to be linearly independent, which means that

$$\det M \neq 0. \quad (13)$$

All possible non-negative integer vectors \vec{m} produce a lattice. For a given \vec{l} all the solutions to Eq. (10) define a slice of this lattice. The subset of them $\{\vec{m}_i\}$ for $i = 1, 2, \dots, n$ entering in M can be chosen in such a way that other solutions can be obtained as their linear combinations with non-negative coefficients. The slice \vec{m} has the property of reflexivity provided that the vector $\vec{1}$ is inner and all other points are on its boundary. For our choice of vectors $\{\vec{m}_i\}$ the slice will be reflexive if $\vec{1}$ can be expanded as a linear combination of $\{\vec{m}_i\}$

$$\vec{1} = \sum_{i=1}^n c_i \vec{m}_i \quad (14)$$

with positive coefficients c_i . Note that according to eq. (10) these coefficients satisfy the constraints

$$\sum_{i=1}^n c_i = 1, \quad c_i > 0 \quad (15)$$

and therefore the vector c_i is analogous to the vector l_j . This implies that we can construct the dual slice $\vec{\bar{m}}$ obeying the equation

$$\vec{\bar{m}} \vec{c} = 1. \quad (16)$$

Due to Eq. (14) the vectors $\tilde{m}_i^{(j)} = m_{ij}$ can be chosen as a basis of the dual lattice. In this case relation (11) due to $l_j > 0$ provides the Batyrev reflexivity condition for the dual polyhedron. For each vector l_j we can construct several matrices M satisfying condition (14) and therefore there is a recurrence procedure to generate new reflexive vectors c_i starting for example from the simplest vector $l_j = 1/n$.

To formulate the reflexivity requirement on \vec{l} it is convenient to extend the matrix M with the matrix elements M_{ij} ($i, j = 1, 2, \dots, n$) adding to it one column and one row composed from units

$$M'_{\mu\nu} = M_{ij} \delta_{\mu i} \delta_{\nu j} + \delta_{\mu 0} + \delta_{\nu 0} - \delta_{\mu 0} \delta_{\nu 0}. \quad (17)$$

For each matrix M we obtain the following set of equations for the reflexive weight vectors l_ν^r and l_μ^s ($\mu, \nu = 0, 1, 2, \dots, n$) with integer positive components

$$M'_{\mu\nu} l_\nu^r = M'_{\mu\nu} l_\mu^s = 0. \quad (18)$$

For their self-consistency we should impose on the matrix elements M_{ij} the condition

$$\det M' = 0. \quad (19)$$

Clearly the existence of solutions of eqs (18) imposes even more severe constraints on M related to the positivity conditions for l_k ($k = 1, 2, \dots, n$). Together with conditions (13) this means that the components l_μ for $\mu = 1, 2, \dots, n$ can be expressed in terms of $l_0 = -1$ in such way that equations (9) and (10) are fulfilled.

For a reflexive slice we can choose the vectors $\vec{\bar{m}}_l$ in such way that one of their projections is zero. For example,

$$\vec{\bar{m}}_l = (m_l^1, m_l^2, \dots, m_l^{n-l} = 0, \dots, m_l^n). \quad (20)$$

Consequently the reflexivity condition imposes an additional constraint on the weight vector \vec{k} : The existence of solutions of the equation

$$\sum_{i \neq n-l} m_l^i k_i = d_k \quad (21)$$

for all $l = 1, 2, \dots, n$.

Let us express M as a product of 2 matrices

$$M = G \lambda, \quad (22)$$

where λ is diagonal, $\lambda_{ij} = \delta_{ij} \lambda_i$, and the matrix elements of G satisfy the requirement

$$\sum_{j=1}^n G_{ij} = 1_i, \quad (23)$$

which means that $\vec{1}$ is an eigenvector of G with its eigenvalue equal to unity. The above-mentioned decomposition of M is unique for any non-degenerate matrix M . Indeed, we have the relation

$$G_{ij} = M_{ij} / \lambda_j, \quad (24)$$

where $1/\lambda_i$ can be found from the system of n linear equations

$$\sum_{j=1}^n M_{ij} (1/\lambda_j) = 1_i. \quad (25)$$

The hyperplane corresponding to the slice generated by \vec{m}_i crosses the coordinate axes at the rational points λ_i . By comparing with the constraint in Eq. (11) we see that the vector components l_i correspond to the inverse of the diagonal elements of the matrix λ

$$l_i = 1/\lambda_i. \quad (26)$$

Note that the relation in Eq. (11) considered as an equation for M , has solutions related by the transformation

$$M \rightarrow G'M, \quad (27)$$

where G' has rational matrix elements satisfying the condition

$$\sum_{j=1}^n G'_{ij} = 1_i. \quad (28)$$

The matrix G' should be chosen in such a way that M remains integer-valued and non-negative. All such matrices produce a group of transmutations corresponding to different choices of the vectors \vec{m}_i on the slice.

Geometrically, the end point of the vector \vec{l} lies on the hyperplane which passes through the points $(1, 0, 0, \dots)$, $(0, 1, 0, \dots)$, $(0, 0, 1, \dots)$, \dots , $(0, 0, 0, \dots, 1)$. The end point of the inversed vector

$$\vec{l}' = \vec{l}/|\vec{l}|^2 \quad (29)$$

lies on the sphere

$$\sum_{i=1}^n |l'_i - \frac{1}{2}|^2 = \frac{n}{4}. \quad (30)$$

The hyperplane corresponding to the slice generated by those vectors \vec{m} satisfying the condition (10) is orthogonal to the vector \vec{l}' and passes through its end point. The intersection of this hyperplane with the sphere (30) is again a sphere of a lower dimension built on the vector $\vec{l}' - \vec{1}$ as on a diameter. On the contrary, for each point \vec{l}' we can construct the above low dimensional sphere belonging to the slice \vec{m} and therefore the developed geometrical picture allows us to relate the reflexive number \vec{k} with the Batyrev polyhedron \vec{m} .

Among the possible sets of vectors \vec{m} for the reflexive polyhedron it is convenient to choose the set of vectors \vec{m}_l ($l = 1, 2, \dots, n$) satisfying conditions (20). The number n_l of such vectors being solutions of the equation $\vec{m}_l \vec{k} = d$ for fixed l is given by the integral

$$n_l = \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{e^{-id\phi} (1 - e^{ik_{n-l}\phi})}{\prod_{r=1}^n (1 - e^{ik_r\phi})} = \int_L \frac{dx}{2\pi i x^{1+d}} \frac{1 - x^{k_{n-l}}}{\prod_{r=1}^n (1 - x^{k_r})}, \quad (31)$$

where the integration contour L is a small circle around the point $x = 0$ drawn anticlockwise. To obtain the above expression for n_l we used the well known representation for the Kronecker symbol

$$\delta_{d, \vec{m} \vec{k}} = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-id\phi} e^{i\vec{m} \vec{k} \phi}. \quad (32)$$

The necessary condition for the reflexivity of the weight vector \vec{k} can be formulated as the set of inequalities

$$n_l \geq 1 \quad (33)$$

valid for all $l = 1, 2, \dots, n$. Of course, it should be combined with the non-degeneracy requirement (13) for the matrix m_{ij} . But even after it the slice could be non-reflexive. The sufficient condition for the reflexivity includes the condition of the existence of such matrix M' (17) constructed from points of the slice which has an eigenvector c_i in the dual space (see (18)).

$$N(\vec{k}) = \sum_{k'_1=1}^{\infty} \dots \sum_{k'_n=1}^{\infty} \prod_{t'=1}^n \int_L \frac{dy_{t'}}{2\pi i y_{t'}^{1+d'}} \prod_{t=1}^n \int_L \frac{dx_t}{2\pi i x_t^{1+d}} \prod_{s=1}^n \prod_{r=1}^n \frac{1}{1 - x_s^{k'_s} y_r^{k'_r}} > 0. \quad (34)$$

The reflexivity of the slice \vec{m} can be verified also by extracting from it the basis \vec{m}_i ($i = 1, 2, \dots, n$) having the maximal value of the determinant

$$T = \max_{\{\vec{m}_i\}} |\det M| \quad (35)$$

and checking the positivity of the coefficients c_k in the expansion (14) of the vector $\vec{1}$. This basis is similar to the basis of simple roots among all roots in the Lie algebras. The vectors \vec{m}_i in this basis are directed almost along the coordinate axes. Below we investigate the so-called simple-laced and quasi-simple laced numbers for which \vec{m}_i have this property.

The total number of solutions of the equation $\vec{m} \cdot \vec{k} = d$, which is equal to the number of moduli $c_{\vec{m}}$ for the Calabi-Yau space, is given by the formula

$$n_d(\vec{k}) = \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{e^{-id\phi}}{\prod_{r=1}^n (1 - e^{ik_r\phi})} = \int_L \frac{dx}{2\pi i} \frac{1}{x^{1+d} \prod_{r=1}^n (1 - x^{k_r})}. \quad (36)$$

All these points except $\vec{m} = \vec{1}$ belong to the polyhedron boundaries \vec{m}_l .

In the case of the higher level Calabi-Yau spaces one should introduce several projective vectors \vec{k}^t ($t = 1, 2, \dots, r$) and construct the intersection of the corresponding slices

$$\vec{m} \vec{k}^t = d^t. \quad (37)$$

In the above geometrical construction it would lead to several points l^t lying on the sphere (30). The hyperplane \vec{m} corresponding to the polyhedron will go through the ends of all vectors \vec{l}^t and will be orthogonal to them. Its intersection with this sphere will be again a sphere with the dimension $n - r$. The obtained polyhedron should satisfy the Batyrev reflexivity property. The number of moduli $c_{\vec{m}}$ for this general case can be obtained from the integral

$$n(\vec{k}^1, \dots, \vec{k}^r) = \int_L \prod_{t=1}^r \frac{dx_t}{2\pi i} \frac{1}{x_t^{1+d^t} \prod_{s=1}^n (1 - \prod_{t=1}^r x_s^{k_r^s})}. \quad (38)$$

The number of the reflexive vectors \vec{k} at the level 1 for the given dimension n and degree d can be expressed in terms of $N(\vec{k})$ (34)

$$N_n(d) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \dots \sum_{k_n=1}^{\infty} \delta_{d, \sum_{r=1}^n k_r} \theta(N(\vec{k})). \quad (39)$$

In the next subsection we develop a physical model based on these relations for Calabi-Yau manifolds.

2.2 Physical interpretation of the Batyrev polynomial

Let us write down Eq. (1) for the Calabi-Yau spaces as the condition

$$\Psi(a_1^+, a_2^+, \dots, a_n^+) = 0 \quad (40)$$

for zeroes of the Schrödinger wave function

$$\Psi(a_1^+, a_2^+, \dots, a_n^+) = \exp(-iEt) \sum_{\vec{m}} c_{\vec{m}} \prod_{r=1}^n (a_r^+)^{m_r} \Psi_0, \quad (41)$$

written for a string-like mechanical system of n harmonic oscillators. Here instead of variables x_r in (1) we introduced the creation operators a_r^+ for the oscillators with frequencies $\omega_r = k_r$ and Ψ_0 is the vacuum state in the diagonal

representation for a_r^+ . The integer components m_r of the vectors \vec{m} coincide with the occupation numbers for these oscillators. The hamiltonian of this quantum mechanical system is

$$H = \sum_{r=1}^n k_r a_r^+ a_r^- . \quad (42)$$

Here n plays the rôle of the dimension D in string theory. The degree $d_k = \sum_{r=1}^n k_r$ corresponds to the total energy of the state

$$d_k = E = \sum_{r=1}^n m_r k_r . \quad (43)$$

In coordinate representation the creation and annihilation operators read

$$a_r^+ = (x_r - \partial_r)/\sqrt{2}, \quad a_r^- = (x_r + \partial_r)/\sqrt{2} \quad (44)$$

respectively, and, therefore, the vacuum state is $\Psi_0 = \exp(-\vec{x}^2/2)$ with $\vec{x}^2 = \sum_{i=1}^n x_i^2$.

Among the degenerate states \vec{m} the simplest one is $\vec{1}$. Other states with the same energy d_k should have at least one occupation number m_l equal to zero, because otherwise $\vec{m} \cdot \vec{k} > d_k$. The reflexivity constraint restricts the number of excited states for the considered harmonic oscillator model. In string theory a similar restriction follows from the Virasoro group. In principle the condition of reflexivity could arise dynamically in a more general theory where oscillator interactions or unharmonic corrections were included in the potential. A similar effect takes place for the fractional quantum Hall effect, where the ground state wave function has a special property: It vanishes as an odd power of small relative distances $x_i - x_j$ [28]. In our model the additional interactions could destroy the degeneracy of the energy levels of the harmonic oscillators leading to the reflexivity constraint for the lowest energy states. Such possibility seems to be related to the fact that the reflexive weights are very special, namely, the number of solutions for the equation $\vec{m} \cdot \vec{k} = d_k$ for reflexive \vec{k} is finite for fixed n in comparison with an infinite number of other vectors. Moreover, the special properties of the reflexive numbers could be connected to a hidden symmetry of the Calabi–Yau spaces appearing in the existence of the Berger graphs which are discussed below.

On the other hand, it would be also interesting to investigate the statistical properties of the above quantum mechanical model for the Batyrev polyhedrons. In particular one can define the micro-canonical ensemble for physical states in which the probability $P(\vec{k})$ to find the mechanical system in the state characterized by the reflexive vector \vec{k} with fixed energy d_k is given by the expression

$$P(\vec{k}) = \frac{1}{n_{d_k}(\vec{k})}, \quad (45)$$

where $n_{d_k}(\vec{k})$ is defined in (38). As usual, we can introduce the thermodynamic potentials for this system and study their physical properties. The investigation of the statistical properties of the physical model related to the Calabi–Yau spaces will be performed in future publications. In the following we focus on a subclass of reflexive Calabi–Yau spaces corresponding to the cases in which the vectors \vec{m}_k are almost collinear with the coordinate axes. Examples of such configurations for simply and quasi-simply-laced polyhedrons are considered below. In particular, it is possible to use recurrent relations for the construction of polyhedrons with increasing dimension n , using the algebraic construction of Ref. [16]. In the next subsection we combine this approach with some geometrical concepts.

2.3 Geometric Relations for Polyhedrons and UCYA

If we assume that all weight vectors \vec{k} for the reflexive slices \vec{m} generated by the vectors \vec{m}_r for all dimensions less than a given number are known, then the Universal Calabi–Yau Algebra (UCYA) [16–18] provides the possibility to calculate similar weight vectors in higher dimensions. We shall illustrate here this method using a simple example of the construction of the $(n + 1)$ -dimensional reflexive polyhedrons containing inside them an n -dimensional reflexive polyhedron.

We introduce the n -dimensional weight vector \vec{k} with dimension $d_k = \sum_{i=1}^n k_i$ and the corresponding polyhedron generated by n integer-valued vectors \vec{m}_r which satisfy the relations

$$\vec{m}_r \cdot \vec{k} = d_k. \quad (46)$$

According to the reflexivity condition the vector $\vec{m} = \vec{1}$ is assumed to be inner, *i.e.*

$$\vec{1} = \sum_{r=1}^n a_r \vec{m}_r, \quad a_r > 0. \quad (47)$$

To generalize this polyhedron to the $(n+1)$ -dimensional space we can add to the components of the vector \vec{m}_r a $(n+1)$ -th component, t , equal to unity

$$\vec{M}_r = (m_{r_1}, m_{r_2}, \dots, m_{r_n}, 1). \quad (48)$$

In this case the constructed polyhedron in the dimension $n+1$ will automatically include inside itself the point $\vec{M} = \vec{1}$ and its reflexivity will follow if in the extended slice there are points \vec{M} with $t = 0$ and $t > 1$.

A weight vector \vec{K} for the polyhedron in the $(n+1)$ -dimensional space should satisfy the equation

$$\vec{M}_r \cdot \vec{K} = d_K \quad (49)$$

for $r = 1, 2, \dots, n+1$, where \vec{M}_{n+1} is a new basis vector. It is natural to choose this vector such that all its components except t are equal to zero, *i.e.*

$$\vec{M}_{n+1} = (0, 0, 0, \dots, 0, \lambda_{n+1}), \quad (50)$$

where $\lambda_{n+1} \geq 2$ is an integer number.

Because the vectors \vec{M}_i ($i = 1, 2, \dots, n$) are known, the first n components of the vectors \vec{k} and \vec{K} should coincide (up to a common factor which can be put equal to unity without loss of generality)

$$k_i = K_i \quad (i = 1, 2, \dots, n). \quad (51)$$

Hence, from the condition that the vector $\vec{1}$ belongs to the constructed slice in the $(n+1)$ -dimensional space, we obtain

$$d_k + K_{n+1} = d_K, \quad (52)$$

where $K_{n+1} = 1, 2, \dots$ is an integer number. This integer number is restricted from above

$$K_{n+1} \leq d_k. \quad (53)$$

Indeed, from Eq. (49) we have that

$$\lambda_{n+1} K_{n+1} = d_K \quad (54)$$

and therefore to obtain the reflexivity for the polyhedron in the $(n+1)$ -dimensional space one should impose the constraint

$$d_K \geq 2 K_{n+1}, \quad (55)$$

because $\lambda_{n+1} \geq 2$, which leads to the inequality $d_k \geq K_{n+1}$ due to Eq. (52).

The above construction is a particular case of UCYA for arity 2. Indeed, in the framework of this method we can take two low-dimensional weight vectors and extend them to the $(n+1)$ -dimensional space in our example as follows

$$\vec{k}^a = (k_1, k_2, \dots, k_n, 0), \quad \vec{k}^b = (0, 0, 0, \dots, 0, 1). \quad (56)$$

It is possible to verify that the intersection of the corresponding slices \vec{m}^a and \vec{m}^b has the property of reflexivity, the reflexive weight vector in the $(n+1)$ -dimensional space can be constructed by taking the following linear combination of \vec{k}^a and \vec{k}^b :

$$\vec{k} = \vec{k}^a + s \vec{k}^b, \quad (57)$$

where $s = 1, 2, \dots$ is an integer number, restricted from above (cf. (53)):

$$s \leq d_{k^a}. \quad (58)$$

In particular, using the UCYA approach, we obtain two weight vectors for $n = 3$,

$$\begin{aligned} \text{a) } (1, 1, 1) &= (1, 1, 0) + (0, 0, 1), \\ \text{b) } (1, 1, 2) &= (1, 1, 0) + 2(0, 0, 1). \end{aligned} \quad (59)$$

The third vector can be constructed by adding two other extended weight vectors,

$$\text{c) } (1, 2, 3) = (1, 0, 1) + 2(0, 1, 1). \quad (60)$$

Here the intersection of the two slices \vec{m}^a and \vec{m}^b consists of the three points $(0, 0, 2)$, $(1, 1, 1)$ and $(2, 2, 0)$ having the reflexivity property.

In the $n = 4$ case starting from the above-mentioned three $n = 3$ vectors we obtain the following weight vectors

$$\begin{aligned} (1, 1, 1, 1) &= a + I, \quad (1, 1, 1, 2) = a + 2I, \quad (1, 1, 1, 3) = a + 3I, \\ (1, 1, 2, 1) &= b + I, \quad (1, 1, 2, 2) = b + 2I, \quad (1, 1, 2, 3) = b + 3I, \quad (1, 1, 2, 4) = b + 4I, \\ (1, 2, 3, 1) &= c + I, \quad (1, 2, 3, 2) = c + 2I, \quad (1, 2, 3, 3) = c + 3I, \\ (1, 2, 3, 4) &= c + 4I, \quad (1, 2, 3, 5) = c + 5I, \quad (1, 2, 3, 6) = c + 6I, \end{aligned} \quad (61)$$

with

$$I = (0, 0, 0, 1), \quad a = (1, 1, 1, 0), \quad b = (1, 1, 2, 0), \quad c = (1, 2, 3, 0). \quad (62)$$

Some of these vectors coincide with others after a transmutation of their components.

It is well-known that using the general UCYA construction for arity 2 we can obtain 90 out of 95 weight vectors for dimension $n = 4$. Other weight vectors can be found using arity 3 and 4. Generally, before adding two or several extended vectors \vec{k}^t ($t = 1, 2, \dots$) with integer coefficients $\vec{k} = \sum_t n_t \vec{k}^t$, one should verify the reflexivity of the slice produced by the solutions of the set of equations $\vec{m} \cdot \vec{k}^t = 0$.

So far we have reviewed the concept of reflexivity in the language of algebra and geometry. In the next subsection we concentrate on arithmetic properties of these numbers making use of some methods common in the study of Feynman diagrams in Quantum Field Theory.

2.4 Simply-Laced Numbers

We will first consider the simplest case with the monomial points $\{\vec{m}_i\}$ satisfying the equation

$$\sum_{r=1}^n \frac{m_r}{s_r} = 1, \quad (63)$$

where s_r are integer numbers obeying the constraint

$$\sum_{r=1}^n \frac{1}{s_r} = 1. \quad (64)$$

Geometrically s_r is the value at which the hyperplane generated by \vec{m}_i crosses the axis r . It is obvious that in the case under consideration the vector $\vec{1}$ is inner in the corresponding polyhedron. The vectors $(1/s_1, 1/s_2, \dots, 1/s_n)$ satisfying the above constraint are known as the ‘‘Egyptian fractions’’ [29]. For the cases $n = 2$ and $n = 3$ all reflexive weight vectors are the ‘‘Egyptian fractions’’:

$$1 = \frac{1}{2} + \frac{1}{2} \quad (65)$$

and

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}. \quad (66)$$

Concerning these decompositions, we note that the plane can be covered by triangles with angles $\alpha = 2\pi/s_1, \beta = 2\pi/s_2, \gamma = 2\pi/s_3$ without mutual overlapping or empty spaces only in these four cases and by strips $a < x < a + \Delta$ in the degenerate case $s_1 = 1, s_2 = s_3 = \infty$.

There are solutions of Eq. (64) with all s_r different. They lead to the polyhedron $\{\vec{m}_k\}$ without any symmetry under the transmutation of m_k . An interesting example of such ‘‘Egyptian fractions’’ is the following decomposition of unity

$$1 = \sum_{k=1}^{r-1} \frac{1}{2^k} + \sum_{k=0}^{r-1} \frac{1}{2^k(2^r - 1)}. \quad (67)$$

In particular, provided that

$$M_r = 2^r - 1 \quad (68)$$

is a prime number $M_r = 3, 7, 31, 127, \dots$ (which can only be for primes $r = 2, 3, 5, 7, \dots$), in the above decomposition $1 = \sum_k 1/s_k$ the integers s_k are all divisors (except of 1) of the degree d being the so-called perfect number

$$d = M_r(M_r + 1)/2. \quad (69)$$

In this case M_r are called ‘‘Mercenna numbers’’ and, according to Euclid and Euler, all even perfect numbers, being the sum of all their divisors d/s_k

$$d = \sum_k \frac{d}{s_k}, \quad (70)$$

can be expressed in terms of Mercenna numbers. Examples of such decomposition are $6=1+2+3$ and $28=1+2+4+7+14$.

In Eq. (61) we found 6 reflexive polyhedrons of the type $\frac{x}{6} + \frac{y}{3} + \frac{z}{2} + \frac{t-1}{\kappa} = 1$ with $\kappa = 1, 6/5, 3/2, 2, 3, 6$ starting from the 3-dimensional polyhedron corresponding to the perfect number $d = 6$ (see (60)). In a similar way one can construct d different $(n + 1)$ -dimensional polyhedrons containing inside them the polyhedrons corresponding to the weight vectors being decompositions of other perfect numbers $d = 28, 496, \dots$ in the sum of all their n divisors. If M_r is not a prime number not all of its divisors enter in the decomposition of d . Note that odd perfect numbers are not known.

The number of ‘‘Egyptian fractions’’ grows rapidly with n (see [29]). To find a recurrence relation for this number one can generalize the decomposition of 1 in Eq. (64) for a general rational number x

$$\sum_{r=1}^n \frac{1}{s_r} = x. \quad (71)$$

We denote the number of such decompositions by $N_n(x)$. For example one can calculate the number of decompositions of unity $N_n(1)$ for several values of n (see [29])

$$N_1(1) = 1, \quad N_2(1) = 1, \quad N_3(1) = 3, \quad N_4(1) = 14, \quad N_5(1) = 147,$$

$$N_6(1) = 3462, \quad N_7(1) = 294314, \quad N_8(1) = 159330691. \quad (72)$$

Let us introduce the symbol $N_n^{(\Lambda)}(x)$ for the number of decompositions of x in the sum of n unit ratios $1/s_r$ (71) satisfying the relations

$$s_r \leq \Lambda, \quad (73)$$

where Λ is a large integer number. One can derive the following recurrent relations for $N_n^{(\Lambda)}(x)$:

$$N_n^{(\Lambda)}(x) = \sum_{t=0}^{\infty} N_{n-t}^{(\Lambda-1)}(x - t/\Lambda) \quad (74)$$

with the following initial conditions

$$N_n^{(0)}(x) = \delta_{x,0}, \quad N_n^{(-k)}(x) = 0 \quad (k = 1, 2, \dots). \quad (75)$$

The generating function for N_n^Λ is given below

$$F_\lambda^{(\Lambda)}(y) = \sum_{n=0}^{\infty} \sum_x \lambda^n y^x N_n^{(\Lambda)}(x) = \prod_{t=1}^{\Lambda} \left(1 - \lambda y^{1/t}\right)^{-1}. \quad (76)$$

The recurrent relation in (74) corresponds to the following equation for this function

$$F_\lambda^{(\Lambda)}(y) = \left(1 - \lambda y^{1/\Lambda}\right)^{-1} F_\lambda^{(\Lambda-1)}(y). \quad (77)$$

Note that $F_\lambda^{(\Lambda)}(y)$ grows rapidly for $\Lambda \rightarrow \infty$, but $N_n^{(\Lambda)}(x)$ tends to a finite limit. For the regularized case the equation $z = F_\lambda^{(\Lambda)}(y)$ defines a Riemann surface with a finite genus.

The inverse relation reads

$$\Phi_n^{(\Lambda)}(y) = \sum_x y^x N_n^{(\Lambda)}(x) = \frac{1}{2\pi i} \int_L \frac{d\lambda}{\lambda^{n+1}} F_\lambda^{(\Lambda)}(y), \quad (78)$$

where a small closed contour of integration, L , is taken anticlockwise around the point $\lambda = 0$.

To express $N_n^{(\Lambda)}(x)$ in terms of $F_\lambda^{(\Lambda)}(y)$ we should perform the additional integration

$$N_n^{(\Lambda)}(x) = \frac{1}{2\pi i m} \int_{l_m} \frac{dy}{y^{1+x}} \Phi_n^{(\Lambda)}(y). \quad (79)$$

Here the closed contour of integration l_m goes m -times around the point $y = 0$ moving through other sheets of the Riemann surface $z = y^{-x} \Phi_n^{(\Lambda)}(y)$. The integer number m is chosen from the condition that the point $y = 0$ on the surface becomes regular in the new coordinate $u = y^{1/m}$.

Let us calculate the asymptotic behavior of $\Phi_n^{(\Lambda)}(y)$ for large n and Λ using the saddle point method. For this purpose we present the generating function in the form

$$\ln F_\lambda^{(\Lambda)}(y) = \ln F_\lambda^{(\Lambda)}(1) + \Delta \ln F_\lambda^{(\Lambda)}(y), \quad (80)$$

where

$$\ln F_\lambda^{(\Lambda)}(1) = -\Lambda \ln(1 - \lambda) \quad (81)$$

and

$$\Delta \ln F_\lambda^{(\Lambda)}(y) = \frac{\lambda}{1 - \lambda} \ln y \sum_{t=1}^n \frac{1}{t} + f_\lambda(y). \quad (82)$$

Here

$$f_\lambda(y) = \sum_{t=1}^{\infty} \left(\ln \frac{1 - \lambda}{1 - \lambda y^{1/t}} - \frac{1}{t} \frac{\lambda}{1 - \lambda} \ln y \right). \quad (83)$$

In the last expression for $f_\lambda(y)$ we pushed Λ to infinity, because the sum over t is convergent.

Now we apply the saddle point method to the calculation of the integral over λ , considering the extremum of the function

$$J_n^{(\Lambda)}(\lambda) = \ln F_\lambda^{(\Lambda)}(1) - n \ln \lambda = \Lambda \ln \frac{1}{1 - \lambda} - n \ln \lambda. \quad (84)$$

From the stationarity condition $\delta J_n^{(\Lambda)}(\lambda) = 0$ one can find the saddle point

$$\tilde{\lambda} = \frac{n}{n + \Lambda} \ll 1 \quad (85)$$

and therefore, with quadratic accuracy in $\delta\lambda = \lambda - \tilde{\lambda}$, we obtain for this function

$$J_n^{(\Lambda)}(\lambda) = \Lambda \ln \frac{n+\Lambda}{\Lambda} + n \ln \frac{n+\Lambda}{n} + \frac{(\delta\lambda)^2}{2} \frac{(n+\Lambda)^3}{n\Lambda} + \dots \quad (86)$$

It is obvious that the contour of integration over $\delta\lambda$ goes through the saddle point in a correct direction parallel to the imaginary axis. Thus, we obtain for $\Phi_n(y)$ the following expression at large n after calculating the Gaussian integral over $\delta\lambda$

$$\Phi_n^{(\Lambda)}(y) = \left(\frac{n+\Lambda}{\Lambda}\right)^\Lambda \left(\frac{n+\Lambda}{n}\right)^n \sqrt{\frac{\Lambda}{2\pi n(n+\Lambda)}} e^{\Delta \ln F_{\tilde{\lambda}}^{(\Lambda)}(y)}. \quad (87)$$

We substituted λ by its saddle point value $\tilde{\lambda}$ in the slowly changing function $\Delta \ln F_{\tilde{\lambda}}^{(\Lambda)}(y)$.

The most interesting case, when $\tilde{\lambda} \ll 1$, is considered in the following. In this limit $f_{\tilde{\lambda}}(y) = 0$ and the result is significantly simplified

$$\Phi_n^{(\Lambda)}(y) = \left(\frac{n+\Lambda}{\Lambda}\right)^\Lambda \left(\frac{n+\Lambda}{n}\right)^n \sqrt{\frac{\Lambda}{2\pi n(n+\Lambda)}} y^x, \quad (88)$$

where $x = \frac{\tilde{\lambda}}{1-\tilde{\lambda}} \sum_{t=1}^{\Lambda} \frac{1}{t}$. As $\sum_{t=1}^{\Lambda} \frac{1}{t} \approx \ln \Lambda - \Psi(1) + \frac{1}{2\Lambda} + \dots$, where $\gamma = -\Psi(1)$ is the Euler constant, we obtain that $N_n(x)$ has a maximum

$$N_n^{(\Lambda)}(x_m) \approx \frac{(n+\Lambda)^n}{n!} \quad (89)$$

at

$$x_m = \frac{n}{\Lambda} \left(\ln \Lambda - \Psi(1) + \frac{1}{2\Lambda} \right). \quad (90)$$

For larger x the above saddle point method should be modified.

The simple-laced numbers in dimension $n = 5$ are constructed in Appendix B. We investigate also their structure in terms of the UCYA construction.

It turns out that when going beyond dimension $n = 3$ not all of the reflexive weight numbers are simply-laced. A large number of them has a different structure in the sense that some of their components k_r are not divisors of the degree d_k . These are what we called non-simply-laced numbers. The simplest case is when each component k_r can be converted in a divisor of the difference of the degree d_k and another component $k_{r'}$. The next subsection is devoted to the study of this class of numbers called ‘‘quasi-simply-laced’’.

2.5 Classification of Quasi-Simply-Laced Numbers

Quasi-simply-laced numbers are important generalizations of the simply-laced ones. For example, in dimension $n = 4$ all 95 polyhedra with a single reflexive vector belong to this class (among them 14 are obtained from the simply-laced numbers). A simple example is the vector $\vec{k} = (1, 2, 3, 5)$ [11], corresponding to the following decomposition of unity in the sum of ratios l_i

$$1 = \frac{1}{11} + \frac{2}{11} + \frac{3}{11} + \frac{5}{11}.$$

For this vector $d_k = 11$ and $d_k/k_1 = 11$, $(d_k - k_1)/k_2 = 5$, $(d_k - k_2)/k_3 = 3$ and $(d_k - k_1)/k_4 = 2$.

Hence we can generalize the diagonal ansatz for the matrix M_{ij} in the case of the simply-laced weight vectors, assuming that the vector \vec{l} satisfies the set of equations

$$s_i l_i + l_{i'} = 1 \quad (91)$$

for $i = 1, 2, \dots, n$ and $i' = i'(i)$ is also one of these numbers. Here s_i are positive integer parameters which will be later chosen from the condition that the vector \vec{l} belongs to the slice of \vec{m} :

$$\sum_{i=1}^n l_i = 1 \quad (92)$$

and the corresponding point in the slice is inner in accordance with the property of reflexivity. In this section we will classify all sets of equations for quasi-simply-laced numbers in such a way that the sets obtained by a transmutation of indices are considered as belonging to the same class.

For this purpose we introduce a diagrammatic representation where the indices i and i' appearing in Eq. (91) are connected by a line with an arrow directed from i to i' . For each different class of sets of equations there is only one “Feynman diagram” related to the function $i'(i)$ in Eq. (91). These Feynman diagrams can be obtained from the “functional” integral $Z(\lambda)$ with the “action” L

$$Z(\lambda) = \int \frac{dx dy}{\pi} e^{-L}, \quad L = |z|^2 - \lambda z^* e^z, \quad z = x + iy \quad (93)$$

by expanding it in the “coupling constant” λ :

$$Z(\lambda) = \sum_{n=0}^{\infty} \lambda^n Z_n, \quad Z_n = \sum_r \frac{1}{G_r}. \quad (94)$$

Here r enumerates different Feynman diagrams in the n^{th} -order of perturbation theory (corresponding to different classes of sets of the equations shown above) and G_r is the number of group elements of the symmetry for the diagram r under permutations of the index i . In agreement with Eq. (91) these diagrams contain all possible vertices V_r ($r = 0, 1, 2, \dots$) in which r particles are absorbed by the field z and only one particle is emitted by the field z^* .

Using the above expression for $Z(\lambda)$ we obtain in the n -th order

$$Z_n = \int \frac{dx dy}{\pi} e^{-|z|^2} \frac{z^{*n}}{n!} e^{nz}. \quad (95)$$

Therefore the number of diagrams of order n weighted with the symmetry factors $1/G_r$ equals

$$Z_n = \sum_r \frac{1}{G_r} = \frac{n^n}{n!}. \quad (96)$$

It is natural to expect that at large n the saddle point configuration for the Feynman diagrams corresponds to an almost constant averaged symmetry factor $\frac{1}{\bar{G}(n)}$ for a subgroup of the permutation group

$$\frac{1}{\bar{G}(n)} = \frac{\sum_r 1/G_r}{\sum_r 1}. \quad (97)$$

In this case the number of different classes of solutions does not grow very rapidly at large n

$$\sum_r 1 \approx \bar{G}(n) \frac{e^n}{\sqrt{2\pi n}} \quad (98)$$

in comparison with the total number of Calabi—Yau spaces. Let us consider the Feynman diagrams for $n = 2, 3, 4, 5, \dots$. For $n = 2$ there is one disconnected and two connected diagrams (see Fig. 1, where the symmetry weights G_r are also indicated).

The corresponding sets of equations are

$$a) \quad s_1 l_1 + l_1 = 1, \quad s_2 l_2 + l_2 = 1, \quad (99)$$

$$b) \quad s_1 l_1 + l_1 = 1, \quad s_2 l_2 + l_1 = 1, \quad (100)$$

$$c) \quad s_1 l_1 + l_2 = 1, \quad s_2 l_2 + l_1 = 1. \quad (101)$$

One can verify in this case the fulfillment of the relation

$$Z_2 = \frac{1}{2!} + 1 + \frac{1}{2!} = \frac{2^2}{2!} = 2.$$

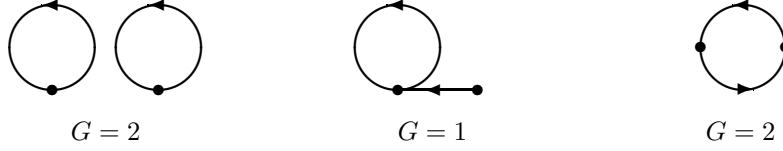


Figure 1: Diagrams for $n = 2$

For $n = 3$ we have 7 different Feynman diagrams and 7 different sets of equations, respectively

$$1) \quad s_1 l_1 + l_1 = 1, \quad s_2 l_2 + l_2 = 1, \quad s_3 l_3 + l_3 = 1; \quad G = 3!; \quad (102)$$

$$2) \quad s_1 l_1 + l_1 = 1, \quad s_2 l_2 + l_2 = 1, \quad s_3 l_3 + l_2 = 1; \quad G = 1; \quad (103)$$

$$3) \quad s_1 l_1 + l_1 = 1, \quad s_2 l_2 + l_1 = 1, \quad s_3 l_3 + l_2 = 1; \quad G = 1; \quad (104)$$

$$4) \quad s_1 l_1 + l_1 = 1, \quad s_2 l_2 + l_1 = 1, \quad s_3 l_3 + l_1 = 1; \quad G = 2!; \quad (105)$$

$$5) \quad s_1 l_1 + l_1 = 1, \quad s_2 l_2 + l_3 = 1, \quad s_3 l_3 + l_2 = 1; \quad G = 2!; \quad (106)$$

$$6) \quad s_1 l_1 + l_3 = 1, \quad s_2 l_2 + l_1 = 1, \quad s_3 l_3 + l_2 = 1; \quad G = 3; \quad (107)$$

$$7) \quad s_1 l_1 + l_2 = 1, \quad s_2 l_2 + l_3 = 1, \quad s_3 l_3 + l_2 = 1; \quad G = 1. \quad (108)$$

The number of diagrams weighted with their symmetry factors is

$$Z_3 = \frac{1}{6} + 1 + 1 + \frac{1}{2!} + \frac{1}{2!} + \frac{1}{3} + 1 = \frac{3^3}{3!} = \frac{9}{2},$$

which agrees with the above relation for Z_n .

For $n = 4$ there are 19 different Feynman diagrams and

$$Z_4 = \frac{4^4}{4!} = \frac{32}{3}.$$

For $n = 5, 6$ and 7 there are respectively 47, 130 and 342 different Feynman diagrams with the corresponding values of Z_n

$$Z_5 = \frac{5^5}{5!} = \frac{625}{24}, \quad Z_6 = \frac{6^6}{6!} = \frac{324}{5}, \quad Z_7 = \frac{7^7}{7!} = \frac{117649}{720}.$$

It is possible to calculate the number of the corresponding diagrams for larger values of n (see [30]). It turns out that the averaged symmetry \overline{G} of the Feynman diagrams grows approximately linearly from $\overline{G} = 1$ for $n = 1$ up to $\overline{G}(n) \approx 10,7$ for $n = 27$.

Looking at these Feynman diagrams we can see that their connected parts contain only one loop. In this way the quasi-classical approximation for the “functional” integral should be exact:

$$Z(\lambda) = \int \frac{dx dy}{\pi} e^{-L} = \frac{1}{1 - z(\lambda)}, \quad (109)$$

where $z(\lambda)$ is the solution of the classical equation $\delta L = 0$:

$$z(\lambda) = \lambda e^{z(\lambda)}. \quad (110)$$

Indeed, solving this equation with the use of perturbation theory we obtain

$$Z(\lambda) = \sum_{n=0}^{\infty} \lambda^n \frac{n^n}{n!}, \quad (111)$$

corresponding to

$$z(\lambda) = 1 - \frac{1}{\sum_{n=0}^{\infty} \lambda^n \frac{n^n}{n!}} = \lambda + \lambda^2 + \frac{3}{2}\lambda^3 + \dots \quad (112)$$

One can obtain a more detailed description of the Feynman diagrams in terms of the number of vertices V_r with a different number $r + 1$ of lines. For this case we should consider the more general action

$$L = |z|^2 - z^* \sum_{r=0}^{\infty} g_r \frac{z^r}{r!}, \quad (113)$$

where g_r are corresponding coupling constants. Here we also obtain that the quasi-classical result is exact

$$Z = \int \frac{dx dy}{\pi} e^{-L} = \frac{1}{1-a}, \quad (114)$$

where a is the solution of the classical equation

$$a = \sum_{r=1}^{\infty} g_r \frac{a^{r-1}}{(r-1)!} \quad (115)$$

and the perturbative expansion for Z reads

$$Z = \sum_{r_0=0}^{\infty} \frac{g_0^{r_0}}{r_0!} \sum_{r_1=0}^{\infty} \frac{g_1^{r_1}}{r_1!(1!)^{r_1}} \dots \sum_{r_{\infty}=0}^{\infty} \frac{g_{\infty}^{r_{\infty}} (\sum_{k=0}^{\infty} r_k)!}{r_{\infty}!(\infty!)^{r_{\infty}}} \delta \left(\sum_{k=0}^{\infty} (k-1)r_k \right). \quad (116)$$

The coefficient in front of the product of $g_k^{r_k}$ coincides with the number of Feynman diagrams (with symmetry factors) having r_k vertices with $k + 1$ lines for each $k = 0, 1, 2, \dots$. At large orders $n = \sum_{k=0}^{\infty} r_k \gg 1$ of perturbation theory there exists a saddle point

$$\tilde{r}_k = n \frac{e^{-1}}{k!}. \quad (117)$$

in the sums over r_k .

In Appendix A we illustrate this classification of the quasi-simply-laced reflexive weight vectors in the case $n = 4$. Here among 19 types of such numbers in 2 cases the reflexivity condition for the polyhedrons is not fulfilled. Moreover, using the freedom to extract from the slice \vec{m} different sets of the vectors \vec{m}_i in the slice \vec{m} we can restrict ourselves to a smaller number of possibilities $k < 17$ corresponding to the above Feynman diagrams containing only closed loops.

After the above analysis of reflexive vectors in particular cases of the Egyptian and quasi-Egyptian fractions with the use of the methods of the Quantum Field Theory we turn now to a more algebraic approach by associating to them some matrices similar to those appearing in the case of the Cartan-Lie algebras.

3 Simply-Laced Numbers as Generators of Berger Graphs

In this Section we investigate the Berger graphs for the simply-laced numbers considered initially in Refs. [24–26]. These graphs are related to the CY_d spaces of the first level described by a single reflexive weight vector. In the case of small d they coincide with the Dynkin diagrams for the roots of the Cartan-Lie algebras A_r , D_r , E_6 , E_7 and E_8 . In the Dynkin diagrams for such root systems the nodes are connected by single lines. Consequently, their Cartan matrices turn out to be symmetric. The Berger matrices for the simply-laced case are also symmetric.

Similar to the Cartan-Lie case the Berger graph for a simply-laced reflexive vector \vec{k} is built by assigning the degree d_k to the central node as its Coxeter label. This will be the maximal Coxeter label in the graph. The number of legs attached to this central node coincides with the dimension n of the vector. In each leg the number of nodes is $d_k/k_i - 1$,

with k_i being the corresponding component of \vec{k} . The Coxeter labels of these nodes are $d_k - k_i, d_k - 2k_i, \dots, k_i$. They decrease along the leg starting from the central node. For example in the primary graph of $(1, 1, 2)[4]$ for dimension 3 there are three legs, the central node has the Coxeter label 4 and in the first and second legs there are additional nodes with the Coxeter labels 3, 2, 1. In the third leg the Coxeter label of the additional node is 2.

The associated Berger matrix B_{ij} for \vec{k} is built from scalar products (α_i, α_j) of the root vectors $\vec{\alpha}_l$ assigned to each node. The scalar product of two vectors of the nodes connected by a line is -1 . Disconnected nodes correspond to orthogonal vectors. The diagonal matrix element B_{ii} is the square of the vector assigned to the corresponding node. For all nodes B_{ii} is 2 with the exception of the central node, where the root square is equal to $n - 1$. The determinant of such a matrix is zero, which is a generalization of the similar result for the affine simply-laced Cartan-Lie matrices. The Coxeter labels assigned to the nodes in a Berger graph coincide with the coefficient c_l in front of the corresponding root in the linear combination of the eigenvector $\sum_l c_l \alpha_l$ corresponding to a vanishing eigenvalue of the Berger matrix. As an example, the primary graph for $\vec{k} = (1, 1)[2]$ has the central node with its Coxeter label equal 2 and $B_{ii} = 1$. Further, each of its two legs has one node with its Coxeter label equal to 1 and $B_{ii} = 2$.

The maximal Coxeter labels for the reflexive simply-laced vectors $(1), (1, 1), (1, 1, 1), (1, 1, 2), (1, 2, 3)$ are 1, 2, 3, 4, 6, respectively, which coincides with the maximal Coxeter labels for the corresponding simply-laced Lie algebras, A_r, D_r, E_6, E_7 and E_8 . In more detail, we have respectively for these primary graphs: one node of A -type with Coxeter label (1), three nodes of D -type with Coxeter labels (1, 2, 1) (one chain), seven nodes of E_6 -type with Coxeter labels (1, 2; 1, 2; 1, 2; 3) (three chains), eight nodes of E_7 -type with Coxeter labels (1, 2, 3; 1, 2, 3; 2; 4) (three chains) and nine nodes of E_8 -type with Coxeter labels (1, 2, 3, 4, 5; 2, 4; 3; 6) (three chains).

It is important to note that the primary graphs for the above cases are generators of generalized Berger graphs in CY_d polyhedra. Namely, such Berger graphs can be built from one or several blocks of the primary graphs. These blocks are connected by the lines appearing in the Cartan graphs for the A_i -series. Namely, on each line there are several nodes with the same Coxeter label equal to the Coxeter label of two nodes to which these lines are attached. Furthermore, the Coxeter labels for all nodes and the matrix elements of the Berger matrix B_{ij} inside each block are universal and coincide with those for the corresponding elementary Cartan graph. Only the square of the root corresponding to the node with the attached lines is changed by adding to it the number l of these lines, i.e., $B_{ii} = 2 \rightarrow B_{ii} = 2 + l$ [24].

Each of the reflexive vectors \vec{k} with n components can be extended to extra dimensions $n \rightarrow p + n$ by adding to it several vanishing components: $\vec{k}_{p+n}^{\text{ext}} = (0, 0, \dots, 0; k_1, \dots, k_n)$. These extended vectors participate in the UCYA r -arity construction leading to new reflexive vectors in higher dimensions. The structure of the Berger graphs for the polyhedrons obtained by this method depends on the number p of zero components for the corresponding extended vectors.

The UCYA r -arity construction can be used to build new Calabi-Yau polyhedra (at level one) as it was discussed in the previous section. In particular, to go from the vectors $(1), (1, 1), (1, 1, 1), (1, 1, 2), (1, 2, 3)$ to $n = 4$ dimensions one should take the extended vectors $(0, 0, 0, 1), (0, 0, 1, 1), (0, 1, 1, 1), (0, 1, 1, 2), (0, 1, 2, 3)$ and those obtained by permutations of their components. Then in the framework of the 2-arity approach we can compose the linear combinations of two of these numbers with integer coefficients. For each pair one should verify that the intersection of two polyhedra corresponding to two extended vectors has the reflexivity property. This condition is fulfilled in the case of the polyhedron corresponding to the eldest vector $\vec{k}_1 + \vec{k}_2$. In the K_3 reflexive polyhedron for each eldest vector one can find the primary graphs of the $A_r^{(1)}, D_r^{(1)}, E_6^{(1)}, E_7^{(1)}$ and $E_8^{(1)}$ types. They are situated at two opposite sides of the polyhedron divided by the above intersection. A similar situation takes place with the reflexive K_3 polyhedron corresponding to the sum of the vectors \vec{k}_1 and \vec{k}_2 with integer coefficients. For each case in the constructed polyhedron one can find generalized graphs corresponding to the primary graphs.

When the 2-arity construction is used for the $n = 5$ dimensional case (CY_3) then the structure of the corresponding graphs becomes more complicated although they are also built from the previous order graphs. The difference is that some of the Berger matrices corresponding to the generalized Dynkin graphs can have $B_{ii} = 3$ instead of the usual $B_{ii} = 2$. This is seen in Fig. 2, where the links between the previous Berger subgraphs belong to the A_l type with modified Coxeter labels $(1, 2, 3, \dots)$ instead of 1). It is remarkable that these five $n = 5$ Berger graphs shown on Fig. 2 generate five infinite series because their structure holds for any l in A_l . This procedure can be generalized by

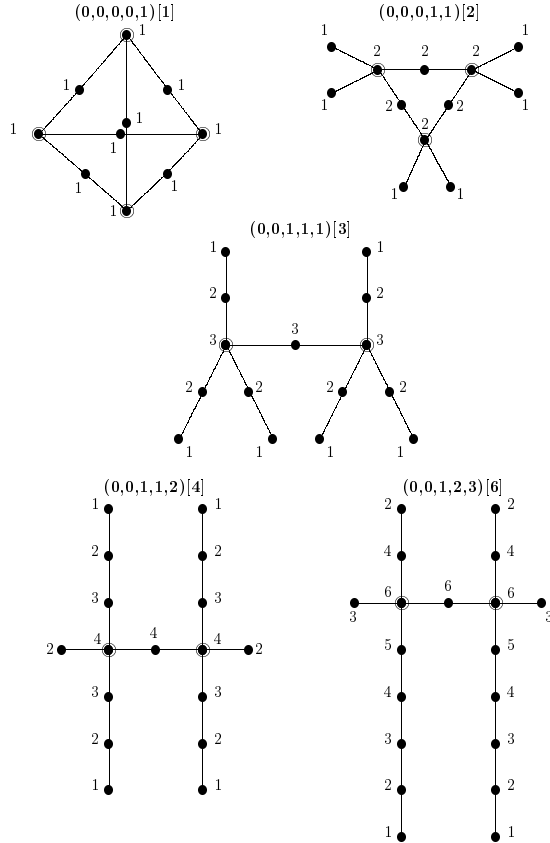


Figure 2: Berger diagrams from a 2-arity construction in the $n = 5$ dimensional case, with the corresponding Coxeter labels at the nodes. They generate five infinite series.

linking with A_l -lines ($B_{ii} = 2$) not only a pair of triple nodes from corresponding primary graphs but also other nodes sharing the same Coxeter label. When this happens the diagonal element $A_{ii} = 2$ in the Cartan matrices is substituted by the matrix element $B_{ii} = 3$. The determinants of the non-affine Berger matrices $(0,0,1,1,1)[3]$, $(0,0,1,1,2)[4]$ and $(0,0,1,2,3)[6]$ are equal to 3^4 , 4^3 and 6^2 independently of the number of nodes along the internal line connecting two primary graphs (see Fig. 2 and table 2). Note that the labels of all the nodes along this line coincide with those of the central nodes of two connected primary graphs, i.e., they are equal to 3, 4 and 6 respectively. Thus, these graphs produce three infinite series analogous to the graphs of the D_r -series generated by the extended reflexive number $(0, 0, 1, 1)$ in the $K3$ polyhedrons. In addition to the infinite series of the Dynkin graphs of A_r and D_r with maximal Coxeter numbers 1 and 2 there appear three new series with maximal Coxeter numbers 3, 4 and 6. This construction of the infinite series of the Berger graphs could lead to a possible generalization of the notion of the direct product of Lie algebras, e.g., $E_8 \times E_8$ in the heterotic string.

Probably the various Berger graphs obtained by the UCYA construction should be placed in the same class. All new graphs contain several initial diagrams joined by different numbers of the A_i -lines. The result depends on the arity 2, 3... in UCYA and on the dimension of the constructed polyhedron. The reflexive polyhedra allow us to build large classes of graphs, among which the set of usual Dynkin diagrams, related to the binary operations, is only a small one, because the Dynkin diagrams and their affine generalizations are in one-to-one correspondence with the well known Cartan-Lie algebras and infinite-dimensional Kac-Moody algebras, respectively. It is then natural to think that the Berger graphs could lead to algebras beyond the Cartan-Lie / Kac-Moody construction and, in particular, could be related to ternary, quaternary, ... generalizations of binary algebras.

To extend the class of Dynkin diagrams we generalize the rules for affine Cartan matrices [24]. Namely the Berger matrices satisfy the following rules:

$$\begin{aligned} \mathbb{B}_{ii} &= 2, \text{ or } 3, \text{ or } 4, \dots, \mathbb{B}_{ij} \leq 0, \mathbb{B}_{ij} = 0 \mapsto \mathbb{B}_{ji} = 0, \\ \mathbb{B}_{ij} &\in \mathbb{Z}, \text{ Det } \mathbb{B} = 0, \text{ Det } \mathbb{B}_{\{(i)\}} > 0. \end{aligned} \quad (118)$$

The constraint of a vanishing determinant is a generalization of the ‘‘affine condition’’ for Kac–Moody algebras. In the above new rules we relax the restriction on the diagonal element $\mathbb{B}_{ii} = 2$, *i.e.*, to satisfy the affine condition we allow for \mathbb{B}_{ii} to be larger: $\mathbb{B}_{ii} = 3, 4, \dots$. These large values can appear in the lattice of reflexive polyhedra starting from CY_3, CY_4, \dots . Below we shall check the coincidence of the graph’s labels indicated on figures with the Coxeter labels obtained from the eigenvalues of the Berger matrices. The proposed prescriptions for the Coxeter labels are universal for all Berger graphs independently from their dimension and arity construction. These prescriptions generalize the Cartan and Kac–Moody rules in a natural way. Note that the value of the diagonal term \mathbb{A}_{ii} of the Cartan matrix can be related to one of the Casimir invariants of the simple Lie algebras. The number of these invariants is equal to the algebra rank. All Cartan–Lie algebras contain the Casimir operator of degree 2, but there are other invariant operators. For example, the exceptional E_6, E_7, E_8 algebras have the following degrees of Casimir invariants:

$$E_6 : \{2, 5, 6, 8, 9, 12\}, E_7 : \{2, 6, 8, 10, 12, 14, 18\}, E_8 : \{2, 8, 12, 14, 18, 20, 24, 30\}. \quad (119)$$

By subtracting 1 from the Casimir operator degree we obtain the so–called ‘‘Coxeter exponents’’ of the corresponding Lie algebra. One can see that the largest degree of the Casimir invariants in this list, 12, 18 and 30, is equal to the Coxeter number of the E_6, E_7 and E_8 algebras, respectively. We note that the diagonal element of the Cartan matrices corresponding to each node on the (extended) Dynkin diagram is always equal to 2 and can be calculated through the Coxeter labels surrounding this node. Let us consider, for example, the A_r series. For each internal node N_i of the corresponding Dynkin diagram the value of the diagonal Cartan element A_{ii} satisfies the relation: $A_{ii} = (C_{i-1} + C_{i+1})/C_i = 2$, where $C_{\dots} = 1$ are Coxeter labels of the nodes N_{i-1}, N_i, N_{i+1} . To generalize this relation to boundary nodes one should consider the extended Dynkin graph of the affine $A_r^{(1)}$ algebra. In this case all nodes are linked by lines with two neighbours. In the Dynkin graphs of the $D_r^{(1)}$ series apart from several nodes with two lines there are two nodes with three lines. The above relation between the Coxeter labels and the diagonal Cartan element can be easily checked for nodes with two and three lines. In particular, for a triple node N_i one obtains $A_{ii} = (C_{i+} + C_{i-1} + C_{i-})C_i = (1 + 2 + 1)/2 = 2$. For this rule to be also valid for a boundary node one can formally add an additional node with a vanishing Coxeter label. The above two examples show that the Coxeter labels 1 or 2 allow one to construct an infinite series of Dynkin diagrams. The larger values $A_{ii} \geq 3$ for triple nodes are allowed only for some special values of the algebra rank, as we can see in the cases of E_6, E_7, E_8 algebras.

The extended reflexive number $(0, 0, 0, 0, 1)$ is the origin of the infinite series of the Berger graphs with a multi–cycle topology. One can compare them with the Kac–Moody case of the $A_r^{(1)}$ infinite series where the graphs have only one cycle. The simplest example of multi–cycle topology corresponds to the tetrahedron Berger graph having 4 closed cycles with the corresponding 4×4 matrix:

$$B_3^{(1)}(00001) = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

This matrix has the eigenvalues $\{0, 4, 4, 4\}$ for the corresponding eigenvectors $(1, 1, 1, 1), (-1, 0, 0, 1), (-1, 0, 1, 0), (-1, 1, 0, 0)$. The Coxeter labels are given by the zero eigenvector $\{1, 1, 1, 1\}$, which provides the well–known relation for the highest root (affine condition): $1 \cdot \alpha_0 + 1 \cdot \alpha_1 + 1 \cdot \alpha_2 + 1 \cdot \alpha_3 = 0$, where $\alpha_i, i = 1, 2, 3$ are the simple roots and $-\alpha_0$ is equal to the highest root α_h . For the non–affine case one should remove one node from the Berger graph. Thus the relation between the affine and non–affine Berger graphs is similar to the relation between the Cartan–Lie and Kac–Moody graphs. In the last case the Cartan–Lie algebra produces the so–called horizontal subalgebra of the Kac–Moody algebra where the highest root participates in the construction of the additional simple root, more exactly,

$-\alpha_h$. Such non-affine Berger matrix is positively defined and $\text{Det } B_3^{(1)}(00001) = 16$. The affine condition and the positive definiteness of the non-affine Berger matrix do not depend on the number of internal nodes with $B_{ii} = 2$. For instance, the non-affine Berger matrix after its inversion offers the possibility to construct three \mathbb{Z}_3 -symmetric fundamental weights:

$$\Lambda_1 = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \right\}, \quad \Lambda_2 = \left\{ \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right\}, \quad \Lambda_3 = \left\{ \frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right\}. \quad (120)$$

Note that in the Cartan-Lie A_r algebras there are two elementary fundamental representations, but in the Berger case one already has three elementary fundamental “representations”. In our example we obtain the \mathbb{Z}_3 symmetry acting in the space of the fundamental representations. One can believe that this new invariance could have some applications to the solution of the family generation problem in the electro-weak theory.

Another example is related to the generalization of the D_r -infinite series of the Cartan-Lie Dynkin graphs. From our point of view the B(011) graph is exceptional. The graphs B(0011) in the K3 2-arity polyhedra produce an infinite series of D_r -Cartan-Lie Dynkin graphs. They are built from two (1, 1)[2] blocks connected by a segment with l -internal nodes having $B_{ii} = 2$ and Coxeter label equal to 2. The Berger graphs B(00011) could have three blocks (1, 1)[2]. Each two of them are connected by a line with the nodes having $B_{ii} = 2$ and the Coxeter labels equal to 2. We illustrate this by the following Berger matrix:

$$B_8^{(1)}(00011) = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & 3 \end{pmatrix}$$

For simplicity we did not include any of the internal nodes. The eigenvalues of this matrix are $\{0, 2, 2, 2, 3, 3 - \sqrt{3}, 3 - \sqrt{3}, 3 + \sqrt{3}, 3 + \sqrt{3}\}$ and the eigenvector with the vanishing eigenvalue is $\{1, 1, 2, 1, 1, 2, 1, 1, 2\}$. To discuss the infinite B(00011) series let us recall the usual $D_r^{(1)}$ Dynkin graphs which are described by the B(0011) graphs in the K3 polyhedra. For example we consider the diagram $D_8^{(1)} = B_8^{(1)}(0011)$ with three internal nodes:

$$B_8^{(1)}(0011) = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

The determinant of this matrix is equal to zero. The affine condition reads $1 \cdot \alpha_0 + 1 \cdot \alpha_1 + 2 \cdot \alpha_2 + 2 \cdot \sum_{i=3}^{i=5} \alpha_i + 2 \cdot \alpha_6 + 1 \cdot \alpha_7 + 1 \cdot \alpha_8 = 0$, with $-\alpha_0$ being the highest root of the D_8 Cartan-Lie algebra. This condition relates the $D_8^{(1)}$ Kac-Moody algebra to the non-affine case of the Cartan-Lie D_8 algebra. One can construct the Cartan matrix

corresponding to this algebra

$$B_8(0011) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

The determinant of this matrix is equal to 4 independently of the internal nodes. The determinant of the non-affine Berger matrix $\text{Det } B(00011)$ in the above example is equal to 48 and depends on the internal nodes. In this case one can also find the fundamental nodes:

$$G_8(00011) = \left(\begin{array}{c|ccc|ccc|ccc} F.W. & \alpha_{a1} & \alpha_{a2} & \alpha_{a3} & \alpha_{a4} & \alpha_{a5} & \alpha_{a6} & \alpha_{a7} & \alpha_{a8} \\ \hline \Lambda_{a1} & 1 & 1 & 1/2 & 1/2 & 1 & 1/2 & 1/2 & 1 \\ \Lambda_{a2} & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \\ \Lambda_{a3} & 1/2 & 1 & 7/6 & 2/3 & 4/3 & 7/12 & 7/12 & 7/6 \\ \Lambda_{a4} & 1/2 & 1 & 2/3 & 7/6 & 4/3 & 7/12 & 7/12 & 7/6 \\ \Lambda_{a5} & 1 & 2 & 4/3 & 4/3 & 8/3 & 7/6 & 7/6 & 7/3 \\ \Lambda_{a6} & 1/2 & 1 & 7/12 & 7/12 & 7/6 & 7/6 & 2/3 & 4/3 \\ \Lambda_{a7} & 1/2 & 1 & 7/12 & 7/12 & 7/6 & 2/3 & 7/6 & 4/3 \\ \Lambda_{a8} & 1 & 2 & 7/6 & 7/6 & 7/3 & 4/3 & 4/3 & 8/3 \end{array} \right)$$

The $B(0011)$ graphs help us to clarify the structure of the Berger graphs determined by the matrices $B(00111)$, $B(00112)$, $B(00123)$ in CY_3 reflexive polyhedra of 2-arity. In the case of $K3$ the $B(0111)$ graph generates one exceptional $E_6^{(1)}$ graph. In the higher dimension $n = 5$ we obtain the graph from an infinite series, constructed from two $E_6^{(1)}$ blocks in which we should change two nodes $A_{ii} = 2 \rightarrow B_{ii} = 3$.

Now we consider the case of the $B(00111)$ Berger graph with one internal node b having the Coxeter label 3 placed between two generalized forms of $E_6^{(1)}$ exceptional graphs having the central nodes $B_{ii} = a$:

$$B_{14}^{(1)}(00111) = \left(\begin{array}{cccccccc|c|cccccccc} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & a & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & b & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & a & -1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{array} \right)$$

with the determinant $\text{Det } B_{14}(00111) = 27^2(a-2)[b(a-2)-2]$. For the choice of parameters $a = 3$ and $b = 2$, the eigenvalues of this Berger matrix read $\{0, 1, 1, 1, 1, 2, 3, 3, 3, 3, 4, 2 - \sqrt{3 - \sqrt{7}}, 2 + \sqrt{3 - \sqrt{7}}, 2 - \sqrt{3 + \sqrt{7}}, 2 + \sqrt{3 + \sqrt{7}}\}$. The eigenvector with zero eigenvalue corresponds to the Coxeter labels: $C_i = \{1, 2, 1, 2, 1, 2, 3, : 3, : 3, 2, 1, 2, 1, 2, 1\}$, which are similar to those for the $E_6^{(1)}$ -diagrams. The link between two parts of the graph can be extended by an arbitrary number of internal nodes with Coxeter labels 3. This choice of labels is supported by

the construction of the corresponding reflexive polyhedron in CY_3 . A different selection of parameters: $a = 2, b = 2$, generates unusual Coxeter labels: $C_i = \{-1, -2, -1, -2, -1, -2, -3, : 0, : 3, 2, 1, 2, 1, 2, 1\}$, with Coxeter number equal to 0. We would like to stress the fact that the solution $a = 3$ and $b_l = 2$ exists for any $l = 0, 1, 2, 3, \dots$, and gives an infinite series of corresponding Berger graphs, $B_l^{(1)}(00111)[3]$.

To obtain non-affine Berger graphs, *i.e.* the analog of the Cartan-Lie case, one should remove one root with the Coxeter label equal to one. This means that the simple roots α_i on the Berger graph define the highest root $\alpha_h = -\alpha_0$ (the affine condition), *i.e.*, $\alpha_0 + \sum_i C_i \cdot \alpha_i = 0$, where C_i are the Coxeter labels. In this case one can check that the determinant of the Berger matrix is equal to 81, a value which does not depend on the number l of the internal nodes. Also, all principal minors are positive-defined in a similar way to the Cartan-Lie case. In a complete analogy with the Cartan case the Berger non-affine graph also defines the fundamental weights which read

$$G_{15}(00111) =$$

F.W.	α_{a1}	α_{a2}	α_{a3}	α_{a4}	α_{a5}	α_{a6}	α_c	α_{b6}	α_{b5}	α_{b4}	α_{b3}	α_{b2}	α_{b1}	α_{b0}
Λ_{a1}	2	1	2	1	2	3	3	3	2	1	2	1	2	1
Λ_{a2}	1	4/3	5/3	2/3	4/3	2	2	2	4/3	2/3	4/3	2/3	4/3	2/3
Λ_{a3}	2	5/3	10/3	4/3	8/3	4	4	4	8/3	4/3	8/3	4/3	8/3	4/3
Λ_{a4}	1	2/3	4/3	4/3	5/3	2	2	2	4/3	2/3	4/3	2/3	4/3	2/3
Λ_{a5}	2	4/3	8/3	5/3	10/3	4	4	4	8/3	4/3	8/3	4/3	8/3	4/3
Λ_{a6}	3	2	4	2	4	6	6	6	4	2	4	2	4	2
Λ_c	3	2	4	2	4	6	7	7	14/3	7/3	14/3	7/3	14/3	7/3
Λ_{b6}	3	2	4	2	4	6	7	8	16/3	8/3	16/3	8/3	16/3	8/3
Λ_{b5}	2	4/3	8/3	4/3	8/3	4	14/3	16/3	38/9	19/9	32/9	16/9	32/9	16/9
Λ_{b4}	1	2/3	4/3	2/3	4/3	2	7/3	8/3	19/9	14/9	16/9	8/9	16/9	8/9
Λ_{b3}	2	4/3	8/3	4/3	8/3	4	14/3	16/3	32/9	16/9	38/9	19/9	32/9	16/9
Λ_{b2}	1	2/3	4/3	2/3	4/3	2	7/3	8/3	16/9	8/9	19/9	14/9	16/9	8/9
Λ_{b1}	2	4/3	8/3	4/3	8/3	4	14/3	16/3	32/9	16/9	32/9	16/9	38/9	19/9
Λ_{b0}	1	2/3	4/3	2/3	4/3	2	7/3	8/3	16/9	8/9	16/9	8/9	19/9	14/9

The Dynkin diagrams for the Cartan-Lie/Kac-Moody algebras can have the nodes with the maximal number of edges equal to 3. For instance, let us take the $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ graphs and consider the vertex-nodes having three edges and the Coxeter labels equal to 3, 4 and 6, respectively. It is known that in the case of the Cartan-Lie algebras the number of Casimir invariants coincides with the algebra rank. The r -degrees of these Casimir take values from 2 up to the maximum equal to the Coxeter number. The important cases correspond to the degrees of invariants for the three above-mentioned algebras equal to the following sums: $Cas_i = C_{i-1} + C_{i+} + C_{i-} = 2 + 2 + 2 = 6$, $Cas = 3 + 3 + 2 = 8$, $Cas = 5 + 4 + 3 = 12$, respectively. The diagonal elements of the Cartan matrices for the nodes in these cases are equal to $A_{ii} = Cas/C_{ii} = 6/3 = 2$, $A_{ii} = Cas/C_{ii} = 8/4 = 2$, $A_{ii} = Cas/C_{ii} = 12/6 = 2$. The relation between the Coxeter labels and B_{ii} is also valid for all nodes with 2, 3, 4, ... edges. Here the diagonal elements of the Berger matrix are $B_{ii} = 3, 4, \dots$

The extension of the $A_r^{(1)}$ series for the $B_{ii} = 3$ case gives a new infinite series $B(00001)$ in which for all triple nodes N_i we have $B_{ii} = (C_{i-1} + C_{i+} + C_{i-})/C_i = 3$, where all $C_{...} = 1$. Hence one can obtain the infinite series of graphs both for the Cartan nodes $A_{ii} = 2$ and for the Berger nodes with $B_{ii} = 3$. Note that all Coxeter labels for the Berger graphs $B(0...01)$ are equal to 1.

A similar extension of the $D_r^{(1)}$ series is $B(00011)$, where apart from the Cartan nodes $A_{ii} = 2$ there appear two nodes $B_{ii} = 3$ with 4 edges. Here one has $B_{ii} = (C_{i-1} + C_{i+1} + C_{i+} + C_{i-})/C_i = (2 + 2 + 1 + 1)/2 = 3$. Note that for the Berger graphs the maximal Coxeter label is equal to 2. Therefore for all Berger graphs we have only two possibilities for Coxeter labels, 1 or 2. In the Cartan-Lie case one obtains just two infinite series of simply-laced Dynkin diagrams, with the maximal Coxeter labels equal to 1 and 2. For other examples of simply-laced Cartan-Lie algebras their maximal values are 3, 4 and 6. This is related to the fact that the corresponding algebras are exceptional.

Apart from the five types of infinite series, which can be interpreted as generalizations of the corresponding Cartan-Lie simply-laced graphs, we also found 14 exceptional completely new graphs (see Fig. 3) corresponding to 14 simply-laced numbers inside the 95 K_3 reflexive numbers as shown in Table 2. As it was mentioned above, the affine

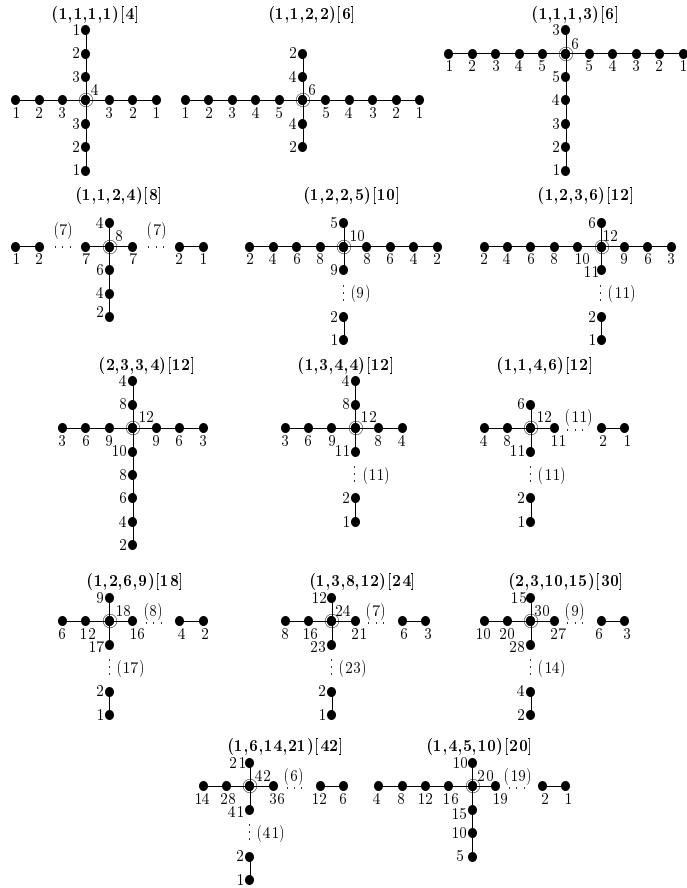


Figure 3: 14 exceptional new Berger diagrams with the Coxeter labels at the nodes.

graphs have symmetric Berger matrices with their determinant equal to zero, which is similar to the Kac–Moody type of infinite-dimensional algebras. Removing one node with a minimal Coxeter label we can obtain the non-affine graph generalizing the Cartan–Lie case. For the corresponding Berger matrix the determinant is positively defined and all principal minors are also positive, see Table 2.

We see that for the Berger graphs one can build the infinite series with the maximal Coxeter labels 3, 4, 6 due to the presence of new nodes with $B_{ii} = 3$. These new nodes lead to the appearance of 14 exceptional simply-laced Berger graphs with their maximal Coxeter labels: 4, 6, ..., 42. When we introduce the new nodes $B_{ii} = 4$ these 14 exceptional examples produce new 14 infinite series of Berger graphs. This could be a key point to understand the nature of exceptional cases in Cartan–Lie and Berger algebras.

The Cartan graphs of dimension 1, 2, 3 are well-known and correspond to the classical Cartan–Lie algebras. The main purpose of the above discussion was to enlarge their list with graphs generated by the reflexive weight vectors of dimension four (corresponding to CY_3). In this dimension, corresponding to $K3$ -sliced CY_3 spaces, we have singled out the following fourteen reflexive weight vectors from the total of 95 $K3$ -vectors $(1,1,1,1)[4]$, $(1,1,2,2)[6]$, $(1,1,1,3)[6]$, $(1,1,2,4)[8]$, $(2,3,3,4)[12]$, $(1,3,4,4)[12]$, $(1,2,3,6)[12]$, $(1,2,2,5)[10]$, $(1,4,5,10)[20]$, $(1,1,4,6)[12]$, $(1,2,6,9)[18]$, $(1,3,8,12)[24]$, $(2,3,10,15)[30]$ and $(1,6,14,21)[42]$. The Coxeter labels for the nodes in the Berger graphs were assigned consistently from the geometrical and algebraic points of view. These Berger matrices have simple properties: they are symmetric and affine. In addition, these graphs and matrices are not extendable, i.e. other graphs and Berger matrices cannot be obtained from them by adding more nodes to any of the legs. In this respect these graphs are “exceptional”. Similarly to classical non-exceptional graphs, one can construct infinite series containing them. Apparently these fourteen vectors are the only set of vectors with symmetric Berger matrices among the total 95 reflexive vectors. We investigate such Berger matrix in the simplest case in Appendix C.

$w_{3,4}$	Rank	n	Casimir (E_n)	Determinant
$(0, 1, 1, 1)[3]$	$6(E_6)$	12	6	3
$(0, 1, 1, 2)[4]$	$7(E_7)$	18	8	2
$(0, 1, 2, 3)[6]$	$8(E_8)$	30	12	1
$(0, 0, 1, 1, 1)[3]$	$2_3 + 10 + l$	$18 + 3(l + 1)$	9	3^4
$(0, 0, 1, 1, 2)[4]$	$2_3 + 13 + l$	$32 + 4(l + 1)$	12	4^3
$(0, 0, 1, 2, 3)[6]$	$2_3 + 15l$	$60 + 6(l - 1)$	18	6^2
$(0, 1, 1, 1, 1)[4]$	$1_3 + 11$	28	12	16
$(0, 2, 3, 3, 4)[12]$	$1_3 + 12$	90	36	8
$(0, 1, 1, 2, 2)[6]$	$1_3 + 13$	48	18	9
$(0, 1, 1, 1, 3)[6]$	$1_3 + 15$	54	18	12
$(0, 1, 1, 2, 4)[8]$	$1_3 + 17$	80	24	8
$(0, 1, 2, 2, 5)[10]$	$1_3 + 17$	100	30	5
$(0, 1, 3, 4, 4)[12]$	$1_3 + 17$	120	36	3
$(0, 1, 2, 3, 6)[12]$	$1_3 + 19$	132	36	6
$(0, 1, 4, 5, 10)[20]$	$1_3 + 26$	290	60	2
$(0, 1, 1, 4, 6)[12]$	$1_3 + 24$	162	36	6
$(0, 1, 2, 6, 9)[18]$	$1_3 + 27$	270	54	3
$(0, 1, 3, 8, 12)[24]$	$1_3 + 32$	420	72	2
$(0, 2, 3, 10, 15)[30]$	$1_3 + 25$	420	90	4
$(0, 1, 6, 14, 21)[42]$	$1_3 + 49$	1092	126	1

Table 2: Rank, Coxeter number h , Casimir depending on B_{ii} and determinants for the non-affine exceptional Berger graphs. The maximal Coxeter labels coincide with the degree of the corresponding reflexive simply-laced vector. The determinants in the last column for the infinite series $(0,0,1,1,1)[3]$, $(0,0,1,1,2)[4]$ and $(0,0,1,2,3)[6]$ are independent from the number l of internal binary $B_{ii} = 2$ nodes. The numbers 1_3 and 2_3 denote the number of nodes with $B_{ii} = 3$.

4 Conclusions, outlook and further possibilities

In this work we have investigated different properties of the weight vectors and graphs related to the CY reflexive polyhedra. The important correspondence between $CY_{3,4}$ spaces and reflexive polyhedra was discovered several years ago by Batyrev [13]. In previous works [16, 24–26] an interesting relation between reflexive vectors and Dynkin and Berger graphs with dimensions $n = 4, 5, \dots$ was established. It was shown with the use of the Universal Calabi–Yau Algebra (UCYA) and the arity construction, that the structure of the reflexive polyhedra can be described in terms of the Berger graphs.

In this paper we studied the reflexivity properties of the Batyrev polyhedra and the UCYA construction using some geometrical and algebraic ideas. The reflexivity condition for a polyhedron corresponds to the existence of one or several dual polyhedra. This condition was formulated as a positivity requirement for an expression depending on the corresponding weight vector. A physical interpretation of the Calabi–Yau spaces in terms of zeroes of the wave function for a string-like harmonic oscillator model was presented. The UCYA approach was illustrated in the problem of constructing the reflexive polyhedra containing the given polyhedron inside it. We extracted large classes of reflexive polyhedra based on the simple-laced and quasi-simple-laced numbers and investigated their properties with the use of number theory, recurrence relations and functional methods of quantum field theory. In particular the simple-laced reflexive vectors were related to the so-called "Egyptian fractions" [29]. We suggested a classification of quasi-simple-laced numbers. Our general approach was illustrated by numerous examples (see Appendices).

We proceeded to study the relation between the reflexive vectors and the structure of the Berger graphs. It was demonstrated that the simply-laced reflexive numbers are generators of the n -dimensional Berger graphs. The well-known Dynkin diagrams for the Cartan–Lie algebras produce a subclass of the full set of the Berger graphs. We assigned the Coxeter labels and weights B_{ii} for the nodes in these graphs. The suggested prescriptions for the corresponding Berger matrices for arbitrary n in agreement with the rules for the Cartan matrices A_{ij} . The constraints for the Cartan case are well known [31–35]:

$$A_{ii} = 2, A_{ij} \leq 0 (i \neq j), A_{ij} = 0 \leftrightarrow A_{ji} = 0, A_{ij} \in \mathbb{Z}, \text{Det } A > 0.$$

Thus, the rank of A is equal to r . For the Kac–Moody case one should modify only the last condition for the determinant A . Indeed, by neglecting the positivity requirement for $\text{Det } A$ one obtains a class of the Kac–Moody algebras. Interesting subclasses of Kac–Moody algebras can be constructed if this restriction on the determinant is

replaced as follows

$$\text{Det } A_i > 0, \quad i = 0, 1, 2, \dots, r, \quad (121)$$

where A_i are the matrices in which the i^{th} row and the i^{th} column are removed. The determinants of A_i are called the principal minors. In a general case the rank of A can be arbitrary, but provided that the new restriction is imposed its rank is r or $r + 1$. For the Cartan–Lie algebras the rank is equal to $r + 1$. For the affine Lie case the Cartan matrices have rank r and all principal minors are positive. In the Berger case we impose the same restrictions on the determinant and principal minors. Then the Berger matrix turns out to be a degenerated semi-definite matrix which is called the affine matrix. We modify the first condition for A_{ij} and allow the diagonal element B_{ii} to be all positive integers, i.e., 2, 3, 4, ...

Summing up the above discussion, we conclude that one can construct not only diagrams for the Cartan (Det $B > 0$) and affine (Det $B = 0$, Det $B_i > 0$) cases, but it is possible to obtain some information from the generalized Berger graphs also about the roots and weights for extended algebraic structures. In particular one can generalize the known simply-laced series A_r and D_r and exceptional simply-laced algebras $E_{6,7,8}$ and $E_{6,7,8}^{(1)}$. The Berger matrices for the new simply-laced graphs in dimensions $n = 4, 5, \dots$ share a number of properties with the Cartan matrices in dimensions $n = 1, 2, 3$.

The interest in the construction of new algebraic structures beyond the Lie algebras started from the investigation of $SU(2)$ -conformal field theories [9,10,36,37] (see also [38]). One can expect that geometrical concepts, in particular, algebraic geometry, are a natural and promising way to discover new algebras. Historically the marriage of algebra and geometry was useful for both branches of mathematics. In particular, to prove the mirror symmetry of the Calabi–Yau spaces the powerful technique of the Newton reflexive polyhedra was used. Furthermore, the ADE -type singularities [39–42] in $K3 = CY_2$ spaces [21,23,43] and their resolution were related to the Dynkin diagrams for the Cartan–Lie algebra. One can formulate these relations in the following way:

1. The algebraic origin of the Cartan–Lie algebras is the Torus and $SU(2)/U(1)$.
2. The geometrical origin of the Cartan–Lie algebras is S^1 and $CP^1 \cong S^2$.

It is possible to continue this correspondence to the Berger graphs

1. The algebraic origin of the Berger graphs is the Torus and $SU(3)/SU(2) \times U(1)$.
2. The geometrical origin of the Berger graphs is S^1 and CP^2 .

Another possibility to understand the origin of the Berger graphs is related to the resolution of the quotient singularities. The Calabi–Yau spaces are defined by their holonomy groups [1,44,45]. Typical quotient singularities C^n/G are characterized by the list of finite subgroups $G \subset H$ of the holonomy groups, $H = SU(2), SU(3), \dots$. For the case of the $SU(2)$ holonomy group there are five well-known Klein–Du–Val singularities of $A-D-E$ type [39–42]. And the crepant resolution of $A-D-E$ -types singularities in $K3$ gives us the corresponding Dynkin diagrams in the $K3$ polyhedra. So it is natural to relate the Berger graphs in CY_3 polyhedra to the resolution of the C^3/G quotient singularities, where G are finite subgroups of $SU(3)$ [46]. Although this relation is not established, we note that the number of finite subgroups of $SU(3)$ is $5 + 12$ (see [46]). Further, the first five finite subgroups are isomorphic to the groups $SU(2)$ and one can guess that they could be the origin of our five Berger graphs $B(00001), B(00011), B(00111), B(00112), B(00123)$ discussed in this paper.

As a final remark, one can assume that the Berger graphs could be linked to new algebras which are realized in quadratic or/and cubic matrices [11]. The main question at this point is how to unify in one approach the Cartan–Lie algebras and new hypothetical (ternary) algebras [6–12].

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A Quasi–Simply–Laced Reflexive Weight Vectors for $n = 4$

To find all possible weight vectors in the $n = 4$ case we firstly construct all expansions of the form

$$d = k_1 + k_2 + k_3 + k_4 \quad (122)$$

which fulfill the simply–laced condition

$$\frac{d}{k_1} = s_1, \quad \frac{d}{k_2} = s_2, \quad \frac{d}{k_3} = s_3, \quad \frac{d}{k_4} = s_4, \quad (123)$$

with $s_1, s_2, s_3, s_4 > 1$ being integer numbers. For this purpose one can use a computer code based on the recurrent relation (74) for the number $N_n(x, 1/s)$ of decompositions of the rational number $x > 0$ in the ratios $1/s_k$.

To search for all possible quasi–simply–laced numbers it is helpful to apply the above classification of the classes of sets of equations. Let us consider in the following the different cases where one, two, three or four numerators of the above ratios are modified.

For one modified numerator there is only one possibility (from now on $s_i \in \mathbb{Z}^+$ and $s_i > 1$)

$$\frac{d}{k_1} = s_1, \quad \frac{d - k_1}{k_2} = s_2, \quad \frac{d}{k_3} = s_3, \quad \frac{d}{k_4} = s_4. \quad (124)$$

For two modified numerators there are four classes

$$\frac{d}{k_1} = s_1, \quad \frac{d - k_1}{k_2} = s_2, \quad \frac{d - k_1}{k_3} = s_3, \quad \frac{d}{k_4} = s_4, \quad (125)$$

$$\frac{d}{k_1} = s_1, \quad \frac{d - k_1}{k_2} = s_2, \quad \frac{d - k_4}{k_3} = s_3, \quad \frac{d}{k_4} = s_4, \quad (126)$$

$$\frac{d}{k_1} = s_1, \quad \frac{d - k_1}{k_2} = s_2, \quad \frac{d - k_2}{k_3} = s_3, \quad \frac{d}{k_4} = s_4, \quad (127)$$

$$\frac{d}{k_1} = s_1, \quad \frac{d - k_3}{k_2} = s_2, \quad \frac{d - k_2}{k_3} = s_3, \quad \frac{d}{k_4} = s_4. \quad (128)$$

In the case of three modified numerators seven classes exist

$$\frac{d}{k_1} = s_1, \quad \frac{d - k_1}{k_2} = s_2, \quad \frac{d - k_1}{k_3} = s_3, \quad \frac{d - k_1}{k_4} = s_4, \quad (129)$$

$$\frac{d}{k_1} = s_1, \quad \frac{d - k_3}{k_2} = s_2, \quad \frac{d - k_1}{k_3} = s_3, \quad \frac{d - k_1}{k_4} = s_4, \quad (130)$$

$$\frac{d}{k_1} = s_1, \quad \frac{d - k_3}{k_2} = s_2, \quad \frac{d - k_2}{k_3} = s_3, \quad \frac{d - k_1}{k_4} = s_4, \quad (131)$$

$$\frac{d}{k_1} = s_1, \quad \frac{d - k_3}{k_2} = s_2, \quad \frac{d - k_4}{k_3} = s_3, \quad \frac{d - k_1}{k_4} = s_4, \quad (132)$$

$$\frac{d}{k_1} = s_1, \quad \frac{d - k_1}{k_2} = s_2, \quad \frac{d - k_2}{k_3} = s_3, \quad \frac{d - k_2}{k_4} = s_4, \quad (133)$$

$$\frac{d}{k_1} = s_1, \quad \frac{d-k_3}{k_2} = s_2, \quad \frac{d-k_2}{k_3} = s_3, \quad \frac{d-k_2}{k_4} = s_4, \quad (134)$$

$$\frac{d}{k_1} = s_1, \quad \frac{d-k_3}{k_2} = s_2, \quad \frac{d-k_4}{k_3} = s_3, \quad \frac{d-k_2}{k_4} = s_4. \quad (135)$$

And, finally, for four numerators six classes should be considered

$$\frac{d-k_2}{k_1} = s_1, \quad \frac{d-k_1}{k_2} = s_2, \quad \frac{d-k_1}{k_3} = s_3, \quad \frac{d-k_1}{k_4} = s_4, \quad (136)$$

$$\frac{d-k_2}{k_1} = s_1, \quad \frac{d-k_3}{k_2} = s_2, \quad \frac{d-k_1}{k_3} = s_3, \quad \frac{d-k_1}{k_4} = s_4, \quad (137)$$

$$\frac{d-k_3}{k_1} = s_1, \quad \frac{d-k_3}{k_2} = s_2, \quad \frac{d-k_1}{k_3} = s_3, \quad \frac{d-k_1}{k_4} = s_4, \quad (138)$$

$$\frac{d-k_4}{k_1} = s_1, \quad \frac{d-k_3}{k_2} = s_2, \quad \frac{d-k_1}{k_3} = s_3, \quad \frac{d-k_1}{k_4} = s_4, \quad (139)$$

$$\frac{d-k_4}{k_1} = s_1, \quad \frac{d-k_3}{k_2} = s_2, \quad \frac{d-k_2}{k_3} = s_3, \quad \frac{d-k_1}{k_4} = s_4, \quad (140)$$

$$\frac{d-k_2}{k_1} = s_1, \quad \frac{d-k_3}{k_2} = s_2, \quad \frac{d-k_4}{k_3} = s_3, \quad \frac{d-k_1}{k_4} = s_4. \quad (141)$$

For the simply-laced case in Eq. (123) we have 14 solutions

$$\begin{aligned} 1 &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}, & 1 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}, & 1 &= \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6}, \\ 1 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}, & 1 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{12}, & 1 &= \frac{1}{2} + \frac{1}{5} + \frac{1}{5} + \frac{1}{10}, \\ 1 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}, & 1 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18}, & 1 &= \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{12}, \\ 1 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{24}, & 1 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20}, & 1 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42}, \\ 1 &= \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6}, & 1 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{15}. \end{aligned} \quad (142)$$

In the class corresponding to Eq. (124) there are 37 new solutions:

$$\begin{aligned} 1 &= \frac{1}{3} + \frac{2}{9} + \frac{1}{3} + \frac{1}{9}, & 1 &= \frac{1}{3} + \frac{2}{15} + \frac{1}{2} + \frac{1}{30}, & 1 &= \frac{1}{3} + \frac{2}{15} + \frac{1}{3} + \frac{1}{5}, \\ 1 &= \frac{1}{3} + \frac{2}{21} + \frac{1}{2} + \frac{1}{14}, & 1 &= \frac{1}{4} + \frac{3}{8} + \frac{1}{3} + \frac{1}{24}, & 1 &= \frac{1}{4} + \frac{3}{8} + \frac{1}{4} + \frac{1}{8}, \\ 1 &= \frac{1}{4} + \frac{3}{16} + \frac{1}{2} + \frac{1}{16}, & 1 &= \frac{1}{4} + \frac{3}{20} + \frac{1}{2} + \frac{1}{10}, & 1 &= \frac{1}{4} + \frac{3}{28} + \frac{1}{2} + \frac{1}{7}, \\ 1 &= \frac{1}{5} + \frac{2}{5} + \frac{1}{3} + \frac{1}{15}, & 1 &= \frac{1}{5} + \frac{2}{5} + \frac{1}{5} + \frac{1}{5}, & 1 &= \frac{1}{5} + \frac{4}{15} + \frac{1}{2} + \frac{1}{30}, \\ 1 &= \frac{1}{5} + \frac{4}{15} + \frac{1}{3} + \frac{1}{5}, & 1 &= \frac{1}{5} + \frac{2}{15} + \frac{1}{2} + \frac{1}{6}, & 1 &= \frac{1}{10} + \frac{9}{20} + \frac{1}{5} + \frac{1}{4}, \\ 1 &= \frac{1}{6} + \frac{5}{12} + \frac{1}{3} + \frac{1}{12}, & 1 &= \frac{1}{6} + \frac{5}{12} + \frac{1}{4} + \frac{1}{6}, & 1 &= \frac{1}{6} + \frac{5}{18} + \frac{1}{2} + \frac{1}{18}, \\ 1 &= \frac{1}{6} + \frac{5}{24} + \frac{1}{2} + \frac{1}{8}, & 1 &= \frac{1}{7} + \frac{2}{7} + \frac{1}{2} + \frac{1}{14}, & 1 &= \frac{1}{7} + \frac{3}{14} + \frac{1}{2} + \frac{1}{7}, \\ 1 &= \frac{1}{8} + \frac{7}{24} + \frac{1}{2} + \frac{1}{12}, & 1 &= \frac{1}{8} + \frac{7}{24} + \frac{1}{3} + \frac{1}{4}, & 1 &= \frac{1}{8} + \frac{7}{40} + \frac{1}{2} + \frac{1}{5}, \end{aligned}$$

$$\begin{aligned}
1 &= \frac{1}{9} + \frac{4}{9} + \frac{1}{3} + \frac{1}{9}, & 1 &= \frac{1}{9} + \frac{2}{9} + \frac{1}{2} + \frac{1}{6}, & 1 &= \frac{1}{10} + \frac{3}{10} + \frac{1}{2} + \frac{1}{10}, \\
1 &= \frac{1}{11} + \frac{5}{66} + \frac{1}{2} + \frac{1}{3}, & 1 &= \frac{1}{12} + \frac{11}{24} + \frac{1}{3} + \frac{1}{8}, & 1 &= \frac{1}{12} + \frac{11}{36} + \frac{1}{2} + \frac{1}{9}, \\
1 &= \frac{1}{15} + \frac{7}{30} + \frac{1}{2} + \frac{1}{5}, & 1 &= \frac{1}{16} + \frac{5}{16} + \frac{1}{2} + \frac{1}{8}, & 1 &= \frac{1}{16} + \frac{5}{48} + \frac{1}{2} + \frac{1}{3}, \\
1 &= \frac{1}{21} + \frac{10}{21} + \frac{1}{3} + \frac{1}{7}, & 1 &= \frac{1}{21} + \frac{5}{42} + \frac{1}{2} + \frac{1}{3}, & 1 &= \frac{1}{28} + \frac{9}{28} + \frac{1}{2} + \frac{1}{7}, \\
1 &= \frac{1}{36} + \frac{5}{36} + \frac{1}{2} + \frac{1}{3}.
\end{aligned} \tag{143}$$

In the classes shown in Eqs. (125)–(128) it is possible to find 32 additional solutions

$$\begin{aligned}
1 &= \frac{1}{7} + \frac{1}{7} + \frac{2}{7} + \frac{3}{7}, & 1 &= \frac{1}{10} + \frac{1}{5} + \frac{3}{10} + \frac{2}{5}, & 1 &= \frac{1}{15} + \frac{2}{15} + \frac{1}{3} + \frac{7}{15}, \\
1 &= \frac{1}{15} + \frac{1}{5} + \frac{4}{15} + \frac{7}{15}, & 1 &= \frac{1}{22} + \frac{3}{22} + \frac{7}{22} + \frac{1}{2}, & 1 &= \frac{1}{16} + \frac{1}{4} + \frac{5}{16} + \frac{3}{8}, \\
1 &= \frac{1}{18} + \frac{2}{9} + \frac{1}{3} + \frac{7}{18}, & 1 &= \frac{1}{22} + \frac{2}{11} + \frac{3}{11} + \frac{1}{2}, & 1 &= \frac{1}{21} + \frac{5}{21} + \frac{1}{3} + \frac{8}{21}, \\
1 &= \frac{1}{26} + \frac{5}{26} + \frac{7}{26} + \frac{1}{2}, & 1 &= \frac{1}{8} + \frac{3}{16} + \frac{1}{4} + \frac{7}{16}, & 1 &= \frac{1}{9} + \frac{1}{6} + \frac{5}{18} + \frac{4}{9}, \\
1 &= \frac{2}{21} + \frac{1}{7} + \frac{1}{3} + \frac{3}{7}, & 1 &= \frac{1}{13} + \frac{3}{26} + \frac{4}{13} + \frac{1}{2}, & 1 &= \frac{1}{11} + \frac{2}{11} + \frac{5}{22} + \frac{1}{2}, \\
1 &= \frac{1}{10} + \frac{1}{4} + \frac{3}{10} + \frac{7}{20}, & 1 &= \frac{1}{13} + \frac{5}{26} + \frac{3}{13} + \frac{1}{2}, & 1 &= \frac{1}{16} + \frac{5}{32} + \frac{9}{32} + \frac{1}{2}, \\
1 &= \frac{1}{6} + \frac{2}{9} + \frac{5}{18} + \frac{1}{3}, & 1 &= \frac{3}{20} + \frac{1}{5} + \frac{1}{4} + \frac{2}{5}, & 1 &= \frac{1}{8} + \frac{1}{6} + \frac{7}{24} + \frac{5}{12}, \\
1 &= \frac{1}{10} + \frac{2}{15} + \frac{1}{3} + \frac{13}{30}, & 1 &= \frac{1}{7} + \frac{5}{21} + \frac{2}{7} + \frac{1}{3}, & 1 &= \frac{1}{11} + \frac{5}{33} + \frac{1}{3} + \frac{14}{33}, \\
1 &= \frac{1}{6} + \frac{5}{24} + \frac{1}{4} + \frac{3}{8}, & 1 &= \frac{1}{8} + \frac{5}{32} + \frac{7}{32} + \frac{1}{2}, & 1 &= \frac{1}{11} + \frac{5}{44} + \frac{13}{44} + \frac{1}{2}, \\
1 &= \frac{2}{27} + \frac{5}{54} + \frac{1}{3} + \frac{1}{2}, & 1 &= \frac{1}{7} + \frac{3}{14} + \frac{1}{4} + \frac{11}{28}, & 1 &= \frac{1}{6} + \frac{1}{5} + \frac{4}{15} + \frac{11}{30}, \\
1 &= \frac{7}{50} + \frac{4}{25} + \frac{1}{5} + \frac{1}{2}, & 1 &= \frac{7}{36} + \frac{2}{9} + \frac{1}{4} + \frac{1}{3}.
\end{aligned} \tag{144}$$

In the sets of Eqs. (130)–(135) there are 10 new solutions, namely

$$\begin{aligned}
1 &= \frac{1}{11} + \frac{2}{11} + \frac{3}{11} + \frac{5}{11}, & 1 &= \frac{1}{13} + \frac{3}{13} + \frac{4}{13} + \frac{5}{13}, & 1 &= \frac{1}{7} + \frac{3}{14} + \frac{2}{7} + \frac{5}{14}, \\
1 &= \frac{2}{27} + \frac{5}{27} + \frac{1}{3} + \frac{11}{27}, & 1 &= \frac{3}{34} + \frac{2}{17} + \frac{5}{17} + \frac{1}{2}, & 1 &= \frac{3}{38} + \frac{5}{38} + \frac{11}{38} + \frac{1}{2}, \\
1 &= \frac{4}{25} + \frac{1}{5} + \frac{7}{25} + \frac{9}{25}, & 1 &= \frac{2}{17} + \frac{3}{17} + \frac{7}{34} + \frac{1}{2}, & 1 &= \frac{5}{27} + \frac{2}{9} + \frac{7}{27} + \frac{1}{3}, \\
1 &= \frac{5}{38} + \frac{3}{19} + \frac{4}{19} + \frac{1}{2}.
\end{aligned} \tag{145}$$

For the last case in Eqs. (136)–(141) we only have 2 new solutions

$$1 = \frac{2}{17} + \frac{3}{17} + \frac{5}{17} + \frac{7}{17}, \quad 1 = \frac{3}{19} + \frac{4}{19} + \frac{5}{19} + \frac{7}{19}. \tag{146}$$

The total number of quasi–simply–laced numbers is therefore 95, which is in agreement with the known number of Calabi–Yau spaces with one reflexive vector.

Matrix	Equation	Matrix	Equation
$\begin{vmatrix} \tilde{s}_1 & 0 & 0 & 0 \\ 1 & s_2 & 0 & 0 \\ 1 & 0 & s_3 & 0 \\ 1 & 0 & 0 & s_4 \end{vmatrix}$	(129)	$\begin{vmatrix} \tilde{s}_1 & 0 & 0 & 0 \\ 0 & \tilde{s}_2 & 0 & 0 \\ 0 & 0 & s_3 & 1 \\ 0 & 0 & 1 & s_4 \end{vmatrix}$	(128)
$\begin{vmatrix} \tilde{s}_1 & 0 & 0 & 0 \\ 1 & s_2 & 0 & 0 \\ 1 & 0 & s_3 & 0 \\ 0 & 1 & 0 & s_4 \end{vmatrix}$	(130)	$\begin{vmatrix} \tilde{s}_1 & 0 & 0 & 0 \\ 0 & s_2 & 1 & 0 \\ 0 & 1 & s_3 & 0 \\ 0 & 1 & 0 & s_4 \end{vmatrix}$	(134)
$\begin{vmatrix} \tilde{s}_1 & 0 & 0 & 0 \\ 1 & s_2 & 0 & 0 \\ 1 & 0 & s_3 & 0 \\ 0 & 0 & 0 & \tilde{s}_4 \end{vmatrix}$	(125)	$\begin{vmatrix} \tilde{s}_1 & 0 & 0 & 0 \\ 0 & s_2 & 1 & 0 \\ 0 & 0 & s_3 & 1 \\ 0 & 1 & 0 & s_4 \end{vmatrix}$	(135)
$\begin{vmatrix} \tilde{s}_1 & 0 & 0 & 0 \\ 1 & s_2 & 0 & 0 \\ 0 & 1 & s_3 & 0 \\ 0 & 1 & 0 & s_4 \end{vmatrix}$	(133)	$\begin{vmatrix} s_1 & 1 & 0 & 0 \\ 1 & s_2 & 0 & 0 \\ 1 & 0 & s_3 & 0 \\ 1 & 0 & 0 & s_4 \end{vmatrix}$	(136)
$\begin{vmatrix} \tilde{s}_1 & 0 & 0 & 0 \\ 1 & s_2 & 0 & 0 \\ 0 & 1 & s_3 & 0 \\ 0 & 0 & 1 & s_4 \end{vmatrix}$	(132)	$\begin{vmatrix} s_1 & 1 & 0 & 0 \\ 1 & s_2 & 0 & 0 \\ 1 & 0 & s_3 & 0 \\ 0 & 1 & 0 & s_4 \end{vmatrix}$	(138)
$\begin{vmatrix} \tilde{s}_1 & 0 & 0 & 0 \\ 1 & s_2 & 0 & 0 \\ 0 & 1 & s_3 & 0 \\ 0 & 0 & 0 & \tilde{s}_4 \end{vmatrix}$	(127)	$\begin{vmatrix} s_1 & 1 & 0 & 0 \\ 1 & s_2 & 0 & 0 \\ 1 & 0 & s_3 & 0 \\ 0 & 0 & 1 & s_4 \end{vmatrix}$	(139)
$\begin{vmatrix} \tilde{s}_1 & 0 & 0 & 0 \\ 1 & s_2 & 0 & 0 \\ 0 & 0 & \tilde{s}_3 & 0 \\ 0 & 0 & 1 & s_4 \end{vmatrix}$	(126)	$\begin{vmatrix} s_1 & 1 & 0 & 0 \\ 1 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 1 \\ 0 & 0 & 1 & s_4 \end{vmatrix}$	(140)
$\begin{vmatrix} \tilde{s}_1 & 0 & 0 & 0 \\ 1 & s_2 & 0 & 0 \\ 0 & 0 & \tilde{s}_3 & 0 \\ 0 & 0 & 0 & \tilde{s}_4 \end{vmatrix}$	(124)	$\begin{vmatrix} s_1 & 1 & 0 & 0 \\ 0 & s_2 & 1 & 0 \\ 1 & 0 & s_3 & 0 \\ 1 & 0 & 0 & s_4 \end{vmatrix}$	(137)
$\begin{vmatrix} \tilde{s}_1 & 0 & 0 & 0 \\ 1 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 1 \\ 0 & 0 & 1 & s_4 \end{vmatrix}$	(131)	$\begin{vmatrix} s_1 & 1 & 0 & 0 \\ 1 & s_2 & 1 & 0 \\ 0 & 1 & s_3 & 1 \\ 1 & 0 & 0 & s_4 \end{vmatrix}$	(141)
$\begin{vmatrix} \tilde{s}_1 & 0 & 0 & 0 \\ 0 & \tilde{s}_2 & 0 & 0 \\ 0 & 0 & \tilde{s}_3 & 0 \\ 0 & 0 & 0 & \tilde{s}_4 \end{vmatrix}$	(123)		

Table 3: Matrix and number of the corresponding equation

The simplest way to find the above-mentioned decompositions is to solve the following system of linear equations

$$l_{r'} + s_r l_r = 1, \quad r, r' = 1, \dots, n \quad (147)$$

$$\sum_{r=1}^n l_r = 1 \quad (148)$$

for all integer s_r . This gives n^n sets of linear equations, which have non-trivial solutions only if the determinant of the extended $(n+1) \times (n+1)$ matrix equals zero

$$J = \begin{vmatrix} s_1 & 0 & \dots & 0 & -1 \\ 0 & s_2 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & s_n & -1 \\ 1 & 1 & \dots & 1 & -1 \end{vmatrix} \equiv 0. \quad (149)$$

Here in every row i the zero in the place i' is substituted by unity. Some of the n^n sets of linear equations are equivalent and are obtained by transmutations of indices i . For example, in the $n = 4$ case there are only 19 non-equivalent sets of equations according to the classification of the quasi-simply-laced numbers given in the previous subsection. The corresponding matrices and numbers of solutions for each matrix are listed in Table 3, where $\tilde{s}_i = s_i + 1$ and one should add to each matrix the row with the numbers “1” and the column with numbers “-1”.

Note that for the ansatz in Eq. (129) the above determinant J is factorized, which means that the corresponding set of equations has an infinite number of solutions. The condition of vanishing for this determinant can be written as follows

$$s_1(s_3 s_4 + s_2 s_4 + s_2 s_3 - s_2 s_3 s_4) = 0 \Leftrightarrow \frac{1}{s_2} + \frac{1}{s_3} + \frac{1}{s_4} - 1 = 0 \quad (150)$$

and in fact, apart from many (non-reflexive) solutions with $s_1 = 0$, we obtain those solutions corresponding to the decompositions of unity for $n = 3$. The same is true for Eq. (136). We should neglect these two classes of sets of equations due to their degeneracy.

B Five-Dimensional Simply-Laced Numbers

As it was discussed above, the arity approach offers the possibility to construct all reflexive vectors for an arbitrary dimension n , but the difficulty for its application is that at each step of the calculations we should verify the reflexivity of the polyhedron obtained as an intersection of the slices corresponding to the extended reflexive vectors in lower dimensions. The simply-laced numbers are a particular case of the reflexive numbers and they can be found by the same method. In Table 4 we list all the 147 simply-laced numbers for $n = 5$ with their corresponding expansion in linear combinations of the extended simply-laced vectors of Table 1. Note that almost all simply-laced numbers have the 2-arity expansion. Only one number $(1, 15, 24, 40, 40)$ [120] cannot be obtained as a sum of two extended simply-laced vectors and it is constructed according to the 3-arity approach:

$$(1, 15, 24, 40, 40) = (1, 0, 0, 1, 1) + 15(0, 1, 0, 1, 1) + 24(0, 0, 1, 1, 1). \quad (151)$$

To find these 147 simply-laced numbers we can also use the recurrent relations (74). Moreover, using these relations, also 3462 simply-laced numbers for $n = 6$ were calculated. However, in the following, we discuss a different method to generate these numbers in a recurrent way. The idea of this recurrent procedure is to present unit fractions in each step of the iteration as a sum of two (or several) unit fractions. We begin with the expansion of unity in the sum of two unit fractions:

$$1 = \frac{1}{2} + \frac{1}{2}. \quad (152)$$

1	$(1, 1, 1, 1, 1) = 1(1, 1, 1, 0, 0) + 1(0, 0, 0, 1, 1)$	2	$(1, 1, 1, 1, 2) = 1(1, 1, 0, 0, 1) + 1(0, 0, 1, 1, 1)$
3	$(1, 1, 1, 1, 4) = 1(1, 1, 0, 0, 2) + 1(0, 0, 1, 1, 2)$	4	$(1, 1, 1, 2, 5) = 1(1, 1, 0, 0, 2) + 1(0, 0, 1, 2, 3)$
5	$(1, 1, 1, 3, 3) = 1(1, 1, 0, 1, 0) + 1(0, 0, 1, 2, 3)$	6	$(1, 1, 1, 3, 6) = 1(1, 0, 0, 1, 2) + 1(0, 1, 1, 2, 4)$
7	$(1, 1, 1, 6, 9) = 1(1, 0, 1, 4, 6) + 1(0, 1, 0, 2, 3)$	8	$(1, 1, 2, 2, 2) = 1(1, 0, 1, 0, 0) + 1(0, 1, 1, 2, 2)$
9	$(1, 1, 2, 2, 6) = 1(1, 0, 0, 1, 2) + 1(0, 1, 2, 1, 4)$	10	$(1, 1, 2, 4, 4) = 1(0, 1, 1, 0, 2) + 1(0, 1, 1, 4, 2)$
11	$(1, 1, 2, 4, 8) = 1(1, 0, 1, 0, 2) + 1(0, 1, 1, 4, 6)$	12	$(1, 1, 2, 8, 12) = 1(1, 0, 1, 4, 6) + 1(0, 1, 1, 4, 6)$
13	$(1, 1, 3, 3, 4) = 1(1, 0, 1, 2, 0) + 1(0, 1, 2, 1, 4)$	14	$(1, 1, 3, 5, 5) = 1(1, 0, 0, 1, 1) + 1(0, 1, 3, 4, 4)$
15	$(1, 1, 3, 10, 15) = 1(1, 0, 0, 2, 3) + 1(0, 1, 3, 8, 12)$	16	$(1, 1, 4, 4, 10) = 1(1, 0, 3, 2, 6) + 1(0, 1, 1, 2, 4)$
17	$(1, 1, 4, 6, 12) = 1(1, 0, 2, 0, 3) + 1(0, 1, 2, 6, 9)$	18	$(1, 1, 4, 12, 18) = 1(1, 0, 2, 6, 9) + 1(0, 1, 2, 6, 9)$
19	$(1, 1, 6, 8, 8) = 1(1, 0, 3, 6, 2) + 1(0, 1, 3, 2, 6)$	20	$(1, 1, 6, 16, 24) = 1(0, 1, 0, 2, 3) + 1(0, 6, 14, 21)$
21	$(1, 1, 8, 10, 20) = 1(0, 1, 4, 5, 10) + 1(0, 4, 5, 10)$	22	$(1, 1, 12, 28, 42) = 1(0, 1, 6, 14, 21) + 1(0, 6, 14, 21)$
23	$(1, 2, 2, 2, 7) = 1(0, 0, 2, 1, 3) + 1(1, 2, 0, 1, 4)$	24	$(1, 2, 2, 3, 4) = 1(1, 0, 2, 0, 0) + 1(0, 1, 2, 1, 4)$
25	$(1, 2, 2, 5, 10) = 1(0, 1, 2, 3, 6) + 1(1, 1, 0, 2, 4)$	26	$(1, 2, 2, 10, 15) = 1(0, 1, 2, 6, 9) + 1(1, 0, 4, 6)$
27	$(1, 2, 3, 3, 3) = 1(1, 0, 2, 3, 0) + 1(0, 2, 1, 0, 3)$	28	$(1, 2, 3, 3, 9) = 1(1, 0, 2, 0, 3) + 1(0, 2, 1, 3, 6)$
29	$(1, 2, 3, 6, 6) = 1(1, 0, 2, 3, 0) + 1(0, 2, 1, 3, 6)$	30	$(1, 2, 3, 6, 12) = 1(1, 0, 2, 0, 3) + 1(0, 2, 1, 6, 9)$
31	$(1, 2, 3, 12, 18) = 1(1, 0, 0, 2, 3) + 1(0, 2, 3, 10, 15)$	32	$(1, 2, 4, 7, 14) = 1(1, 0, 2, 6, 9) + 1(0, 2, 2, 1, 5)$
33	$(1, 2, 6, 6, 15) = 1(0, 1, 2, 6, 9) + 1(1, 1, 4, 0, 6)$	34	$(1, 2, 6, 9, 18) = 1(1, 0, 3, 8, 12) + 1(0, 2, 3, 1, 6)$
35	$(1, 2, 6, 18, 27) = 1(0, 1, 6, 14, 21) + 1(1, 1, 0, 4, 6)$	36	$(1, 2, 9, 12, 12) = 1(0, 1, 3, 12, 8) + 1(1, 1, 6, 0, 4)$
37	$(1, 2, 9, 24, 36) = 1(1, 0, 6, 14, 21) + 1(0, 2, 3, 10, 15)$	38	$(1, 2, 12, 15, 30) = 1(0, 2, 6, 1, 9) + 1(1, 0, 6, 14, 21)$
39	$(1, 2, 18, 42, 63) = 1(1, 0, 6, 14, 21) + 2(0, 1, 6, 14, 21)$	40	$(1, 3, 3, 3, 5) = 1(1, 1, 0, 0, 1) + 1(0, 2, 3, 3, 4)$
41	$(1, 3, 3, 7, 7) = 1(1, 3, 0, 4, 4) + 3(0, 0, 1, 1, 1)$	42	$(1, 3, 3, 14, 21) = 1(0, 1, 3, 8, 12) + 1(1, 2, 0, 6, 9)$
43	$(1, 3, 4, 4, 12) = 1(0, 1, 1, 4, 6) + 1(1, 2, 3, 0, 6)$	44	$(1, 3, 4, 8, 8) = 1(1, 0, 3, 2, 6) + 1(0, 3, 1, 6, 2)$
45	$(1, 3, 4, 16, 24) = 1(1, 0, 3, 8, 12) + 1(0, 3, 1, 8, 12)$	46	$(1, 3, 5, 6, 15) = 1(0, 1, 2, 6, 9) + 1(1, 2, 3, 0, 6)$
47	$(1, 3, 6, 6, 8) = 1(1, 2, 3, 0, 6) + 1(0, 1, 3, 6, 2)$	48	$(1, 3, 6, 10, 10) = 1(1, 2, 0, 6, 9) + 1(0, 1, 6, 4, 1)$
49	$(1, 3, 6, 20, 30) = 1(0, 1, 6, 14, 21) + 1(1, 2, 0, 6, 9)$	50	$(1, 3, 8, 12, 24) = 1(1, 0, 6, 2, 9) + 1(0, 3, 2, 10, 15)$
51	$(1, 3, 8, 24, 36) = 1(1, 0, 6, 14, 21) + 1(0, 3, 2, 10, 15)$	52	$(1, 3, 12, 16, 16) = 1(1, 0, 3, 4, 4) + 3(0, 1, 3, 4, 4)$
53	$(1, 3, 12, 32, 48) = 1(1, 1, 0, 4, 6) + 2(0, 1, 6, 14, 21)$	54	$(1, 3, 24, 56, 84) = 1(1, 0, 6, 14, 21) + 3(0, 1, 6, 14, 21)$
55	$(1, 4, 4, 9, 18) = 1(1, 0, 3, 8, 12) + 1(0, 4, 1, 1, 6)$	56	$(1, 4, 5, 5, 5) = 1(1, 0, 2, 4, 1) + 1(0, 4, 3, 1, 4)$
57	$(1, 4, 5, 10, 20) = 1(0, 2, 3, 10, 15) + 1(1, 2, 2, 0, 5)$	58	$(1, 4, 5, 20, 30) = 1(1, 0, 3, 8, 12) + 2(0, 2, 1, 6, 9)$
59	$(1, 4, 10, 15, 30) = 1(1, 0, 8, 3, 12) + 2(0, 2, 1, 6, 9)$	60	$(1, 4, 15, 20, 20) = 1(1, 0, 3, 4, 4) + 4(0, 1, 3, 4, 4)$
61	$(1, 4, 15, 40, 60) = 1(1, 0, 3, 8, 12) + 4(0, 1, 3, 8, 12)$	62	$(1, 4, 20, 25, 50) = 1(1, 0, 4, 5, 10) + 4(0, 1, 4, 5, 10)$
63	$2(1, 5, 9, 15, 15) = 1(2, 0, 3, 15, 10) + 5(0, 2, 3, 3, 4)$	64	$(1, 5, 9, 30, 45) = 1(1, 2, 0, 6, 9) + 3(0, 1, 3, 8, 12)$
65	$(1, 5, 12, 12, 30) = 1(1, 3, 8, 0, 12) + 2(0, 1, 2, 6, 9)$	66	$(1, 5, 24, 30, 60) = 1(1, 0, 4, 5, 10) + 5(0, 1, 4, 5, 10)$
67	$(1, 6, 6, 26, 39) = 1(1, 0, 2, 6, 9) + 2(0, 3, 2, 10, 15)$	68	$(1, 6, 7, 7, 21) = 1(1, 0, 4, 1, 6) + 3(0, 2, 1, 2, 5)$
69	$(1, 6, 7, 14, 14) = 1(1, 0, 1, 2, 2) + 6(0, 1, 1, 2, 2)$	70	$(1, 6, 7, 28, 42) = 1(1, 0, 3, 8, 12) + 2(0, 3, 2, 10, 15)$
71	$(1, 6, 14, 21, 42) = 1(1, 0, 2, 3, 6) + 6(0, 1, 2, 3, 6)$	72	$(1, 6, 14, 42, 63) = 1(1, 0, 6, 2, 9) + 1(0, 3, 2, 6, 9)$
73	$(1, 6, 21, 28, 28) = 1(1, 0, 3, 4, 4) + 6(0, 1, 3, 4, 4)$	74	$(1, 6, 21, 56, 84) = 1(1, 0, 3, 8, 12) + 6(0, 1, 3, 8, 12)$
75	$(1, 6, 42, 98, 147) = 1(1, 0, 6, 14, 21) + 6(0, 1, 6, 14, 21)$	76	$(1, 7, 48, 112, 168) = 1(1, 0, 6, 14, 21) + 7(0, 1, 6, 14, 21)$
77	$(1, 8, 9, 18, 36) = 1(1, 2, 6, 0, 9) + 3(0, 2, 1, 6, 9)$	78	$(1, 8, 27, 72, 108) = 1(1, 0, 3, 8, 12) + 8(0, 1, 3, 8, 12)$
79	$(1, 9, 20, 60, 90) = 1(1, 0, 2, 6, 9) + 9(0, 1, 2, 6, 9)$	80	$(1, 10, 10, 14, 35) = 1(1, 0, 5, 4, 10) + 5(0, 2, 1, 2, 5)$
81	$(1, 10, 22, 22, 55) = 1(1, 0, 2, 2, 5) + 10(0, 1, 2, 2, 5)$	82	$(1, 10, 44, 55, 110) = 1(1, 0, 4, 5, 10) + 10(0, 1, 4, 5, 10)$
83	$(1, 12, 12, 15, 20) = 1(1, 4, 10, 5, 0) + 2(0, 4, 1, 5, 10)$	84	$(1, 12, 13, 52, 78) = 1(1, 0, 1, 4, 6) + 12(0, 1, 4, 6)$
85	$(1, 12, 26, 39, 78) = 1(1, 0, 2, 3, 6) + 12(0, 1, 2, 3, 6)$	86	$(1, 12, 39, 52, 52) = 1(1, 0, 3, 4, 4) + 12(0, 1, 3, 4, 4)$
87	$(1, 12, 39, 104, 156) = 1(1, 0, 3, 8, 12) + 12(0, 1, 3, 8, 12)$	88	$(1, 14, 20, 35, 70) = 1(1, 4, 0, 5, 10) + 10(0, 1, 2, 3, 6)$
89	$(1, 14, 90, 210, 315) = 1(1, 0, 6, 14, 21) + 14(0, 1, 6, 14, 21)$	90	$(1, 15, 20, 24, 60) = 1(1, 5, 0, 4, 10) + 10(0, 1, 2, 2, 5)$
91	$(1, 15, 24, 40, 40) = 1(1, 0, 0, 1, 1) + 15(0, 1, 0, 1, 1) + 24(0, 0, 1, 1, 1)$	92	$(1, 15, 24, 80, 120) = 1(1, 3, 0, 8, 12) + 12(0, 1, 2, 6, 9)$
93	$(1, 18, 38, 114, 171) = 1(1, 0, 2, 6, 9) + 18(0, 1, 2, 6, 9)$	94	$(1, 20, 84, 105, 210) = 1(1, 0, 4, 5, 10) + 20(0, 1, 4, 5, 10)$
95	$(1, 21, 132, 308, 462) = 1(1, 0, 6, 14, 21) + 21(0, 1, 6, 14, 21)$	96	$(1, 24, 75, 200, 300) = 1(1, 0, 3, 8, 12) + 24(0, 1, 3, 8, 12)$
97	$(1, 42, 258, 602, 903) = 1(1, 0, 6, 14, 21) + 42(0, 1, 6, 14, 21)$	98	$(2, 2, 2, 3, 3) = 1(1, 2, 0, 3, 0) + 1(1, 0, 2, 0, 3)$
99	$(2, 2, 2, 3, 9) = 1(1, 2, 0, 0, 3) + 1(1, 0, 2, 3, 6)$	100	$(2, 2, 3, 14, 21) = 1(1, 0, 3, 8, 12) + 1(1, 2, 0, 6, 9)$
101	$(2, 2, 5, 6, 15) = 1(1, 0, 2, 6, 9) + 1(1, 2, 3, 0, 6)$	102	$(2, 3, 3, 4, 12) = 1(1, 0, 1, 4, 6) + 1(1, 3, 2, 0, 6)$
103	$(2, 3, 3, 8, 8) = 1(1, 3, 0, 2, 6) + 1(1, 0, 3, 6, 2)$	104	$(2, 3, 3, 16, 24) = 1(1, 3, 0, 8, 12) + 1(1, 0, 3, 8, 12)$
105	$(2, 3, 4, 9, 18) = 1(1, 3, 0, 8, 12) + 1(1, 0, 4, 1, 6)$	106	$(2, 3, 5, 5, 15) = 1(0, 1, 4, 5, 10) + 1(2, 2, 1, 0, 5)$
107	$(2, 3, 5, 10, 10) = 1(0, 1, 4, 5, 10) + 1(2, 2, 1, 5, 0)$	108	$(2, 3, 5, 20, 30) = 1(2, 0, 3, 10, 15) + 1(0, 3, 2, 10, 15)$
109	$(2, 3, 6, 22, 33) = 1(1, 0, 6, 14, 21) + 1(1, 3, 0, 8, 12)$	110	$(2, 3, 10, 15, 30) = 1(2, 3, 0, 10, 15) + 5(0, 0, 2, 1, 3)$
111	$(2, 3, 10, 30, 45) = 1(2, 0, 1, 6, 9) + 3(0, 1, 3, 8, 12)$	112	$(2, 3, 15, 20, 20) = 1(2, 3, 0, 15, 10) + 5(0, 0, 3, 1, 2)$
113	$(2, 3, 15, 40, 60) = 2(1, 0, 3, 8, 12) + 3(0, 1, 3, 8, 12)$	114	$(2, 3, 30, 70, 105) = 2(1, 0, 6, 14, 21) + 3(0, 1, 6, 14, 21)$
115	$(2, 4, 4, 5, 5) = 1(2, 0, 1, 4, 1) + 1(0, 4, 3, 1, 4)$	116	$(2, 4, 9, 9, 12) = 1(1, 0, 3, 8, 12) + 1(1, 4, 6, 1, 0)$
117	$(2, 5, 5, 8, 20) = 1(1, 0, 5, 4, 10) + 1(1, 5, 0, 4, 10)$	118	$(2, 5, 14, 14, 35) = 2(1, 0, 2, 5, 5) + 5(0, 1, 2, 2, 5)$
119	$(2, 5, 28, 35, 70) = 2(1, 0, 4, 5, 10) + 5(0, 1, 4, 5, 10)$	120	$(2, 6, 6, 7, 21) = 1(0, 6, 2, 1, 9) + 2(1, 0, 2, 3, 6)$
121	$(2, 7, 12, 21, 42) = 1(2, 1, 0, 3, 6) + 6(0, 1, 2, 3, 6)$	122	$(2, 7, 54, 126, 189) = 2(1, 0, 6, 14, 21) + 7(0, 1, 6, 14, 21)$
123	$(2, 9, 22, 66, 99) = 2(1, 0, 2, 6, 9) + 9(0, 1, 2, 6, 9)$	124	$(2, 10, 15, 18, 45) = 2(1, 2, 6, 0, 9) + 3(0, 2, 1, 6, 9)$
125	$(2, 12, 21, 21, 28) = 1(1, 6, 21, 0, 14) + 1(1, 6, 0, 21, 14)$	126	$(2, 21, 138, 322, 483) = 2(1, 0, 6, 14, 21) + 21(0, 1, 6, 14, 21)$
127	$(2, 33, 42, 154, 231) = 2(1, 6, 0, 14, 21) + 21(0, 1, 2, 6, 9)$	128	$(3, 3, 4, 6, 8) = 1(1, 0, 3, 6, 2) + 1(2, 3, 1, 0, 6)$
129	$(3, 3, 4, 20, 30) = 1(3, 0, 2, 10, 15) + 1(0, 3, 2, 10, 15)$	130	$(3, 4, 7, 28, 42) = 1(3, 0, 1, 8, 12) + 2(0, 2, 3, 10, 15)$
131	$(3, 4, 14, 21, 42) = 1(3, 0, 8, 1, 12) + 2(0, 2, 3, 10, 15)$	132	$(3, 4, 21, 28, 28) = 3(1, 0, 3, 4, 4) + 4(0, 1, 3, 4, 4)$
133	$(3, 4, 21, 56, 84) = 3(1, 0, 3, 8, 12) + 4(0, 1, 3, 8, 12)$	134	$(3, 5, 6, 6, 10) = 1(1, 2, 0, 6, 9) + 1(2, 3, 6, 0, 1)$
135	$(3, 5, 10, 12, 30) = 1(1, 5, 0, 4, 10) + 2(1, 0, 5, 4, 10)$	136	$(3, 5, 12, 20, 20) = 1(3, 0, 2, 15, 10) + 5(0, 1, 2, 1, 2)$
137	$(3, 5, 12, 40, 60) = 2(0, 1, 6, 14, 21) + 3(1, 1, 0, 4, 6)$	138	$(3, 7, 60, 140, 210) = 3(1, 0, 6, 14, 21) + 7(0, 1, 6, 14, 21)$
139	$(3, 8, 33, 88, 132) = 3(1, 0, 3, 8, 12) + 8(0, 1, 3, 8, 12)$	140	$(3, 10, 12, 15, 20) = 1(1, 10, 4, 5, 0) + 2(1, 0, 4, 5, 10)$
141	$(3, 14, 102, 238, 357) = 3(1, 0, 6, 14, 21) + 14(0, 1, 6, 14, 21)$	142	$(3, 22, 30, 110, 165) = 1(3, 2, 0, 10, 15) + 10(0, 2, 3, 10, 15)$
143	$(4, 5, 6, 15, 30) = 2(2, 1, 0, 6, 9) + 3(0, 1, 2, 1, 4)$	144	$(4, 5, 36, 45, 90) = 4(1, 0, 4, 5, 10) + 5(0, 1, 4, 5, 10)$
145	$(4, 6, 15, 15, 20) = 1(2, 3, 15, 0, 10) + 1(2, 3, 0, 15, 10)$	146	$(6, 7, 78, 182, 273) = 6(1, 0, 6, 14, 21) + 7(0, 1, 6, 14, 21)$
147	$(6, 14, 15, 70, 105) = 2(3, 2, 0, 10, 15) + 5(0, 2, 3, 10, 15)$		

Table 4: Expansion of the $n = 5$ reflexive simply-laced numbers on the $n = 4$ reflexive simply-laced numbers. 146 are expanded using 2-arity, the vector with number 91 is expanded using 3-arity.

If in the fraction $1/p$ the number p is a prime number, we only have two of such expansions

$$\frac{1}{p} = \frac{1}{2p} + \frac{1}{2p} = \frac{1}{p+1} + \frac{1}{p(p+1)}. \quad (153)$$

If the number in its denominator is a product of two (or several) prime numbers $p_1 p_2 \dots$, the fraction has more unit expansions. For example,

$$\begin{aligned} \frac{1}{p_1 p_2} &= \frac{1}{2p_1 p_2} + \frac{1}{2p_1 p_2} = \frac{1}{p_1 p_2 + 1} + \frac{1}{p_1 p_2 (p_1 p_2 + 1)} = \\ \frac{1}{p_2 (p_1 + 1)} + \frac{1}{p_1 p_2 (p_1 + 1)} &= \frac{1}{p_1 (p_2 + 1)} + \frac{1}{p_1 p_2 (p_2 + 1)} = \frac{1}{p_2 (p_1 + p_2)} + \frac{1}{p_1 (p_1 + p_2)}. \end{aligned} \quad (154)$$

Using the above relations we obtain for $n = 3$ from the unit expansion for $n = 2$:

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}. \quad (155)$$

The following expansion is obvious and easily generalized for an arbitrary n :

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}. \quad (156)$$

In the $n = 4$ case the corresponding unity expansions obtained from the above 3 decompositions for $n = 3$ are given below

$$\begin{aligned} 1 &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} = \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} \\ &= \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42} = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18} = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{24} \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{15} = \frac{1}{2} + \frac{1}{5} + \frac{1}{5} + \frac{1}{10}. \end{aligned} \quad (157)$$

Only the last unity decomposition cannot be obtained by the use of the above relations. It is necessary to generalize this procedure by using the expansion of a unit fraction in the sum of three unit fractions. Namely, we should apply the relation

$$\frac{1}{p_1 p_2 p_3} = \frac{1}{p_1 (p_1 p_2 + p_1 p_3 + p_2 p_3)} + \frac{1}{p_2 (p_1 p_2 + p_1 p_3 + p_2 p_3)} + \frac{1}{p_3 (p_1 p_2 + p_1 p_3 + p_2 p_3)}. \quad (158)$$

The generalization to a higher number of terms in the sum is trivial and one can verify that indeed the simply-laced numbers for $n = 5$ written in Table 5 in the form of the unit fractions can be obtained by this method from the corresponding expansions for $n = 2, 3, 4$.

It is important to note that among all unit decompositions for each n there is a decomposition with the maximal denominator corresponding to the maximal dimension $d(n) = \max d_k$. This number satisfies the simple recurrence relation

$$d(n+1) = d(n)(d(n)+1). \quad (159)$$

It grows very rapidly

$$d(1) = 1, \quad d(2) = 2, \quad d(3) = 6, \quad d(4) = 42, \quad d(5) = 1806 \dots \quad (160)$$

Note that the numbers $d(n) + 1$ are not prime because $1807 = 13 \cdot 139$.

The above recurrence relation can be written as the finite difference equation

$$d(n+1) - d(n) = (d(n))^2. \quad (161)$$

Note, that it can not be substituted by the differential equation $\tilde{d}' = \tilde{d}^2$ for large n , because its solution in this case would have a pole $\tilde{d} = 1/(n_0 - n)$ absent in the solution of the recurrence relation.

1	(1/5, 1/5, 1/5, 1/5, 1/5)	2	(1/6, 1/6, 1/6, 1/6, 1/3)	3	(1/8, 1/8, 1/8, 1/8, 1/2)
4	(1/10, 1/10, 1/10, 1/5, 1/2)	5	(1/9, 1/9, 1/9, 1/3, 1/3)	6	(1/12, 1/12, 1/12, 1/4, 1/2)
7	(1/18, 1/18, 1/18, 1/3, 1/2)	8	(1/8, 1/8, 1/4, 1/4, 1/4)	9	(1/12, 1/12, 1/6, 1/6, 1/2)
10	(1/12, 1/12, 1/6, 1/3, 1/3)	11	(1/16, 1/16, 1/8, 1/4, 1/2)	12	(1/24, 1/24, 1/12, 1/3, 1/2)
13	(1/12, 1/12, 1/4, 1/4, 1/3)	14	(1/15, 1/15, 1/5, 1/3, 1/3)	15	(1/30, 1/30, 1/10, 1/3, 1/2)
16	(1/20, 1/20, 1/5, 1/5, 1/2)	17	(1/24, 1/24, 1/6, 1/4, 1/2)	18	(1/36, 1/36, 1/9, 1/3, 1/2)
19	(1/24, 1/24, 1/4, 1/3, 1/3)	20	(1/48, 1/48, 1/8, 1/3, 1/2)	21	(1/40, 1/40, 1/5, 1/4, 1/2)
22	(1/84, 1/84, 1/7, 1/3, 1/2)	23	(1/14, 1/7, 1/7, 1/7, 1/2)	24	(1/12, 1/6, 1/6, 1/4, 1/3)
25	(1/20, 1/10, 1/10, 1/4, 1/2)	26	(1/30, 1/15, 1/15, 1/3, 1/2)	27	(1/12, 1/6, 1/4, 1/4, 1/4)
28	(1/18, 1/9, 1/6, 1/6, 1/2)	29	(1/18, 1/9, 1/6, 1/3, 1/3)	30	(1/24, 1/12, 1/8, 1/4, 1/2)
31	(1/36, 1/18, 1/12, 1/3, 1/2)	32	(1/28, 1/14, 1/7, 1/4, 1/2)	33	(1/30, 1/15, 1/5, 1/5, 1/2)
34	(1/36, 1/18, 1/6, 1/4, 1/2)	35	(1/54, 1/27, 1/9, 1/3, 1/2)	36	(1/36, 1/18, 1/4, 1/3, 1/3)
37	(1/72, 1/36, 1/8, 1/3, 1/2)	38	(1/60, 1/30, 1/5, 1/4, 1/2)	39	(1/126, 1/63, 1/7, 1/3, 1/2)
40	(1/15, 1/5, 1/5, 1/5, 1/3)	41	(1/21, 1/7, 1/7, 1/3, 1/3)	42	(1/42, 1/14, 1/14, 1/3, 1/2)
43	(1/24, 1/8, 1/6, 1/6, 1/2)	44	(1/24, 1/8, 1/6, 1/3, 1/3)	45	(1/48, 1/16, 1/12, 1/3, 1/2)
46	(1/30, 1/10, 1/6, 1/5, 1/2)	47	(1/24, 1/8, 1/4, 1/4, 1/3)	48	(1/30, 1/10, 1/5, 1/3, 1/3)
49	(1/60, 1/20, 1/10, 1/3, 1/2)	50	(1/48, 1/16, 1/6, 1/4, 1/2)	51	(1/72, 1/24, 1/9, 1/3, 1/2)
52	(1/48, 1/16, 1/4, 1/3, 1/3)	53	(1/96, 1/32, 1/8, 1/3, 1/2)	54	(1/168, 1/56, 1/7, 1/3, 1/2)
55	(1/36, 1/9, 1/9, 1/4, 1/2)	56	(1/20, 1/5, 1/4, 1/4, 1/4)	57	(1/40, 1/10, 1/8, 1/4, 1/2)
58	(1/60, 1/15, 1/12, 1/3, 1/2)	59	(1/60, 1/15, 1/6, 1/4, 1/2)	60	(1/60, 1/15, 1/4, 1/3, 1/3)
61	(1/120, 1/30, 1/8, 1/3, 1/2)	62	(1/100, 1/25, 1/5, 1/4, 1/2)	63	(1/45, 1/9, 1/5, 1/3, 1/3)
64	(1/90, 1/18, 1/10, 1/3, 1/2)	65	(1/60, 1/12, 1/5, 1/5, 1/2)	66	(1/120, 1/24, 1/5, 1/4, 1/2)
67	(1/78, 1/13, 1/13, 1/3, 1/2)	68	(1/42, 1/7, 1/6, 1/6, 1/2)	69	(1/42, 1/7, 1/6, 1/3, 1/3)
70	(1/84, 1/14, 1/12, 1/3, 1/2)	71	(1/84, 1/14, 1/6, 1/4, 1/2)	72	(1/126, 1/21, 1/9, 1/3, 1/2)
73	(1/84, 1/14, 1/4, 1/3, 1/3)	74	(1/168, 1/28, 1/8, 1/3, 1/2)	75	(1/294, 1/49, 1/7, 1/3, 1/2)
76	(1/336, 1/48, 1/7, 1/3, 1/2)	77	(1/72, 1/9, 1/8, 1/4, 1/2)	78	(1/216, 1/27, 1/8, 1/3, 1/2)
79	(1/180, 1/20, 1/9, 1/3, 1/2)	80	(1/70, 1/7, 1/7, 1/5, 1/2)	81	(1/110, 1/11, 1/5, 1/5, 1/2)
82	(1/220, 1/22, 1/5, 1/4, 1/2)	83	(1/60, 1/5, 1/5, 1/4, 1/3)	84	(1/156, 1/13, 1/12, 1/3, 1/2)
85	(1/156, 1/13, 1/6, 1/4, 1/2)	86	(1/156, 1/13, 1/4, 1/3, 1/3)	87	(1/312, 1/26, 1/8, 1/3, 1/2)
88	(1/140, 1/10, 1/7, 1/4, 1/2)	89	(1/630, 1/45, 1/7, 1/3, 1/2)	90	(1/120, 1/8, 1/6, 1/5, 1/2)
91	(1/120, 1/8, 1/5, 1/3, 1/3)	92	(1/240, 1/16, 1/10, 1/3, 1/2)	93	(1/342, 1/19, 1/9, 1/3, 1/2)
94	(1/420, 1/21, 1/5, 1/4, 1/2)	95	(1/924, 1/44, 1/7, 1/3, 1/2)	96	(1/600, 1/25, 1/8, 1/3, 1/2)
97	(1/1806, 1/43, 1/7, 1/3, 1/2)	98	(1/6, 1/6, 1/6, 1/4, 1/4)	99	(1/9, 1/9, 1/9, 1/6, 1/2)
100	(1/21, 1/21, 1/14, 1/3, 1/2)	101	(1/15, 1/15, 1/6, 1/5, 1/2)	102	(1/12, 1/8, 1/8, 1/6, 1/2)
103	(1/12, 1/8, 1/8, 1/3, 1/3)	104	(1/24, 1/16, 1/16, 1/3, 1/2)	105	(1/18, 1/12, 1/9, 1/4, 1/2)
106	(1/15, 1/10, 1/6, 1/6, 1/2)	107	(1/15, 1/10, 1/6, 1/3, 1/3)	108	(1/30, 1/20, 1/12, 1/3, 1/2)
109	(1/33, 1/22, 1/11, 1/3, 1/2)	110	(1/30, 1/20, 1/6, 1/4, 1/2)	111	(1/45, 1/30, 1/9, 1/3, 1/2)
112	(1/30, 1/20, 1/4, 1/3, 1/3)	113	(1/60, 1/40, 1/8, 1/3, 1/2)	114	(1/105, 1/70, 1/7, 1/3, 1/2)
115	(1/10, 1/5, 1/5, 1/4, 1/4)	116	(1/18, 1/9, 1/4, 1/4, 1/3)	117	(1/20, 1/8, 1/8, 1/5, 1/2)
118	(1/35, 1/14, 1/5, 1/5, 1/2)	119	(1/70, 1/28, 1/5, 1/4, 1/2)	120	(1/21, 1/7, 1/7, 1/6, 1/2)
121	(1/42, 1/12, 1/7, 1/4, 1/2)	122	(1/189, 1/54, 1/7, 1/3, 1/2)	123	(1/99, 1/22, 1/9, 1/3, 1/2)
124	(1/45, 1/9, 1/6, 1/5, 1/2)	125	(1/42, 1/7, 1/4, 1/4, 1/3)	126	(1/483, 1/46, 1/7, 1/3, 1/2)
127	(1/231, 1/14, 1/11, 1/3, 1/2)	128	(1/8, 1/8, 1/6, 1/4, 1/3)	129	(1/20, 1/20, 1/15, 1/3, 1/2)
130	(1/28, 1/21, 1/12, 1/3, 1/2)	131	(1/28, 1/21, 1/6, 1/4, 1/2)	132	(1/28, 1/21, 1/4, 1/3, 1/3)
133	(1/56, 1/42, 1/8, 1/3, 1/2)	134	(1/10, 1/6, 1/5, 1/5, 1/3)	135	(1/20, 1/12, 1/6, 1/5, 1/3)
136	(1/20, 1/12, 1/5, 1/3, 1/3)	137	(1/40, 1/24, 1/10, 1/3, 1/2)	138	(1/140, 1/60, 1/7, 1/3, 1/2)
139	(1/88, 1/33, 1/8, 1/3, 1/2)	140	(1/20, 1/6, 1/5, 1/4, 1/3)	141	(1/238, 1/51, 1/7, 1/3, 1/2)
142	(1/110, 1/15, 1/11, 1/3, 1/2)	143	(1/15, 1/12, 1/10, 1/4, 1/2)	144	(1/45, 1/36, 1/5, 1/4, 1/2)
145	(1/15, 1/10, 1/4, 1/4, 1/3)	146	(1/91, 1/78, 1/7, 1/3, 1/2)	147	(1/35, 1/15, 1/14, 1/3, 1/2)

Table 5: The 147 simply-laced number for $n = 5$ written in the form of unit fractions.

C A case study: The $B(01111)$ Berger Graph

For illustrative purposes we now consider the affine Berger graph $B(01111)$ as a case study among the 14 exceptional simply-laced graphs. This graph can be found in the four dimensional polyhedron determined by the reflexive number $\vec{k}_5 = (1, 1, 1, 1, 1)[5]$, constructed by 2-arity from the two simply-laced numbers $\vec{k}_4 = (1, 1, 1, 1)[4]$ and $\vec{k}_1 = (1)[1]$. In this case the Berger graph $B(01111)$ coincides with the primary $B(1111)$ graph (see Fig. 4). This graph looks like a natural generalization of the extended Dynkin graph for E_6 . It is then possible to propose the next generalizations of such graphs in dimensions $n = 6, 7, \dots$. This proposal can be confirmed directly by looking for these graphs in the polyhedra determined by the 2-arity expansion: $\vec{k}_{n+1} = (1, \dots, 1)[n+1] = (1, \dots, 1, 0)_{n+1} + (0, \dots, 0, 1)_{n+1}$. Thus the reflexive simply-laced numbers $(1, 1, \dots, 1)[n]$ determine the set of primary graphs, which have the form of stars with n -legs each of them having $(n-1)$ nodes, with Coxeter labels $1, 2, \dots, (n-1)$. The center of the star has the maximal

$$\begin{aligned}
a_0 &= e_3 - e_2 - e_9 \\
a_1 &= e_6 - e_5 \\
a_2 &= e_5 - e_4 \\
a_3 &= e_4 - e_3 \\
a_4 &= e_7 + e_8 \\
a_5 &= -(e_1 + e_2 + e_7 + e_8 - e_9 - e_{10} - e_{11} - e_{12})/2 \\
a_6 &= e_1 + e_2 \\
a_7 &= e_{11} - e_{12} \\
a_8 &= e_{10} - e_{11} \\
a_9 &= e_9 - e_{10} \\
a_{10} &= e_7 - e_8 \\
a_{11} &= -(-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 - e_8)/2 \\
a_{12} &= e_2 - e_1
\end{aligned}$$

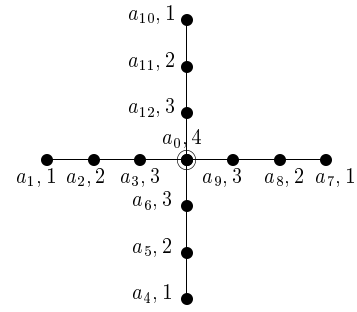


Figure 4: Berger graph for the B(01111) vector with Coxeter labels and simple roots in orthogonal basis.

Coxeter label equals to n .

On such a case it is possible to extract the full information from the affine graph, *i.e.*, Berger matrices for affine and non-affine cases, Coxeter labels, simple roots, fundamental weights, etc. The corresponding affine Berger matrix can be constructed by the canonical way:

$$B(01111) = \left(\begin{array}{ccc|ccc|ccc|ccc||c}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
\hline
-1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 3
\end{array} \right)$$

The generalization of this graph $B(01111)$ to $B((01\dots 1)_n)$ is straightforward. Instead of the maximal diagonal element being $B_{ii} = 3$ one can change it to $B_{ii} = n$. For all these cases the determinant cancels. The “simple roots” for this graph in an orthonormal basis $\{e_i\}, i = 1, \dots, 12$ are written in Fig 4. They obey our restrictions:

$$\begin{aligned}
\langle \hat{\alpha}_0 \cdot \hat{\alpha}_0 \rangle &= 3, \\
\langle \alpha_i \cdot \alpha_i \rangle &= 2, \quad i = 1, 2, 3, \dots, 12, \\
\langle \alpha_i \cdot \alpha_j \rangle &= -1, \quad j = i \pm 1, \\
\langle \alpha_i \cdot \alpha_j \rangle &= 0, \quad |j - i| > 1.
\end{aligned} \tag{162}$$

The following linear combination of the “roots” satisfies the affine condition:

$$4\hat{\alpha}_0 + 3\alpha_{a1} + 2\alpha_{a2} + \alpha_{a3} + 3\alpha_{b1} + 2\alpha_{b2} + \alpha_{b3} + 3\alpha_{c1} + 2\alpha_{c2} + \alpha_{c3} + 3\alpha_{d1} + 2\alpha_{d2} + \alpha_{d3} = 0.$$

The last affine link gives a possibility to find a non-affine graph from an affine Berger graph. For this it is enough to remove one zero node with Coxeter label 1. Correspondingly, removing, for example, the first row and column from the affine Berger matrix one can obtain a non-affine Berger matrix having the determinant equal to 16. One can write this number as 4×4 , where one coefficient 4 could be related to the number of nodes having Coxeter label one plus 1, and the other coefficient 4 could be related to a \mathbb{Z}_4 symmetry of the affine Berger graph. Let us recall that in the Cartan–Lie case the determinant of Cartan simply-laced matrices is equal to the number of non-equivalent representations. For example, in the A_r, D_r, E_6, E_7, E_8 cases the number of non-equivalent representations is $r + 1, 4, 3, 2, 1$, respectively. For any n of the non-affine $B(01\dots 1)$ matrix its determinant is equal to n^{n-2} , $n > 3$. In the Cartan case from the inverse Cartan matrix one can find the set of r fundamental weights. Also, from the non-affine

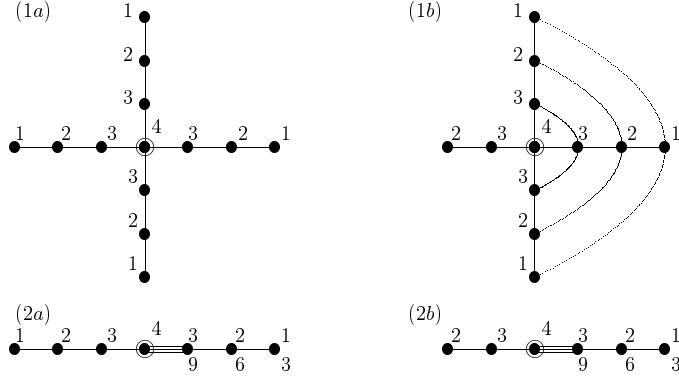


Figure 5: Relation between symmetric simply-laced B(01111) and non-simply-laced graphs.

Berger matrix one can obtain all fundamental weights:

$$G = \left(\begin{array}{c|cccc|cccc|cccc|c} F.W. & \alpha_{a1} & \alpha_{a2} & \alpha_{b1} & \alpha_{b2} & \alpha_{b3} & \alpha_{c1} & \alpha_{c2} & \alpha_{c3} & \alpha_{d1} & \alpha_{d2} & \alpha_{d3} & \alpha_0 \\ \hline \Lambda_{a1} & 6 & 3 & 6 & 4 & 2 & 6 & 4 & 2 & 6 & 4 & 2 & 8 \\ \Lambda_{a2} & 3 & 2 & 3 & 2 & 1 & 3 & 2 & 1 & 3 & 2 & 1 & 4 \\ \hline \Lambda_{b1} & 6 & 3 & 15/2 & 5 & 5/2 & 27/4 & 9/2 & 9/4 & 27/4 & 9/2 & 9/4 & 9 \\ \Lambda_{b2} & 4 & 2 & 5 & 4 & 2 & 9/2 & 3 & 3/2 & 9/2 & 3 & 3/2 & 6 \\ \Lambda_{b3} & 2 & 1 & 5/2 & 2 & 3/2 & 9/4 & 3/2 & 3/4 & 9/4 & 3/2 & 3/4 & 3 \\ \hline \Lambda_{c1} & 6 & 3 & 27/4 & 9/2 & 9/4 & 15/2 & 5 & 5/2 & 27/4 & 9/2 & 9/4 & 9 \\ \Lambda_{c2} & 4 & 2 & 9/2 & 3 & 3/2 & 5 & 4 & 2 & 9/2 & 3 & 3/2 & 6 \\ \Lambda_{c3} & 2 & 1 & 9/4 & 3/2 & 3/4 & 5/2 & 2 & 3/2 & 9/4 & 3/2 & 3/4 & 3 \\ \hline \Lambda_{d1} & 6 & 3 & 27/4 & 9/2 & 9/4 & 27/4 & 9/2 & 9/4 & 15/2 & 5 & 5/2 & 9 \\ \Lambda_{d2} & 4 & 2 & 9/2 & 3 & 3/2 & 9/2 & 3 & 3/2 & 5 & 4 & 2 & 6 \\ \Lambda_{d3} & 2 & 1 & 9/4 & 3/2 & 3/4 & 9/4 & 3/2 & 3/4 & 5/2 & 2 & 3/2 & 3 \\ \hline \Lambda_0 & 8 & 4 & 9 & 6 & 3 & 9 & 6 & 3 & 9 & 6 & 3 & 12 \end{array} \right)$$

Finally we briefly discuss the non-simply-laced case. Apart from the calculation of the reflexive numbers and the particular case of the quasi-simply-laced ones, one can attempt to construct the non-simply-laced graphs from the above-investigated simply-laced graphs. For this purpose we shall return again to the Cartan-Lie algebras and use the known method to construct the root system of the non-simply-laced B_r- , C_r- , F_4- , G_2- algebras from the simply-laced $D_{r+1}, A_{2r-1}, E_6, D_4$ root systems, respectively. The root systems of these simply-laced algebras have the following diagram automorphisms f :

1. D_{r+1} : $f(\alpha_i) = \alpha_i$ for $1 \leq r \leq r-1$, $f(\alpha_r) = \alpha_{r+1}$, $f(\alpha_{r+1}) = \alpha_r$: $D_{r+1} \rightarrow B_r$
2. A_{2r-1} : $f(\alpha_i) = \alpha_{2r-i}$, $f(\alpha_{2r-i}) = \alpha_i$, for $1 \leq i \leq r-1$, and $f(\alpha_r) = \alpha_r$: $A_{2r-1} \rightarrow C_r$;
3. E_6 : $f(\alpha_1) = \alpha_6$, $f(\alpha_2) = \alpha_5$, $f(\alpha_3) = \alpha_3$, $f(\alpha_5) = \alpha_2$, $f(\alpha_6) = \alpha_1$, $f(\alpha_4) = \alpha_4$: $E_6 \rightarrow F_4$;
4. D_4 : $f(\alpha_1) = \alpha_3$, $f(\alpha_2) = \alpha_2$, $f(\alpha_3) = \alpha_4$, $f(\alpha_4) = \alpha_1$: $D_4 \rightarrow G_2$.

Let us remark that the automorphisms in the first three cases are of order 2, and the last case is of order 3. We can use a similar algorithm to build a non-simply-laced Berger graph from the simply-laced B(01111) graph (see Fig. 5). The corresponding non-affine Berger graph has a \mathbb{Z}_3 symmetry. Acting as in the Cartan case and using this symmetry we obtain from the simply-laced Berger graph of rank 12 a new non-simply and non-affine graph with rank 6. On Fig. 5 three consecutive steps to construct a new affine non-simply-laced graph are shown: The first step, $1a \rightarrow 1b$, corresponds to removing in the affine B(01111) graph one zero node having the Coxeter label equal to 1. In the second step, $1b \rightarrow 2b$, one obtains the non-simply-laced graph. The third step $2b \rightarrow 2a$ corresponds to adding the zero node with the Coxeter label 1. From the resulting new non-simply-laced graph one can construct the corresponding Berger

matrix with the determinant equal to zero:

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

with the set of eigenvalues $\{0, 2, 3, 2 - \sqrt{2}, 2 + \sqrt{2}, 3 - \sqrt{3}, 3 + \sqrt{3}\}$. Two zero eigenvectors corresponding to the Berger matrix B_{ij} and its transposed matrix B_{ji} give us two sets of Coxeter labels:

$$\vec{C} = \{1, 2, 3, 4, 3, 2, 1\}, \quad \vec{\tilde{C}} = \{1, 2, 3, 4, 9, 6, 3\}. \quad (163)$$

The corresponding non-affine Berger matrix has the determinant equal to 1 similar to the G_2 graph in the Cartan case ($D_4 \rightarrow G_2$). This value for the determinant agrees with our previous arguments, when we discussed the determinants for non-affine $B(01111)$ matrices and also for the Cartan simply-laced cases.

In this way, using the symmetry of the $B(0, \dots, 0, 1)$, $B(0, \dots, 0, 1, 1)$, $B(0, \dots, 0, 1, 1, 1)$, $B(0, \dots, 0, 1, 1, 2)$, $B(0, \dots, 0, 1, 2, 3)$, $B(0, 1, 1, 2, 2)$, $B(0, 1, 1, 1, 3)$, $B(0, 1, 2, 2, 5)$, $B(0, 2, 3, 3, 4)$ and $B(0, 1, 3, 3, 4)$ Berger graphs one can get new infinite series and some exceptional non-simply-laced graphs. From this construction one can see that the generation of non-simply-laced Berger graphs takes place as in the $K3$ case. There we had just $A-D-E$ types of singularities but in $K3$ polyhedra one can also obtain non-simply-laced Dynkin graphs [16, 20, 21]. Moreover, in the arity-dimension approach in the $K3$ case we used only simply-laced numbers (1)[1], (1, 1)[2], (1, 1, 1)[3], (1, 1, 2)[4], (1, 2, 3)[6] but we also have non-simply-laced Dynkin diagrams. In this approach the non-simply-laced graphs appear as subgraphs of simply-laced ones. To understand this more deeply it would be needed to use some additional dynamics [22]. However in CY_d cases with $d > 2$ we have a new principle as we have briefly discussed here: We should take into consideration not only simply-laced numbers but also the set of non-simply-laced ones.

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