Hilly poor noncrossing partitions and $(2,3)$-Motzkin paths

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Abstract. A hilly poor noncrossing partition is a noncrossing partition with the properties : (1) each block has at most two elements, (2) in its linear representation, any isolated vertex is covered by some arc. This paper defines basic pairs as a combinatorial object and gives the number of hilly poor noncrossing partitions with n blocks, which is closely related to Maximal Davenport-Schinzel sequences. Authors introduce a class of generalized Motzkin paths called (i,j)-Motzkin paths, and present a bijection between hilly poor noncrossing partitions and $(2,3)$-Motzkin paths. Specialization of the bijection deduces various results regarding 3-colored Motzkin paths, Catalan numbers, Motzkin numbers and Riordan numbers.

Keywords: Hilly poor noncrossing partition, basic pair, $(2,3)$-Motzkin path, 3-colored Motzkin path, Catalan number, Motzkin number, Riordan number.

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1. Introduction and notation

A *partition* P of $[l] := \{1, 2, \ldots, l\}$ is a collection $\{B_1, B_2, \ldots, B_n\}$ of nonempty disjoint subsets of $[l]$, called *blocks*, whose union is $[l]$ and which are listed in the increasing order of their minimal elements. A partition is called m-*regular*, $m \geq 1$, if for any $x, y \in B_i$, implies $|x - y| \geq m$. A partition is a *noncrossing* partition if there are no four distinct numbers $a < b < c < d$ and no two distinct blocks B_i, B_j such that $a, c \in B_i$ and $b, d \in B_j$. A partition is *poor* if each block has at most two elements. See [1], [6], [8] and [11] for more information and references on partition.

There are various ways to represent a partition. For our purpose we will use a *linear representation* and a *canonical sequential form*. In the linear representation, l vertices are arranged on the line, the i-th vertex is labelled by j if $i \in B_j$, and successive vertices with the same label are joined by an arc. Specially, the sequence consisting of labels is called its canonical sequential form. The noncrossing property of a partition corresponds to the facts that the arcs do not intersect in its linear representation and that its canonical sequential form contains no any $\ldots a \ldots b \ldots a \ldots b \ldots$ subsequence. Fig.1 is the linear representation of $P = \{\{1, 5\}, \{2, 4\}, \{3\}, \{6\}\}$ and its canonical sequential form is $a = 123214$.

Let $\mathcal{P}(n)$ be the set of poor noncrossing partitions with n blocks and $\mathcal{P}_2(n)$

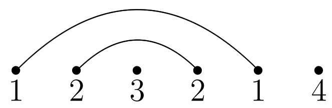

Figure 1: A linear presentation and its canonical sequential form.

be the subset of $\mathcal{P}(n)$ in which each block has two elements. It is easy to check that the cardinality of $\mathcal{P}_2(n)$ is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ by a simple bijection matching such partitions with Dyck paths of length $2n$.

For convenience, in the linear representations of a poor noncrossing partition, if two vertices of an arc are labelled by i, we call it arc i and do not distinguish arc i with block B_i. An arc is *void* if there is no vertex under it. Otherwise, it is a non-void arc. An arc is *big* if and only if there is no arc above it. By an *interval* $I(i)$ we mean an interval in a canonical sequential form which begins and ends with an i-occurrence. In other words, $I(i)$ is a subgraph covered by arc i including arc i. We say that block B_i is *covered* by block B_j if all vertices with label i are covered by arc j.

This paper focuses on hilly poor noncrossing partitions and $(2,3)$-Motzkin paths. In Section 2, we give a positive definition of hilly poor noncrossing partitions and count its cardinality using generating function. In Section 3, we characterize the parameter k by refining hill poor noncrossing partitions with basic pairs, and obtain its enumerative formula by constructing a mapping. In Section 4, we define (i,j)-Motzkin paths and show a bijection π between $(2,3)$-Motzkin paths of length n and hilly poor noncrossing partitions of $n+1$ blocks. In Section 5, specialization of the bijection π figures out a combinatorial explanation that $\sum_{k=1}^{n}\binom{n}{k}C_{k+1}$ counts the number of 3-colored Motzkin paths of length n, which is derived by Ferrari, Pergola, Pinzani and Rinaldi in [5], and leads to byproducts which are related to Catalan numbers, Motzkin numbers and Riordan numbers.

2. Hilly poor noncrossing partitions

Definition 2.1 *A hilly poor noncrossing partition is a poor noncrossing partition satisfying that each isolated vertex is covered by some arc in its linear presentation.*

Denote $\mathcal{H}(n)$ the set of hilly poor noncrossing partitions with n blocks. Let h_n and p_n be the cardinality of $\mathcal{H}(n)$ and $\mathcal{P}(n)$, respectively. By definitions, it is clear that $\mathcal{H}(n) \subset \mathcal{P}(n)$ and $p_n = 2h_n$ for $n \geq 1$ by changing the last uncovered isolated vertex into an arc whose another vertex is a new added vertex on the leftmost position. Suppose that $p(x)$ and $h(x)$ are their generating functions, respectively. It is trivial to derive their relations as follows:

$$p(x) = 1 + xp(x) + xp^2(x), \quad h(x) = 1 + xp(x)h(x).$$

So the generating function of h_n is

$$h(x) = \frac{1 + x - \sqrt{x^2 - 6x + 1}}{4x}. \tag{2.1}$$

It is interesting to find that $xh(x)$ is exactly the generating function of *Maximal Davenport-Schinzel sequences* over n symbols which have been studied by [9] and [7]. We know that a Maximal Davenport-Schinzel sequence is the canonical sequential form of a 2-regular noncrossing partition satisfying that the least and largest elements are in the same block. In fact, the reduction algorithm in [1] gave a bijection between Maximal Davenport-Schinzel sequences and hilly poor noncrossing partitions. We briefly describe the reduction algorithm as follows without proof:

The reduction algorithm: For a m-regular ($m \geq 2$) partition with n ($n \geq 1$) blocks, in its linear representation, for each arc joining the i-th vertex with the j-th vertex, replace it by a new arc joining the i-th vertex with the $j-1$-th vertex. In other words, the second vertex of each arc is changed into its frontal vertex. As a result, we obtain a linear representation of a $m-1$-regular partition with $n-1$ blocks by deleting the last vertex. For example, given $u = 1232141$, the reduction algorithm is demonstrated in Fig.2:

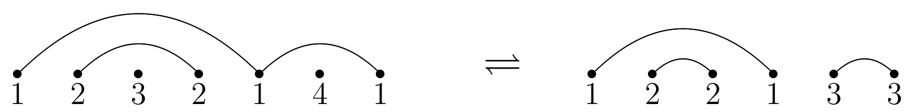

Figure 2: The reduction algorithm.

Using the generating function and algebraic methods in [9], Millin and Stanton derive an enumerative formula for Maximal Davenport-Schinzel sequences over n symbols, i.e.,

$$f_n = \sum_{0 \leq k \leq \lfloor n/2 \rfloor - 1} 3^{n-2-2k} 2^k \binom{n-2}{2k} C_k, \quad \text{for} \quad n \geq 2, \tag{2.2}$$

which is strictly related to the small schröder numbers: $1, 1, 3, 11, 45, 197, \ldots$ (A001003 in [10]). If we apply the reduction algorithm to 2-regular noncrossing partitions with n blocks whose canonical sequential forms are Maximal Davenport-Schinzel sequences over $n (n \geq 2)$ symbols, then the partitions obtained are hilly poor nocrossing partitions with $n-1$ blocks. Thus we can also obtain

$$h_n = f_{n+1} = \sum_{0 \leq k \leq \lfloor (n-1)/2 \rfloor} 3^{n-1-2k} 2^k \binom{n-1}{2k} C_k. \tag{2.3}$$

3. Refinement of hilly poor noncrossing partitions

In the above section, we derive the number of hilly poor noncrossing partitions with n blocks as the sum. A natural problem arising from the formula (2.3) is the character of parameter k. We hope to address k's combinatorial explanation in this section.

Definition 3.1 *Given a poor noncrossing partition $P = \{B_1, B_2, \ldots, B_n\}$. For any $j \in [n]$, let $I(i_1), I(i_2), \ldots, I(i_k)$ be all intervals satisfying that j is the largest number in each of them and $i_1 < i_2 < \cdots < i_k < j$. If $I(i_1)$ is included by another interval, then (B_{i_1}, B_j) forms a basic pair. Otherwise, (B_{i_2}, B_j) is called a basic pair if $k \geq 2$.*

If (B_i, B_j) is a basic pair, we call B_i an *out block* and B_j a *rightmost block*. It is obvious that B_j must be an isolated vertex or a void arc. Let $\mathcal{H}(n,k)$ and $h_{n,k}$ be the set of hilly poor noncrossing partitions of n blocks with k basic pairs and its cardinality, respectively, where $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$.

Definition 3.2 *In the linear representation of a poor noncrossing partition P, we delete all the blocks which do not form basic pairs, then the remaining graph is called the basic structure of P.*

Given a partition $P \in \mathcal{H}(n,k)$, its basic structure has exactly k non-void arcs. If we relabel vertices of the basic structure with $[2k]$, it is changed into a partition in $\mathcal{P}(2k)$. For example in Fig.3, (B_3, B_9) and (B_4, B_7) are basic pairs where B_3, B_4 are out blocks and B_9, B_7 are the corresponding rightmost blocks respectively. Whereas, (B_1, B_{10}), (B_6, B_7) and (B_{11}, B_{12}) are not basic pairs. $\{\{1,7\},\{2,4\},\{3\},\{5,6\}\}$ is the corresponding partition of its basic structure in $\mathcal{P}(4)$.

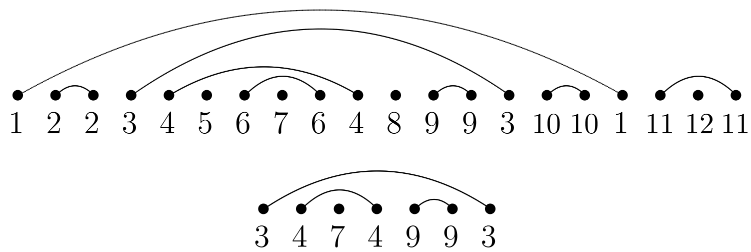

Figure 3: A poor noncrossing partition and its basic structure

Let $\mathcal{S}(k)$ be the set of basic structures of $\mathcal{H}(n,k)$, we construct a map η from $\mathcal{H}(n,k)$ onto $\mathcal{S}(k)$ which deduces the expression of $h_{n,k}$.

Theorem 3.3 *There exists a mapping η: $\mathcal{H}(n,k) \longmapsto \mathcal{S}(k)$ satisfying that for any $S \in \mathcal{S}(k)$, S is the basic structure of a partition P in $\mathcal{H}(n,k)$ if and only if $P \in \eta^{-1}(S)$. Moreover, $h_{n,k} = 3^{n-1-2k}|\mathcal{S}(k)|$.*

Proof. Given a partition $P \in \mathcal{H}(n,k)$, let $\eta(P)$ be its basic structure, and it is unique in $\mathcal{S}(k)$.

Now we construct the inverse mapping η^{-1}. Given a basic structure $S \in \mathcal{S}(k)$, suppose that labels of non-void arcs are $a_1, a_2, \ldots, a_k$ where $1 < a_i < a_{i+1}$, in which $a_{i_1}, a_{i_2}, \ldots, a_{i_s}$ are labels of big arcs and put $a_0 = 1$ and $a_{k+1} = n+2$. Let b_i be the label of the rightmost block corresponding to the non-void arc a_i. We determine every block's position and type from left to right. Draw an arc with label 1. Suppose that positions of the first $m-1$ blocks have been determined and b is the label of the last big arc. Now we consider the position of B_m:

If $m = a_{i_j}$ for some $j \in [s]$, we add the interval $I(m)$ of S to the inside and rightmost position of arc b. Since relative positions of blocks in S covered by arc m are also determined, we omit the case of $m = a_j$ or $m = b_j$ where $a_j \neq a_{i_t}$ for all $t \in [s]$.

If $a_j < m < a_{j+1}$ for some $j \in \{0, 1, 2, \ldots, k\}$, we consider the choices for B_m as follows: (i) if there does not exist any $l \in [k]$ such that $a_l \leq a_j < m < b_l$, we add an isolated vertex or a void arc to the inside and rightmost position of arc b, or draw an arc after arc b, which can be a big arc in the subsequent construction. Otherwise, let p be the greatest index such that $a_p \leq a_j < m < b_p$. (ii) If $b_p < a_{j+1}$, we draw an isolated vertex or a void arc before the block B_{b_p} or an arc covering the block B_{b_p}. (iii) If $b_p > a_{j+1}$, we draw an arc covering the block B_{b_p}, or draw an isolated vertex or a void arc before arc a_{j+1}.

Repeat above process until n blocks are all determined. From the above procedure, it is easy to see that any isolated vertex must be covered by at least one arc. Moreover, in each step determining positions and types of blocks, in order not to destroy its basic structure, each block whose label is neither a_i nor b_i has exactly 3 choices except for the first block. So we have just 3^{n-1-2k} partitions in $\mathcal{H}(n,k)$ having the same basic structure S. ∎

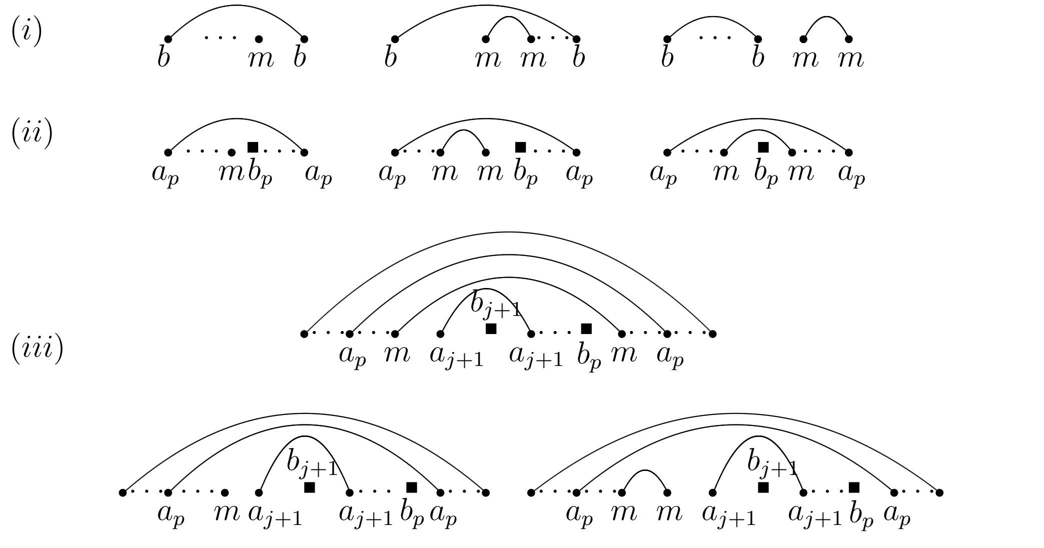

Figure 4: Three choices for the m-th block.

Remark 1: “A little black square” represents a rightmost block in Fig.4.

Remark 2: “...” represents some possible existed labelled vertices in Fig.4.

It is clear that a basic structure $S \in \mathcal{S}(k)$ is derived from $\mathcal{P}_2(k)$: given a partition $P \in \mathcal{P}_2(k)$, for each arc in the linear representation of P, add an isolated vertex or a void arc to the inside and rightmost position of the arc, and choose $2k$ elements from the set $\{2, 3, \ldots, n\}$ to label these blocks from left to right in increasing order. Thus we obtain a basic structure S, and $|\mathcal{S}(k)| = 2^k \binom{n-1}{2k} C_k$ is derived. From the constructive mapping in Theorem 3.3, we can use basic pairs to characterize the parameter k in Formula (2.3).

Theorem 3.4 *The number of hilly poor noncrossing partitions of n blocks with k basic pairs is*

$$h_{n,k} = 3^{n-1-2k} 2^k \binom{n-1}{2k} C_k, \quad \text{for} \quad n \geq 1. \tag{3.1}$$

So we give a combinatorial proof of Formula (2.2) combining Theorem 3.4 with the reduction algorithm. From Theorem 3.4, we deduce

Corollary 3.5 *For $n \geq 1$, $3^{n-1-2k}\binom{n-1}{2k} C_k$ counts the number of hilly poor noncrossing partitions of n blocks with k basic pairs satisfying that all rightmost blocks are of the same type.*

We can use another method to enumerate the number of hilly poor noncrossing partitions of n blocks satisfying that all rightmost blocks are void arcs.

Lemma 3.6 *The number of partitions in $\mathcal{H}(n+1)$ satisfying that all rightmost blocks are arcs is*

$$\sum_{k=0}^{n} \binom{n}{k} C_{k+1}.$$

Proof. Given such a partition P, suppose that there are exactly $k+1$ arcs in its linear representation, then the $k+1$ arcs correspond to the linear representation of a partition in $\mathcal{P}_2(k+1)$.

Conversely, given a partition P in $\mathcal{P}_2(k+1)$, we insert $n-k$ isolated vertices into positions between arcs to consist a partition in $\mathcal{H}(n+1)$ with $k+1$ arcs such that each rightmost block is an arc. In fact, these $n-k$ isolated vertices only can be inserted into $k+1$ positions which are the inside and rightmost positions of big arcs and the frontal positions of the first vertices of non big arcs. So there are $\binom{n}{k}$ methods to choose positions of isolated vertices. Summing over k, the lemma is proved. ∎

Theorem 3.7 *For $n \geq 0$,*

$$\sum_{k=0}^{n} \binom{n}{k} C_{k+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 3^{n-2k} \binom{n}{2k} C_k. \tag{3.2}$$

4. A bijection π between (2, 3)-Motzkin paths and hilly poor noncrossing partitions

It is well known that Motzkin paths are lattice paths in the plane using up steps $(1,1)$ (U), down steps $(1,-1)(D)$, and horizontal steps $(1,0)$ (H) and running from $(0,0)$ to $(n,0)$ that never pass below x-axis [11]. Motzkin paths of length n are counted by $M_n = \sum_{0\le k\le\lfloor n/2\rfloor}\binom{n}{2k}C_k$ [4]. If (i,j) and $(i+1,j+1)$ are the coordinates of an up step, we say the up step is at *level* j. Similarly, a horizontal step whose coordinates are (i,j) and $(i+1,j)$ is said to be at *level* j, and a down step whose coordinates are (i,j) and$(i+1,j-1)$ is called to be at *level* $j-1$. For each up step, its *corresponding down step* is the first down step at the same level to the right.

In this section we consider a class of generalized Motzkin paths, which is closely related to hilly poor noncrossing partitions.

Definition 4.1 *A Motzkin path is a (i,j)-Motzkin path if each up step is colored by one letter of alphabet $\{u_1,u_2,\ldots,u_i\}$, and each horizontal step is colored by one letter of alphabet $\{h_1,h_2,\ldots,h_j\}$.*

Denote $M_{i,j}(n)$ the number of (i,j)-Motzkin paths of length n. Note that $(1,1)$-Motzkin paths are ordinary Motzkin paths, $(1,2)$-Motzkin paths are 2-Motzkin paths [3], and $(1,3)$-Motzkin paths are 3-colored Motzkin paths [12]. It is trivial to derive $M_{i,j}(n)$ and its generating function $M_{i,j}(x)$:

$$M_{i,j}(n) = \sum_{0\le k\le\lfloor\frac{n}{2}\rfloor} i^k j^{n-2k}\binom{n}{2k}C_k,$$

$$M_{i,j}(x) = 1 + jxM_{i,j}(x) + ix^2M_{i,j}^2(x),$$

$$M_{i,j}(x) = \frac{1 - jx - \sqrt{1-2jx+j^2x^2-4ix^2}}{2ix^2}.$$

In particular,

$$M_{2,3}(x) = \frac{1-3x-\sqrt{x^2-6x+1}}{4x^2}, \tag{4.1}$$

$$M_{2,3}(n) = \sum_{0\le k\le\lfloor\frac{n}{2}\rfloor} 2^k3^{n-2k}\binom{n}{2k}C_k. \tag{4.2}$$

Comparing Formula (2.1) with (4.1) and (2.3) with (4.2), we find that $h(x) = 1+xM_{2,3}(x)$ and $h_{n+1} = M_{2,3}(n)$. A natural problem is to find a bijection between $(2,3)$-Motzkin paths of length n and hilly poor noncrossing partitions with $n+1$ blocks.

Theorem 4.2 *There exists a bijection π between $(2,3)$-Motzkin paths of length n and $\mathcal{H}(n+1)$. If a $(2,3)$-Motzkin path P has k up steps, $\pi(P)$ has exactly k basic pairs.*

Proof. Given a $(2,3)$-Motzkin path M of length n with k up steps, we can define its image $\pi(M)$ as follows: label the steps with $2, 3, \ldots, n+1$ from left to right. Suppose that labels of up steps are $a_1, a_2, \ldots, a_k$ which are listed in increasing order and labels of corresponding down steps are $b_1, b_2, \ldots, b_k$. Assume that the horizontal steps at level 0 colored by h_1 are labelled by $c_1, c_2, c_3, \ldots, c_t$ where $c_i < c_{i+1}$ and $c_{t+1} = n+2$. For our convenience, step $\{x, y\}$ denotes that the step is colored by x and labelled by y. Replace each up step $\{u_1, a_i\}$ and its corresponding down step with a basic pair (B_{a_i}, B_{b_i}) such that B_{b_i} is a void arc and B_{a_i} covers all the blocks B_j where $a_i < j \leq b_i$. Replace each up step $\{u_2, a_i\}$ and its corresponding down step with a basic pair (B_{a_i}, B_{b_i}) such that B_{b_i} is an isolated vertex and B_{a_i} covers all the blocks B_j where $a_i < j \leq b_i$. Replace each horizontal step $\{h_2, i\}$ with an isolated vertex B_i. Replace each horizontal step $\{h_3, i\}$ with a void arc B_i. For all $1 \leq j \leq t$, replace each horizontal step $\{h_1, c_j\}$ with a big arc which covers all the blocks B_i where $c_j < i < c_{j+1}$. For each horizontal step $\{h_1, i\}$ at level $l+1$, there exists a largest index j such that $a_j < i < b_j$ and the step labelled by a_j is at level l. We replace each such step with an arc which covers the block B_{b_j} preserving its noncrossing property. Draw the first block as a big arc which covers all the blocks B_i where $1 < i < c_1$. Thus we get a partition $\pi(M) \in \mathcal{H}(n+1, k)$.

Conversely, for each basic pair (B_i, B_j) of $P \in \mathcal{H}(n+1, k)$, if B_j is a void arc, then B_i corresponds to an up step colored by u_1 and B_j corresponds to a down step. Otherwise, B_i corresponds to an up step colored by u_2 and B_j corresponds to a down step. For the remaining blocks, an isolated vertex corresponds to a horizontal step colored by h_2; a void arc covered by at least one arc corresponds to a horizontal step colored by h_3; a big arc or a non-void arc corresponds to a horizontal step colored by h_1. Deleting the first step and encoding all the steps from left to right, we get a $(2,3)$-Motzkin path $\pi^{-1}(P)$ of length n with k up steps. ∎

For instance, $p = 122345647731898$ and the corresponding $(2,3)$-Motzkin path is shown in Fig. 5.

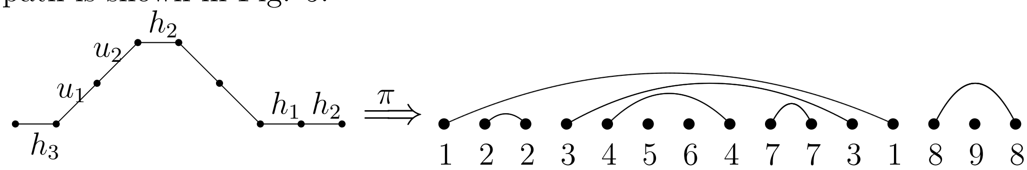

Figure 5: An example of the bijection π.

5. Specialization of the bijection π

In this section, applying the bijection π to some specialization of $(2,3)$-Motzkin paths, some known results are derived, and we give another combinatorial explanation for 3-colored Motzkin paths, Catalan numbers, Motzkin numbers and Riordan numbers, respectively.

Corollary 5.1 ([3]) *There exists a bijection π between 2-Motzkin paths of length n and $\mathcal{P}_2(n+1)$. Thus the number of 2-Motzkin paths of length n is C_{n+1}.*

Proof. Since $\mathcal{P}_2(n+1) \subset \mathcal{H}(n+1)$, if we apply the bijection π^{-1} to $\mathcal{P}_2(n+1)$, we get the set of all 2-Motzkin paths of length n colored by u_1, h_1 or h_3. ∎

For a $(2,3)$-Motzkin path, if each up step is colored by u_1 and each horizontal step is colored by h_1, h_2 or h_3, we obtain a 3-colored Motzkin path. The first numbers of 3-colored Motzkin paths are $1, 3, 10, 36, 137, 543, 2219, 9285, \ldots$ (A002212 in [10]).

Corollary 5.2 *There exists a bijection π between 3-colored Motzkin paths of length n and hilly poor noncrossing partitions with $n+1$ blocks satisfying that all rightmost blocks are void arcs. Moverover, the number of 3-colored Motzkin paths of length n is $\sum_{k=0}^{n} \binom{n}{k} C_{k+1}$.*

By the above corollary, the identity (3.2) can also be proved. Note that the fact that $\sum_{k=0}^{n} \binom{n}{k} C_{k+1}$ counts the number of 3-colored Motzkin paths of length n has been proved by [5] using generating function.

View 2-Motzkin paths as $(1,2)$-Motzkin paths colored by u_2, h_1 or h_2. Similar to the bijection between Dyck paths of length $2n+2$ and 2-Motzkin paths of length n in [3], a bijection between Dyck paths of length $2n$ and 2-Motkzin paths of length n without horizontal steps colored by h_1 at x-axis is the following: for a given 2-Motzkin path M we replace U by UU, D by DD, a horizontal step colored with h_1 by DU, and a horizontal step colored with h_2 by UD. Note that M has no horizontal steps colored by h_1 at x-axis if and only if the obtained path is a Dyck path of length $2n$. Applying π to such 2-Motzkin paths and deleting the first arc, we obtain

Corollary 5.3 ([6]) *The n-th Catalan number counts the number of all the 2-regular poor noncrossing partitions with n blocks.*

Applying π to an ordinary Motzkin path M of length n colored by u_2 and h_2, and deleting the first block, we get a 2-regular poor noncrossing partitions with n blocks satisfying that if b, c are in the same block, then $c+1$ must be the minimal element in some block. This leads to a new combinatorial interpretation of Motzkin numbers.

Corollary 5.4 *The n-th Motzkin number counts the number of all the 2-regular poor noncrossing partitions with n blocks such that if b, c are in the same block and $b < c$, then $c+1$ must be the minimal element in some block.*

Riordan numbers count the numbers of Motzkin paths without horizontal steps on the x-axis [2]. The first Riordan numbers are $1, 0, 1, 1, 3, 6, 15, 36 \ldots$ (A005043 in [10]). Hence, we lead to another combinatorial explanation of Riordan number by applying π to such Motzkin paths.

Corollary 5.5 *The n-th Riordan number counts the number of all the 2-regular hilly poor noncrossing partitions with n blocks such that if b, c are in the same block and $b < c$, then $c + 1$ must be the minimal element in some block.*

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