TWO INVOLUTIONS ON VERTICES OF ORDERED TREES

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Abstract. We will give two natural involutions on the set of pointed ordered trees. These involutions strongly rely on the fact that the pointed ordered trees are embedded in the plane. Using these involutions, we provide combinatorial bijections between the terminal and the internal vertices of ordered trees, and then we deal with some counting problems on the pointed ordered trees.

Résumé. Nous donnons deux involutions naturelles sur l'ensemble des arbres pointés planaires. Ces deux involutions reposent fortement sur le fait que les arbres planaires pointés sont plongés dans le plan. En utilisant ces involutions, nous donnons des bijections entre les sommets internes et externes des arbres planaires. Nous donnons des résultats sur certains de leurs énumérations.

1. Introduction

An *ordered tree* is a rooted tree in which children of each vertex are ordered. A *pointed ordered tree* is an ordered tree one of whose vertex is pointed, where 'pointed' means distinguishing one vertex. For an ordered tree $\mathbf{o}$ and a vertex v in $\mathbf{o}$, let $\mathbf{o}^v$ denote the pointed ordered tree with v pointed. Let $\mathcal{O}_n$ be the set of all ordered trees with n edges and $\mathcal{O}_n^\bullet$ be the set of all pointed ordered trees with n edges. Figure 1 shows all the elements of $\mathcal{O}_3$ with roots at the top.

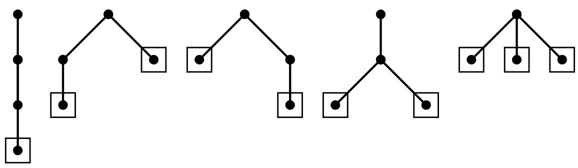

Figure 1. Ordered trees with 3 edges

A pointed ordered tree $\mathbf{o}^v$ is called *terminal*, if v is a *leaf* in $\mathbf{o}^v$, otherwise, *internal*. Let $\mathcal{O}_n^-$ be the set of all terminal pointed ordered trees in $\mathcal{O}_n^\bullet$ and $\mathcal{O}_n^+$ the set of all internal pointed ordered trees. Then these two sets have the same cardinality, as illustrated in Figure 1, where there are the same number of boxed and unboxed vertices.

Many researchers proved $|\mathcal{O}_n^+| = |\mathcal{O}_n^-|$ with various combinatorial methods. Dasarathy and Yang [3] gave combinatorial explanations by Knuth natural correspondence [7, pp. 332–333]. Chauve [2], Deutsch [5], and Seo [8] introduced proofs in Dyck path version independently. These proofs used binary trees, Dyck paths, or some other objects, but no proof has been built directly on ordered trees.

In this paper, we will give two *natural* bijections on $\mathcal{O}_n^\bullet$. These bijections strongly rely on the fact that pointed ordered trees are embedded in the plane. We will analyze the set $\mathcal{O}_n^\bullet$ and find some interesting formulas.

2. Preliminary

We adopt conventional terminology on trees as in [1], [6]. Let $\mathcal{T}_n$ be the set of all trees with n edges and $\mathcal{T}_n^\bullet$ the set of pointed trees with n edges. Every tree can be embedded in a plane without edge-crossing. A tree embedded in the plane is called a *plane tree*. Two plane trees are considered to be the same, if they can be made identical by an orientation preserving homeomorphism defined

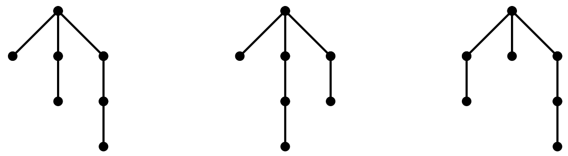

FIGURE 2. Trees, Plane Trees, Ordered Trees

on the plane. Let $\mathcal{P}_n$ be the set of all plane trees with n edges and $\mathcal{P}_n^\bullet$ the set of all pointed plane trees with n edges.

An *ordered tree* is a plane tree which has a distinguished vertex (called the *root*) and a distinguished edge (called the *leftmost edge*) which is incident with the root. A tree consisting of one vertex is considered to be an ordered tree. This definition of ordered tree is equivalent to the original definition which mentioned in previous section. Two ordered trees are the same, if there is an orientation preserving homeomorphism which preserves the root and the leftmost edge. Let $\mathcal{O}_n$ be the set of all ordered trees with n edges and $\mathcal{O}_n^\bullet$ the set of all pointed ordered trees with n edges.

In Figure 2, we illustrate three different ordered trees with roots at the top. As plane trees, two are the same, the second and the third, and all three are the same as trees.

Let $\mathbf{o}$ be an ordered tree. For each vertex v of $\mathbf{o}$, let $\mathrm{p}(v)$ be the unique path from the root to v. For the non-root vertex v, the *parent* of v is the neighbor of v in $\mathrm{p}(v)$, and any neighbor of v in $\mathbf{o}$, which is not the parent of v, is called a *child* of v. A vertex v in an ordered tree $\mathbf{o}$ is called a *terminal vertex or leaf*, if it has no child, and an *internal vertex*, otherwise. The vertex in the ordered tree consisting of one vertex is considered as a terminal vertex. We call $\mathbf{o}^v$ *terminal*, if v is a terminal vertex in $\mathbf{o}$, and *internal*, if v is an internal vertex. Let $\mathcal{O}_n^+$ denote the set of all internal pointed ordered trees, and let $\mathcal{O}_n^-$ denote the set of all terminal pointed ordered trees, i.e. $\mathcal{O}_n^- = \mathcal{O}_n^\bullet \setminus \mathcal{O}_n^+$. A vertex w in an ordered tree $\mathbf{o}$ is called a *descendant* of a vertex v, if $\mathrm{p}(v) \subsetneq \mathrm{p}(w)$ holds.

Given a pointed ordered tree $\mathbf{o}^v$, let $\mathrm{d}(\mathbf{o}^v)$, the *descendant subtree of* $\mathbf{o}^v$, denote the induced subtree of $\mathbf{o}^v$ consisting of all descendants of v and v itself as a root, and let $\bar{\mathrm{d}}(\mathbf{o}^v)$, the *non-descendant subtree of* $\mathbf{o}^v$, denote the induced subtree of $\mathbf{o}^v$ consisting of the vertices which are not descendants of v.

Let $\mathbf{o}^v$ be a pointed ordered tree in which v is not the root of $\mathbf{o}$, and u be the parent of v. Define $\mathrm{l}(\mathbf{o}^v)$ (resp. $\mathrm{r}(\mathbf{o}^v)$), the *left (resp. right)-descendant subtree of* $\mathbf{o}^v$ by the subtree of $\mathrm{d}(\mathbf{o}^u)$ consisting of u as a root and all descendants of u to the left (resp. right) of the edge $\{u, v\}$. Finally, let $\bar{\mathrm{l}}(\mathbf{o}^v)$ (resp. $\bar{\mathrm{r}}(\mathbf{o}^v)$) be the union of $\mathrm{r}(\mathbf{o}^v)$ (resp. $\mathrm{l}(\mathbf{o}^v)$), $\bar{\mathrm{d}}(\mathbf{o}^u)$, $\mathrm{d}(\mathbf{o}^v)$ and the edge $\{u, v\}$ (see Figure 3). In fact, $\mathrm{d}(\mathbf{o}^v)$ (resp. $\mathrm{l}(\mathbf{o}^v)$, $\mathrm{r}(\mathbf{o}^v)$) is the edge complement of $\bar{\mathrm{d}}(\mathbf{o}^v)$ (resp. $\bar{\mathrm{l}}(\mathbf{o}^v)$, $\bar{\mathrm{r}}(\mathbf{o}^v)$).

3. INVOLUTIONS L AND R ON POINTED ORDERED TREES

In this section, we assume that $n > 0$. So if $\mathbf{o}^v \in \mathcal{O}_n^+$, then v has at least one child, and if $\mathbf{o}^v \in \mathcal{O}_n^-$, then v has a parent.

We can define a mapping $L^\mp : \mathcal{O}_n^- \to \mathcal{O}_n^+$ as follows (Figure 4): Given $\mathbf{o}^v \in \mathcal{O}_n^-$, where u is the parent of v, we first cut $\mathrm{l}(\mathbf{o}^v)$ from $\mathbf{o}^v$, and paste it to $\bar{\mathrm{l}}(\mathbf{o}^v)$ by identifying $u \in \mathrm{l}(\mathbf{o}^v)$ and $v \in \bar{\mathrm{l}}(\mathbf{o}^v)$. Let $\widetilde{\mathbf{o}}^u$ be the resulting pointed ordered tree. Note that u is an internal vertex in $\widetilde{\mathbf{o}}^u$, so $\widetilde{\mathbf{o}}^u \in \mathcal{O}_n^+$. Set $L^\mp(\mathbf{o}^v) = \widetilde{\mathbf{o}}^u$.

Conversely, we can also define a mapping $L^\pm : \mathcal{O}_n^+ \to \mathcal{O}_n^-$ as follows: Given $\mathbf{o}^v \in \mathcal{O}_n^+$, where w is the leftmost child of v, we first cut $\mathrm{d}(\mathbf{o}^w)$ from $\mathbf{o}^v$, and paste it to $\bar{\mathrm{d}}(\mathbf{o}^w)$ by identifying $w \in \mathrm{d}(\mathbf{o}^w)$ and $v \in \bar{\mathrm{d}}(\mathbf{o}^w)$, and put $\mathrm{d}(\mathbf{o}^w)$ on the left of the edge $\{v, w\} \in \bar{\mathrm{d}}(\mathbf{o}^w)$. Let $\widehat{\mathbf{o}}^w$ be the resulting pointed ordered tree. Note that w is a terminal vertex in $\widehat{\mathbf{o}}^w$, so $\widehat{\mathbf{o}}^w \in \mathcal{O}_n^-$. Set $L^\pm(\mathbf{o}^v) = \widehat{\mathbf{o}}^w$. Now define $L : \mathcal{O}_n^\bullet \to \mathcal{O}_n^\bullet$ by

$$L(\mathbf{o}^v) = \begin{cases} L^\pm(\mathbf{o}^v), & \text{if } \mathbf{o}^v \in \mathcal{O}_n^+, \\ L^\mp(\mathbf{o}^v), & \text{if } \mathbf{o}^v \in \mathcal{O}_n^-. \end{cases}$$

Figure 5 shows how L maps $\mathcal{O}_3^-$ to $\mathcal{O}_3^+$.

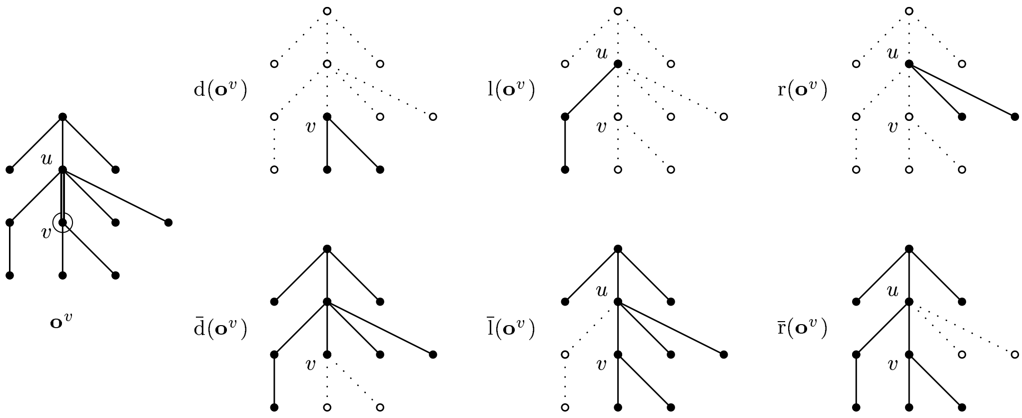

FIGURE 3. Decomposition of $\mathbf{o}^v$

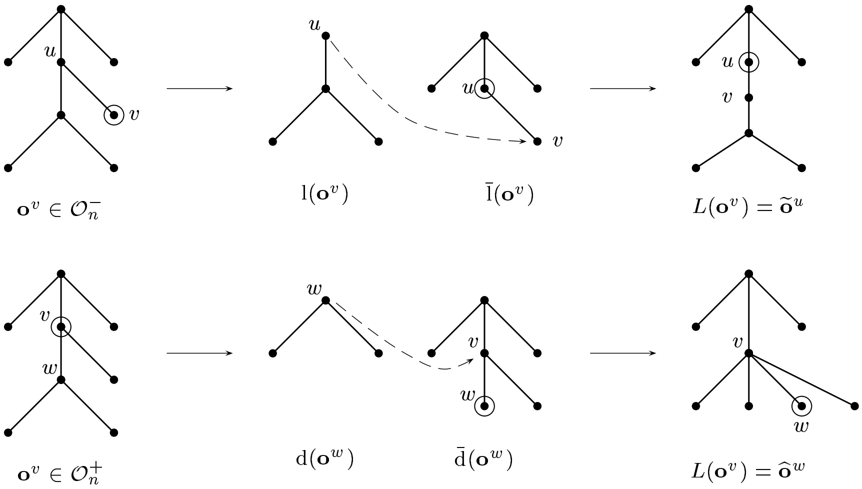

FIGURE 4. L map on $\mathcal{O}_n^-$ and $\mathcal{O}_n^+$

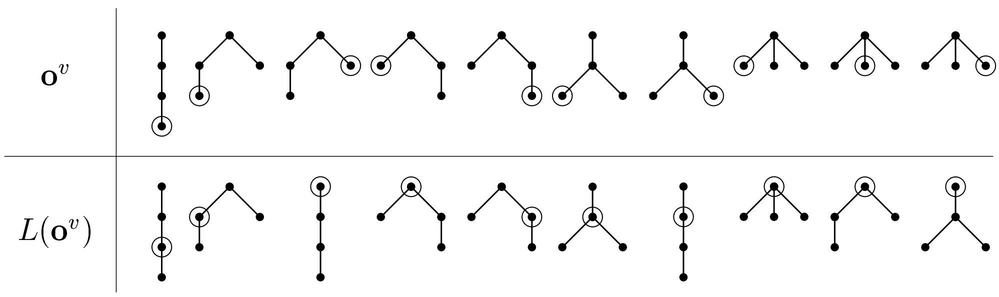

FIGURE 5. Correspondence in $\mathcal{O}_3^\bullet$ by L

By replacing left with right in the definition of L, we can define $R : \mathcal{O}_n^\bullet \to \mathcal{O}_n^\bullet$ similarly.

Theorem 1. *The maps L and R are* involutions *in $\mathcal{O}_n^\bullet$ with $L(\mathcal{O}_n^-) = \mathcal{O}_n^+$ and $R(\mathcal{O}_n^-) = \mathcal{O}_n^+$. So L and R are bijections from $\mathcal{O}_n^-$ to $\mathcal{O}_n^+$.*

Proof. It suffices to show that $(L^\mp)^{-1} = L^\pm$. Let $\mathbf{o}^v$ be an arbitrary tree in $\mathcal{O}_n^-$, u be the parent of v, and $\widetilde{\mathbf{o}}^u$ be the image of $\mathbf{o}^v$ under $L^\mp$. Then, from the definition of $L^\mp$, we obtain $\mathrm{l}(\mathbf{o}^v) = \mathrm{d}(\widetilde{\mathbf{o}}^v)$ and $\bar{\mathrm{l}}(\mathbf{o}^v) = \bar{\mathrm{d}}(\widetilde{\mathbf{o}}^v)$. Observe that v is the leftmost child of u in $\widetilde{\mathbf{o}}^u$. Let $L^\pm(\widetilde{\mathbf{o}}^u) = \widehat{\mathbf{o}}^v$. Then, from the definition of $L^\pm$, we obtain $\mathrm{d}(\widetilde{\mathbf{o}}^v) = \mathrm{l}(\widehat{\mathbf{o}}^v)$ and $\bar{\mathrm{d}}(\widetilde{\mathbf{o}}^v) = \bar{\mathrm{l}}(\widehat{\mathbf{o}}^v)$. So we get $\mathrm{l}(\mathbf{o}^v) = \mathrm{l}(\widehat{\mathbf{o}}^v)$ and $\bar{\mathrm{l}}(\mathbf{o}^v) = \bar{\mathrm{l}}(\widehat{\mathbf{o}}^v)$, which yield $\mathbf{o}^v = \widehat{\mathbf{o}}^v$, and so $L^\pm \circ L^\mp$ is the identity. □

Moreover, from Theorem 1, we can easily get the following result. The statement about the average level previously appeared in [4].

Corollary 2. *For $\mathbf{o}^v \in \mathcal{O}_n^\bullet$, define the* level *of $\mathbf{o}^v$, denoted by $\rho(\mathbf{o}^v)$, to be the number of edges in the path $\mathrm{p}(v)$. If $\rho(\mathbf{o}^v) = m$, then*

$$\rho(L(\mathbf{o}^v)) = \rho(R(\mathbf{o}^v)) = \begin{cases} m+1, & \text{if } \mathbf{o}^v \text{ is internal,} \\ m-1, & \text{if } \mathbf{o}^v \text{ is terminal.} \end{cases}$$

Consequently, the average level of terminal $\mathbf{o}^v$'s is greater than the average level of internal $\mathbf{o}^v$'s by 1.

4. Various Enumerations on $\mathcal{O}_n^\bullet$

4.1. A group action on $\mathcal{O}_n^\bullet$.

For simplicity, we define L and R on $\mathcal{O}_0^\bullet$ to be the identity map. Let $\mathbf{G}$ be the group generated by L and R with composition as the operation. Since L and R are involutions, $\mathbf{G}$ has the following presentation.

$$\mathbf{G} = \langle L, R : L^2 = 1,\ R^2 = 1 \rangle.$$

The group $\mathbf{G}$ acts on $\mathcal{O}_n^\bullet$ by

$$G \cdot \mathbf{o}^v = G(\mathbf{o}^v) \quad \text{for all } \mathbf{o}^v \in \mathcal{O}_n^\bullet \text{ and all } G \in \mathbf{G}.$$

From now on, we write $G\mathbf{o}^v$ for $G(\mathbf{o}^v)$. $\mathcal{O}_n^\bullet$ is partitioned into G-orbits, we are interested in finding *the number of distinct* $\mathbf{G}$*-orbits* and *the size of each orbit*.

Clearly, every orbit has an even number of elements and exactly half of them are in $\mathcal{O}_n^+$. For any element $\mathbf{o}^v$ in $\mathcal{O}_n^\bullet$, let $[\mathbf{o}^v]$ denote the $\mathbf{G}$-orbit of $\mathbf{o}^v$, and set $[\mathbf{o}^v]^+ = [\mathbf{o}^v] \cap \mathcal{O}_n^+$ and $[\mathbf{o}^v]^- = [\mathbf{o}^v] \cap \mathcal{O}_n^-$. Since $[\mathbf{o}^v]^+$ and $[\mathbf{o}^v]^-$ are equinumerous, $|[\mathbf{o}^v]| = 2|[\mathbf{o}^v]^+|$. Let $\mathbf{H}$ be the subgroup of $\mathbf{G}$ generated by RL, i.e. $\mathbf{H} = \langle RL \rangle$. If $\mathbf{o}^v$ is chosen from $\mathcal{O}_n^+$, then we can easily see that

$$[\mathbf{o}^v]^+ = \{H\mathbf{o}^v \mid H \in \mathbf{H}\}.$$

And the number of $\mathbf{H}$-orbits in $\mathcal{O}_n^+$ equals the number of $\mathbf{G}$-orbits in $\mathcal{O}_n^\bullet$. So instead of the $\mathbf{G}$-action on $\mathcal{O}_n^\bullet$, we will discuss the $\mathbf{H}$-action on $\mathcal{O}_n^+$. The following proposition shows how RL acts on $\mathbf{o}^v \in \mathcal{O}_n^+$ more intuitively.

Proposition 3. *For each $\mathbf{o}^v \in \mathcal{O}_n^+$, $RL\mathbf{o}^v$ is obtained as follows: Let w be the leftmost child of v. First cut $\mathrm{d}(\mathbf{o}^v)$ from $\mathbf{o}^v$, exchange labels v and w in $\mathrm{d}(\mathbf{o}^v)$, and finally identify v in $\mathrm{d}(\mathbf{o}^v)$ and the original v in $\bar{\mathrm{d}}(\mathbf{o}^v)$. The resulting pointed ordered tree is $RL\mathbf{o}^v$.*

Proof. This is apparent from the definition of L and R. See Figure 6. □

4.2. Enumeration of H-orbits.

The Proposition 3 enable us to compute the number of distinct $\mathbf{H}$-orbits in $\mathcal{O}_n^+$. Recall that $\mathcal{P}_n$ is the set of all plane trees with n edges. Let p_n denote the cardinality of $\mathcal{P}_n$.

Theorem 4. *Let orb_n be the number of distinct $\mathbf{H}$-orbits in $\mathcal{O}_n^+$. Then,*

$$orb_n = p_n + \sum_{k=1}^{n-1} \binom{2k-1}{k} p_{n-k}. \tag{4.1}$$

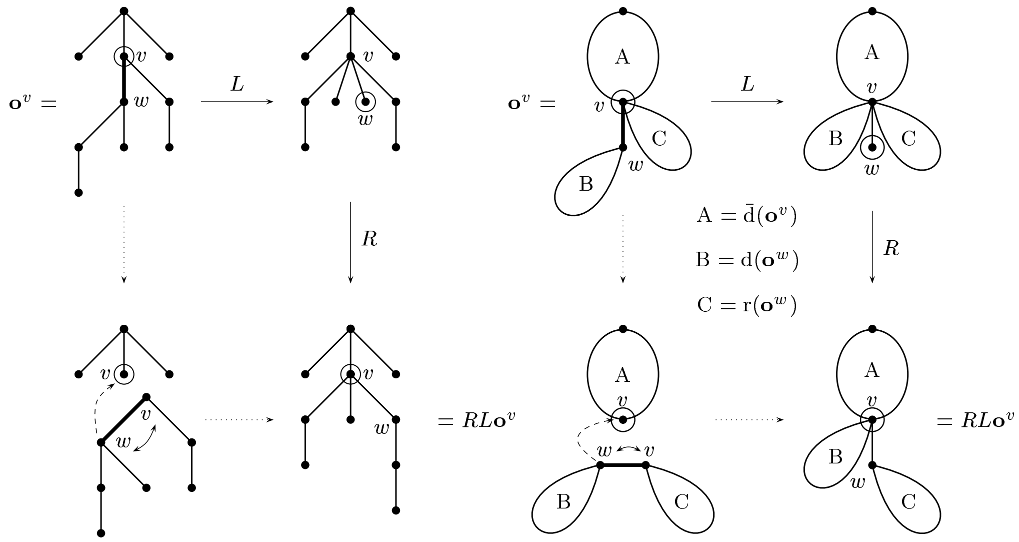

FIGURE 6. How RL acts on a given $\mathbf{o}^v \in \mathcal{O}_n^+$

Proof. Let U be the natural map from $\mathcal{O}_n$ to $\mathcal{P}_n$ by just *forgetting* the root and the initial edge. By Proposition 3, $\bar{\mathrm{d}}(RL\mathbf{o}^v) = \bar{\mathrm{d}}(\mathbf{o}^v)$ and $U(\mathrm{d}(RL(\mathbf{o}^v))) = U(\mathrm{d}(\mathbf{o}^v))$. So two orbits $[\mathbf{o}^v]^+$ and $[\check{\mathbf{o}}^w]^+$ are different if and only if $\bar{\mathrm{d}}(\mathbf{o}^v) \neq \bar{\mathrm{d}}(\check{\mathbf{o}}^w)$ or $U(\mathrm{d}(\mathbf{o}^v)) \neq U(\mathrm{d}(\check{\mathbf{o}}^w))$. Assume that $\bar{\mathrm{d}}(\mathbf{o}^v)$ has k edges. Since v has no descendants in $\bar{\mathrm{d}}(\mathbf{o}^v)$, $\bar{\mathrm{d}}(\mathbf{o}^v)$ is an element of $\mathcal{O}_k^-$, and clearly $U(\mathrm{d}(\mathbf{o}^v))$ is an element of $\mathcal{P}_{n-k}$. Then the number of distinct orbits is the sum

$$\sum_{k=0}^{n-1} |\mathcal{O}_k^-| \cdot |\mathcal{P}_{n-k}|.$$

Now (4.1) follows, since $|\mathcal{O}_k| = c_k = \frac{1}{k+1}\binom{2k}{k}$, where c_k denotes the k-th Catalan number, and for $k > 0$, $|\mathcal{O}_k^-| = \frac{1}{2}(k+1)|\mathcal{O}_k| = \binom{2k-1}{k}$, and $|\mathcal{O}_0^-| = 1$. □

Let $\mathcal{P}(x)$ denote the ordinary generating function for p_n, and $\mathcal{O}(x)$ for Catalan number c_n. Then by dissymmetry Theorem for trees [1, Theorem 4.1.1],

$$\mathcal{P}(x) = 1 + \sum_{n\geq 1} \frac{\phi(n)}{n} \log \frac{1}{1 - x^n \mathcal{O}(x^n)} + \frac{x}{2}\left(\mathcal{O}(x^2) - \mathcal{O}^2(x)\right)$$

and

$$p_n = \frac{1}{2n}\sum_{d|n} \phi\left(\frac{n}{d}\right)\binom{2d}{d} - \frac{1}{2}c_n + \frac{1}{2}\chi_{\mathrm{odd}}(n)\, c_{\frac{n-1}{2}}, \tag{4.2}$$

where ϕ is the Euler's totient function and χ_{odd} is the characteristic function of odd integers. From (4.1) and (4.2), we can have the summation form of orb_n, but we can not find a simple formula. The sequence $\{orb_n\}_{n=0}^{\infty}$ starts with 1, 1, 2, 6, 18, 60, 210, 754, 2766, 10280, 38568, … , and it does not appear in On-Line Encyclopedia of Integer Sequences [9].

4.3. **Counting the cardinality of an orbit.**

Given $\mathbf{o}^v \in \mathcal{O}_n^+$, we compute the size of $[\mathbf{o}^v]^+$. For a plane tree $\mathbf{p}$ having at least one edge, we define $c(\mathbf{p})$, the *center* of $\mathbf{p}$, which appears in [1], as follows :

(1) Delete all the leaves in $\mathbf{p}$.
(2) Iterate (1), until the resulting plane tree has at most one internal vertex.
(3) If the resulting plane tree has no internal vertex (one edge), then the edge is $c(\mathbf{p})$.

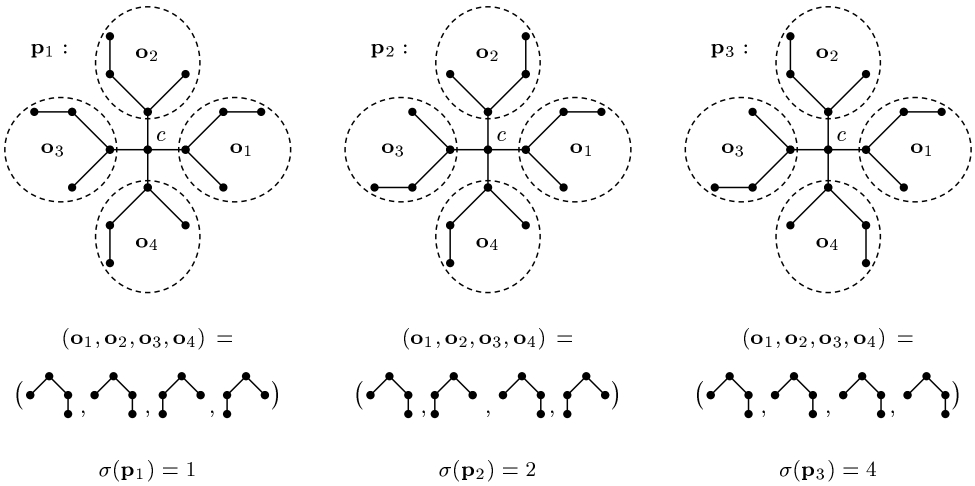

FIGURE 7. $\sigma(\mathbf{p})$: When $c(\mathbf{p})$ is a vertex.

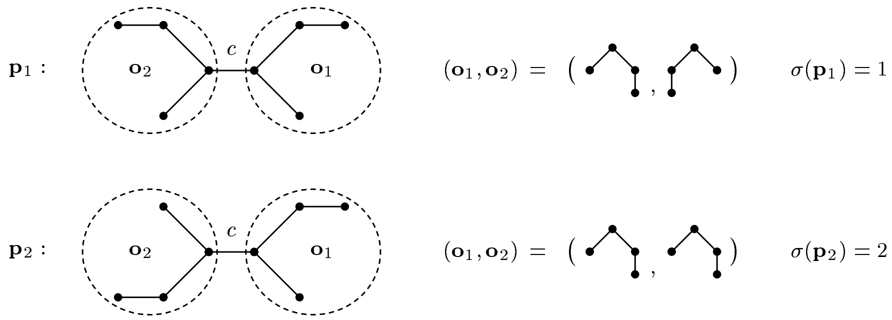

FIGURE 8. $\sigma(\mathbf{p})$: When $c(\mathbf{p})$ is an edge.

(4) If the resulting tree has a unique internal vertex (star shape), then the internal vertex is $c(\mathbf{p})$.

If we delete[1] $c(\mathbf{p})$ from $\mathbf{p}$, $\mathbf{p}\setminus c(\mathbf{p})$ is decomposed into several components. In fact, each component is an ordered tree whose root is a neighbor of the center. We fix one component $\mathbf{o}_1$ and label other components counterclockwise $\mathbf{o}_2, \dots, \mathbf{o}_i$, where i is the number of components of $\mathbf{p} \setminus c(\mathbf{p})$. Let j be the smallest positive integer such that $(\mathbf{o}_{j+1}, \dots, \mathbf{o}_{j+i}) = (\mathbf{o}_1, \dots, \mathbf{o}_i)$, where indices are read in mod i. Define $\sigma(\mathbf{p})$ the *symmetry number of* $\mathbf{p}$ to be the value i/j (see Figures 7, 8).

The symmetry number plays an important role in obtaining the size of an orbit as follows:

Theorem 5. *Given* $\mathbf{o}^v \in \mathcal{O}_n^+$*, the cardinality of* $[\mathbf{o}^v]^+$ *is*

$$|[\mathbf{o}^v]^+| = \frac{2\,\epsilon(\mathbf{p})}{\sigma(\mathbf{p})}, \tag{4.3}$$

where $\mathbf{p} = U(\mathrm{d}(\mathbf{o}^v))$*, and* $\epsilon(\mathbf{p})$ *is the number of edges in* $\mathbf{p}$.

Proof. Since $\bar{\mathrm{d}}(RL\mathbf{o}^v) = \bar{\mathrm{d}}(\mathbf{o}^v)$ and $U(\mathrm{d}(RL\mathbf{o}^v)) = U(\mathrm{d}(\mathbf{o}^v))$, the size of $[\mathbf{o}^v]^+$ is determined by $\mathbf{p}$. More precisely, the size of $[\mathbf{o}^v]^+$ equals the number of ways of identifying a vertex in $w \in \mathbf{p}$ with $v \in \bar{\mathrm{d}}(\mathbf{o}^v)$. For each vertex w in $\mathbf{p}$, since $\mathbf{p}$ is embedded in the plane, we have $\deg(w)$ distinct ways of

[1] If $c(\mathbf{p})$ is a vertex, then it is a vertex deletion and if $c(\mathbf{p})$ is an edge, then it is an edge deletion.

identifying w with $v \in \mathrm{d}(\mathbf{o}^v)$. So, if we allow repetition, the number of all possible ways of attaching $\mathbf{p}$ to $\bar{\mathrm{d}}(\mathbf{o}^v)$ is $\sum_{w \in \mathbf{p}} \deg(w) = 2\epsilon(\mathbf{p})$. But by the definition of the symmetry number, each pattern occurs exactly $\sigma(\mathbf{p})$ times. This yields (4.3). □

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