An Introduction to Digit Product Sequences

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A problem [1] and two articles [4,5] in this *Journal* have considered properties of digit sum sequences - sequences generated by iterating the function $g(n) = n + \Sigma(n)$, where $\Sigma(n)$ is the sum of the digits of a natural number n. None, though, have considered the similar idea of a digit product sequence.[1] Let n be a natural number, written in base 10, and let $\Pi(n)$ equal the product of the nonzero digits of n. Let $f(n) = n + \Pi(n)$ and define a sequence recursively by $a_{n,0} = n$, $a_{n,k+1} = f(a_{n,k})$ for $k \geq 0$. Following the notation of [5], we let $\overline{n}$ denote the sequence generated by n. The first such sequence is $\overline{1} = 1, 2, 4, 8, 16, 22, 26, 38, 62, \ldots$. The first natural number not in $\overline{1}$ is 3; it generates the sequence $\overline{3} = 3, 6, 12, 14, 18, 26, \ldots$. Similarily, $\overline{5} = 5, 10, 11, 12, \ldots$, and $\overline{7} = 7, 14, \ldots$. Each of these sequences joins a previously generated sequence, which in turn joins $\overline{1}$. This joining is illustrated by the tree in Fig. 1, in which $a \to b$ if $f(a) = b$. Thus a natural question arises: does every sequence join the main sequence $\overline{1}$?

Conjecture 1. For any natural number n, the sequence $\overline{n}$ merges with $\overline{1}$. That is, given n, there exist nonnegative integers i, j such that $f^i(1) = f^j(n)$.

In 1991 I wrote a C program to confirm this conjecture for $n \leq 1{,}000{,}000$. In 2002, Tim Smith, a Bloomsburg University undergraduate, wrote a C++ program that extends this to 10^8 and provides the other numerical evidence used in this paper. The first "stubborn" sequence is $\overline{63}$, which merges with $\overline{1}$ at $150{,}056$, its 323rd term (and the 262nd of $\overline{1}$).

In [5] natural numbers that don't appear in any previously generated sequence are called *starters*; here, we call them *unattainables*, since they cannot be attained by applying f to any other number. (Unattainables are the ends of the branches in Fig. 1.) We also call n a *descendent* of m if n is in $\overline{m}$; that is, if $f^j(m) = n$ for some nonnegative integer j.

Theorem 1. *There are infinitely many unattainables.*

Proof. Suppose there are a finite number k of unattainables. Then let l be an integer so that $9^l > k$ and let $m = 9\ldots90\ldots0$ be the $2l$ digit number consisting of l 9's followed by l 0's. Now consider the $k+1$ integers $m, m+1, \ldots, m+k$. For any $0 \leq i \leq m$ and $j \geq 1$, $f^j(m+i) \geq f(m+i) \geq m+9^l > m+k$, so none

[1] The only previous appearance of these sequences has been in [6], submitted by the author and Neal Sloane.

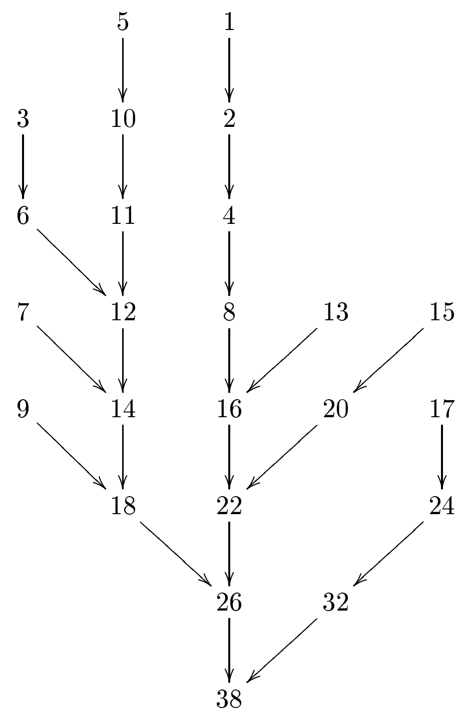

Fig. 1. Sequence tree in base 10

of these $k+1$ integers is a descendent of another. Clearly, every positive integer is either an unattainable or a descendent of an unattainable; thus these integers must correspond to $k+1$ distinct unattainables, a contradiction.

This is our only rigorous result. The rest is a series of questions, heuristic calculations, and numerical evidence.

Question 1. How common are the unattainables?

Let $u(n)$ = the number of unattainables $\leq n$. It is natural to ask if $u(n)/n$ tends to some finite limit. The following table suggests that this is the case.

n	$u(n)/n$
10	.5
100	.44
1000	.429
10^4	.4069
10^5	.39433
10^6	.388459
10^7	.3855173
10^8	.38374875

Question 2. What about bases other than 10?

In base 2, $\Pi(n) = 1$, so $f(n) = n + 1$ for any n. Thus 1 is the only unattainable and the tree has only a single branch. As the base increases, values for $\Pi(n)$ become larger, the sequences move more quickly, there are more unattainables, and the trees are more branched. One could eventually ask for a universal result on unattainables, which would find $\lim_{n\to\infty} u(n)/n$ as a function of the base b.

Question 3. How fast do the sequences grow?

We can approximate the growth of the main base 10 sequence $\{a(n)\} = \{f^{n-1}(1)\}$. On average, an n digit number has $\log_{10} n$ digits. Since a 0 is treated like a 1, the geometric mean of the 10 possible digits is $\sqrt[10]{9!} \approxeq 3.5973$. Thus

$$\Pi(n) \approxeq (\sqrt[10]{9!})^{\log_{10} n} = n^{\log_{10}(\sqrt[10]{9!})}.$$

Letting $k = \log_{10}(\sqrt[10]{9!})$, we can now write $a(n+1) \approxeq a(n) + a(n)^k$, or $a(n+1) - a(n) \approxeq a(n)^k$. This is a difference equation, which we can for the moment treat like a differential equation and write $\frac{da}{dn} = a(n)^k$. Integrating, we have $\int a^{-k} da = \int dn$, and hence

$$a(n) \approxeq [(1-k)(n+c)]^{\frac{1}{1-k}}.$$

Since $a(1) = 1$, solving for c we find $c = \frac{k}{k-1}$. Putting it all together, we have the following asymptotic approximation for $a(n)$:

$$a(n) \approxeq [(1-k)n - k]^{\frac{1}{1-k}} \text{ with } k = \log_{10} \sqrt[10]{9!} \approxeq .55598.$$

This approximation for $a(n)$ works reasonably well, as the following table shows.

n	actual $a(n)$	approx $a(n)$
100	21428	4987
10^4	5.06×10^8	1.64×10^8
10^6	1.02×10^{13}	5.23×10^{12}
10^8	1.02×10^{17}	1.67×10^{17}

It should be noted that under f even numbers tend to stay even - they can only become odd when the last digit is 0 and all others are odd - and odd numbers tend to become even. As a result, the assumption that all digits occur with equal frequency isn't entirely accurate, but as n increases the importance of the terminal digit decreases.

Question 4. How many preimages under f, or “immediate predecessors”, can an integer have?

The number 12 is the least with more than one direct predecessor (6 and 11), while 102 has three: 66, 74, and 101; 116 has four: 68, 84, 108, and 116; and 1474 has five: 898, 1366, 1393, 1426, and 1442. The number 11474 will have 6 direct predecessors, 8786 and 5 others found by adding 10000 to the direct predecessors of 1474. The number $1,011,474$ has seven direct predecessors; it should be possible to construct a sequence of numbers with a strictly increasing number of direct predecessors.

Question 5. How many sequences will contain a given number?

Looking back at Fig. 1, note that the sequences beginning at 20 of the integers less than or equal to 26 pass through 26. If we let $j(n)$ be the proportion of numbers less than or equal to n that pass through n, then $j(26) = \frac{20}{26} \approxeq .769$. What happens to the maximum values of $j(n)$ as n increases? In other words, what is $\limsup_{n\to\infty} j(n)$? It would seem natural that $j(n)$ values would decrease, but does the limsup have a nonzero lower bound?

And, lastly,

Question 6. Do digit-product sequences occur in nature?

In 1989 I was an undergraduate at Wabash College trying to devise sequences with unpredictable growth properties. It was a calculation error that first brought the merging property to my attention. It is a pleasure to thank Bonnie Gold, Ganesh Sundaram, Bill Calhoun, John Riley, and Neal Sloane, who have listened, made comments, and shared ideas since then, Tim Smith, for the programming, Bob Montante, for making that connection, and Steve Krantz, who during a summer REU in 1991 helped get these sequences rolling.

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